BOUNDED GENERATION OF STEINBERG GROUPS OVER DEDEKIND RINGS OF ARITHMETIC TYPE

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Abstract. First, we establish a definitive result which almost completely closes the problem of bounded elementary generation for Chevalley groups over arbitrary Dedekind rings R of arithmetic type with uniform bounds. Namely, for every reduced irreducible root system $\Phi \neq \mathsf{A}_1$, B_2 there exists a universal bound $L = L(\Phi)$ such that the simply connected Chevalley groups $\mathrm{G}(\Phi, R)$ have elementary width $\leq L$ for all Dedekind rings of arithmetic type R.

This result, together with the deep arithmetic results on K_2 is then applied to derive the main new result of the present paper, bounded elementary generation of the corresponding Steinberg groups $\mathrm{St}(\Phi, R)$ for simply laced root systems Φ of rank ≥ 2. Also, we prove bounded generation of $\mathrm{St}(\Phi, \mathbb{F}_q[t, t^{-1}])$ for all root systems Φ, and bounded generation of $\mathrm{St}(\Phi, \mathbb{F}_q[t])$ for all root systems $\Phi \neq \mathsf{A}_1$.

Introduction

In the present paper, we consider simply-connected Chevalley groups $G = \mathrm{G}_{\mathrm{sc}}(\Phi, R)$, and the corresponding Steinberg groups $\mathrm{St}(\Phi, R)$ over Dedekind rings of arithmetic type. G is generated by the elementary root unipotents $x_\alpha(r)$, $\alpha \in \Phi$, $r \in R$, and we are interested in the classical problem of estimating the width of G with respect to the generators $x_\alpha(r)$. The width is defined as the smallest possible m such as every element of G is representable as a product of m generators $x_\alpha(r)$. If there is no such m, we say that the width is infinite. If the width is finite, we say that G is **boundedly elementarily generated**.

The problem of bounded generation has attracted considerable attention over the last 40 years or so. We refer the reader to [KPV] containing a survey of this long activity as well as some applications to Kac–Moody groups and model theory.

To make a long story short, given a reduced irreducible root system Φ of rank ≥ 2, a Dedekind ring of arithmetic type R, and a Chevalley group $G = \mathrm{G}_{\mathrm{sc}}(\Phi, R)$, until now it was known that G is boundedly elementarily generated, and apart from

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several particular cases, two kinds of general upper estimates for the elementary width of G were available:

• explicit estimates depending on Φ and the fraction field of R;

• estimates depending on Φ alone in the case $\Phi = \mathsf{A}_l$.

Combining the methods of [KPV] and [Tr22], we are now able to come up with a complete solution in the general case. An important — and unexpected! — aspect of this work is the existence of *explicit uniform* bounds in the function case. In the number case the bounds are also uniform, but if we wish to cover all R and not just those with infinite multiplicative group $R^\times$ not explicit. For symplectic groups the following result is new even in the number case.

Theorem A. *Let Φ be a reduced irreducible root system $\neq \mathsf{A}_1$, B_2. Then there exists a constant $L = L(\Phi)$, depending on Φ alone, such that for any Dedekind ring of arithmetic type R, any element in $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$ is a product of at most L elementary root unipotents.*

Note that uniform estimates, being interesting in their own right, are indispensable for some applications, e.g. for estimating Kazhdan constants of arithmetic groups, see [Had].

Roughly, the ingredients of the proof are as follows. An amazing aspect of this argument is that the analysis of all types of rank ≥ 2, with the sole exception of B_2 is reduced to A_2. The case B_2 requires serious special analysis and we postpone it, as also the explicit bounds and some further details, to our next paper [KPV2].

• For the **number case**, when $R^\times$ is infinite there is a definitive result for $\mathrm{SL}(2, R)$ by Morgan, Rapinchuk and Sury [MRS], with a small uniform bound $L \leq 9$, which can be improved [and was improved!] in some cases. A recent paper by Kunyavskii, Morris and Rapinchuk [KMR] improves the uniform bound in the number case to $L \leq 7$ and establishes a similar result with the bound $L \leq 8$ in the **function case**.

Some uniform bound when $R^\times$ is infinite can be now easily derived by a version of the usual Tavgen's trick [Tav, Theorem 1], as described and generalised in [VSS, SSV] and [KPV].

• The uniform bound for $\mathrm{SL}(3, R)$ over imaginary quadratic rings was obtained by [Mor], see also [Tr21, Tr22]. Using the rank reduction methods based on Tavgen's lemma *and* stability, as in [KPV], we can reduce the analysis of $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$ for all *non-symplectic* root systems to $\mathrm{SL}(3, R)$.

• This leaves us with the analysis of $\mathrm{Sp}(2l, R)$, $l \geq 3$. What has been overlooked in [KPV] is that bounded generation in this case also reduces to $\mathrm{SL}(3, R)$, with the help of the symplectic lemmas on switching long and short roots [KPV]. (It is also worth noting here that similar lemmas have been used by Kairat Zakiryanov [Zak], and this reference should have been included in [KPV].)

• For the **function case** we have to rely on the reduction to rank two systems from the very start. In this case, for $\mathrm{SL}(3, R)$ an explicit uniform bound $L = 65$

is given by Trost [Tr22]. Again with the help of reduction lemmas from [KPV] it provides uniform bounds for all other groups of rank ≥ 2, with the sole exception of $\mathrm{Sp}(4, R)$.

At this point it is natural to ask whether Theorem A or maybe its weaker forms can be generalized to Steinberg groups. This question was explicitly raised by Alexei Myasnikov, at the conference GAGTA 2022. The reason was that bounded generation would have important model theoretic applications. It plays a crucial role in such problems as first order rigidity, elementary equivalence of groups, Diophantine theory, etc. So, having as a base Theorem A for Chevalley groups one can think about the similar properties for their coverings.

Again, we are interested in the bounded generation of Steinberg groups in the bounded generation in terms of the set

$$X = \{x_\alpha(r) \mid \alpha \in \Phi,\ r \in R\}$$

of elementary generators (which we continue to denote by the same letter).

However, this case turned out to be much more challenging. Apart from the bounded generation of the Chevalley groups themselves, it depends on the deep results on the finiteness of the (linear) K_2-functor, and on bunch of other difficult results of K-theory, such as stability theorem for K_2, centrality of K_2, etc.

It is not clear how one could get *uniform* bounds in this case. Even with the bounds that depend on R so far we could only prove it for the case when the root system Φ is *simply-laced*, i.e., $\Phi = \mathsf{A}_l, \mathsf{D}_l, \mathsf{E}_l$.

Theorem B. *Let Φ be a reduced irreducible simply laced root system of rank ≥ 2, and let R be a Dedekind ring of arithmetic type. If $\Phi = \mathsf{A}_2$ assume additionally that $R^\times$ is infinite. Then* $\mathrm{St}(\Phi,\ R)$ *is boundedly elementarily generated.*

The idea is to derive this result from Theorem A. It suffices to establish that the kernel $\mathrm{K}_2(\Phi, R)$ of the projection $\mathrm{St}(\Phi, R) \to \mathrm{G}_{\mathrm{sc}}(\Phi, R)$ is finite and central, and thus bounded elementary generation of $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$ implies that of $\mathrm{St}(\Phi, R)$. Here are the main sources on which we rely in this proof.

• The *stable* linear $\mathrm{K}_2(R)$ is finite, for the function case this is proven by Hyman Bass and John Tate [BaTa] and for the number case by Howard Garland [Gar]. (These finiteness results were generalised to higher K-theory by Daniel Quillen and Günter Harder, see the survey by Chuck Weibel [W]).

• However, we need similar results for the unstable K_2-functors $\mathrm{K}_2(\Phi, R)$. For the *linear* case $\mathrm{SL}(n, R)$ there is a definitive stability theorem by Andrei Suslin and Marat Tulenbaev [SuTu]. However, injective stability for Dedekind rings only starts with $n \geq 4$, so that for $\mathrm{SL}(3, R)$ one has to refer to Wilberd van der Kallen [Ka81] instead, which accounts for the extra-condition in this case.

• For other embeddings there are no stability theorems in the form we need them and starting where we want them to start. For instance, in the even orthogonal case the theorem of Ivan Panin [Pan] starts with $\mathrm{Spin}(10, R)$, whereas we would like to cover also $\mathrm{Spin}(8, R)$. In any case, there are no similar results for the exceptional embeddings.

Thus, we have to prove a comparison theorem relating $\mathrm{K}_2(\Phi, R)$ to $\mathrm{K}_2(\mathsf{A}_3, R)$. This is obtained as a corollary of partial stability results for Dedekind rings developed by Hideya Matsumoto [Mat] and *surjective* stability of K_2 for some embeddings, established by Michael Stein [St78] and the third author [Pl91, Pl98].

We also remark that the centrality of K_2 for all Chevalley groups over arbitrary rings is accomplished by the second author, Sergey Sinchuk and Egor Voronetsky, also in collaboration [Lav, Sin, LS17, LS20, Vor, LSV].

• An essential obstacle in the symplectic case is that $\mathrm{K}_2(\mathsf{C}_l, R)$ is the Milnor–Witt $\mathrm{K}_2^{\mathrm{MW}}$, rather than the usual Milnor $\mathrm{K}_2^{\mathrm{M}}$, as for all other cases (compare [Sus] for an explicit connection between $\mathrm{K}_2\mathrm{Sp}(R)$ and $\mathrm{K}_2(R)$). As is well known, it may fail to be finite, which means that our approach does not work at all in this case. This does not mean that the result itself fails, but the proof would require an entirely different idea.

But even for non-symplectic multiply laced systems, where our approach could theoretically work, we were unable to overcome occurring technical difficulties related to the K_2-stability and comparison theorems. At least, as yet.

In contrast to Theorem B, in our second generalisation of Theorem A, we put the restrictions on the ring rather than the root system.

Namely, using specific calculations of $\mathrm{K}_2(\Phi, \mathbb{F}_q[t])$ and $\mathrm{K}_2(\Phi, \mathbb{F}_q[t, t^{-1}])$ by Eiichi Abe, Jun Morita, Jürgen Hurrelbrink and Ulf Rehmann [AbMo, Hur, MoRe, Reh] we were able to establish similar results over $\mathbb{F}_q[t]$ and $\mathbb{F}_q[t,\, t^{-1}]$ also for the multiply laced systems, even the symplectic ones.

Theorem C. *Let Φ be a reduced irreducible root system, and $R = \mathbb{F}_q[t,\, t^{-1}]$ or $R = \mathbb{F}_q[t]$. In the latter case assume additionally that $\Phi \neq \mathsf{A}_1$. Then $\mathrm{St}(\Phi,\, R)$ is boundedly elementarily generated.*

The paper is organised as follows. In § 1 we recall notation and collect some preliminary results. In § 2 we reproduce the proof of Theorem A. In § 3 we recall some basic facts concerning $K_2(\Phi, R)$ and in § 4 collect the necessary facts concerning finiteness of K_2 in the arithmetic case. In § 5 we prove comparison theorems for $\mathrm{K}_2(\Phi, R)$ in the case of simply laced Φ, and thus prove Theorem B. In § 6 we recall computation of K_2 of polynomial rings, which imply Theorem C. Finally, in § 7 we mention some further generalisations.

1. Notation and preliminaries

In this section we briefly recall the notation that will be used throughout the paper and some background facts. For more details on Chevalley groups over rings see [Va91] or [VaPl], where one can find many further references.

1.1. **Chevalley groups.** Let Φ be a reduced root system and $W = W(\Phi)$ be its Weyl group. In our main results, Φ will be assumed irreducible, though in some proofs one has to use subsystems that are not. As usual, we choose an order on Φ and let Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding sets of positive, negative and fundamental roots, respectively. Further, we consider a lattice $\mathcal{P}$ intermediate between the root lattice $\mathcal{Q}(\Phi)$ and the weight lattice $\mathcal{P}(\Phi)$. Finally, let R be a commutative ring with 1, with the multiplicative group $R^\times$.

These data determine the Chevalley group $G = \mathrm{G}_{\mathcal{P}}(\Phi, R)$, of type $(\Phi, \mathcal{P})$ over R. It is usually constructed as the group of R-points of the Chevalley–Demazure group scheme $\mathrm{G}_{\mathcal{P}}(\Phi, -)$ of type $(\Phi, \mathcal{P})$. In the case $\mathcal{P} = \mathcal{P}(\Phi)$ the group G is called simply connected and is denoted by $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$. In another extreme case $\mathcal{P} = \mathcal{Q}(\Phi)$ the group G is called adjoint and is denoted by $\mathrm{G}_{\mathrm{ad}}(\Phi, R)$.

Many results do not depend on the lattice $\mathcal{P}$ and hold for all groups of a given type Φ. In all such cases, or when $\mathcal{P}$ is determined by the context, we omit any reference to $\mathcal{P}$ in the notation and denote by $\mathrm{G}(\Phi, R)$ *any* Chevalley group of type Φ over R. However in some cases specific bounds may depend on $\mathcal{P}$. Usually, we work with a simply connected group, but in some cases it is convenient to work with the adjoint group, which is then reflected in the notation.

In what follows, we also fix a split maximal torus $T = \mathrm{T}(\Phi, R)$ in $G = \mathrm{G}(\Phi, R)$ and identify Φ with $\Phi(G, T)$. This choice uniquely determines the unipotent root subgroups, X_α, $\alpha \in \Phi$, in G, elementary with respect to T. As usual, we fix maps $x_\alpha \colon R \mapsto X_\alpha$, so that $X_\alpha = \{x_\alpha(r) \mid r \in R\}$, and require that these parametrisations are interrelated by the Chevalley commutator formula with integer coefficients, see [C], [S]. The above unipotent elements $x_\alpha(r)$, where $\alpha \in \Phi$, $r \in R$, elementary with respect to $\mathrm{T}(\Phi, R)$, are also called [elementary] unipotent root elements or, for short, simply root unipotents.

Further,

$$\mathrm{E}(\Phi, R) = \big\langle x_\alpha(r) \mid \alpha \in \Phi,\ r \in R \big\rangle$$

denotes the *absolute* elementary subgroup of $\mathrm{G}(\Phi, R)$, spanned by all elementary root unipotents, or, what is the same, by all [elementary] root subgroups X_α, $\alpha \in \Phi$. For $\epsilon \in \{+, -\}$ denote

$$U^\epsilon(\Phi,\, R) = \langle x_\alpha(r) \mid \alpha \in \Phi^\epsilon,\ r \in R \rangle \le \mathrm{E}(\Phi,\, R).$$

1.2. **Steinberg groups.** Denote by $\mathrm{St}(\Phi,\, -)$ the *Steinberg group* functor corresponding to Φ. For Φ that does not have irreducible components $\cong \mathsf{A}_1$, and a commutative ring R the Steinberg group $\mathrm{St}(\Phi,\, R)$ is a group defined by the set of generators

$$\{x_\alpha(r) \mid \alpha \in \Phi,\ r \in R\}$$

subject to the *Steinberg relations*

- Additivity

$$x_\alpha(r) x_\alpha(s) = x_\alpha(r + s) \quad \text{for } \alpha \in \Phi,\ r, s \in R,$$

• Chevalley commutator formula

$$[x_\alpha(r),\, x_\beta(s)] = \prod_{\substack{i,j\in\mathbb{N}\setminus 0\\ i\alpha+j\beta\in\Phi}} x_{i\alpha+j\beta}(N_{\alpha\beta ij}\, r^i s^j) \quad \text{for } \alpha,\beta\in\Phi,\ \beta\neq-\alpha,\ r,s\in R,$$

where, as usual, $[g,\, h] = ghg^{-1}h^{-1}$ denotes the left normed commutator, whereas $N_{\alpha\beta ij}\in\mathbb{Z}$ are the structure constants of the Chevalley group $\mathrm{G}_{\mathrm{sc}}(\Phi,\, R)$.

The choice of the structure constants $N_{\alpha\beta ij}\in\mathbb{Z}$ and the order of factors in (1.2) are not unique, and we fix any possible choice, see [VaPl, Va08] for many more details and further references. It is not a problem to specify signs for classical cases, see [B]. On the other hand in [Va01] one can find specific choice of the structure constants $N_{\alpha\beta}$ for E_6, E_7 and E_8, corresponding to a positive Chevalley base (in this case automatically $i=j=1$, so that $N_{\alpha\beta 11}=N_{\alpha\beta}$ are just the structure constants of the corresponding Lie algebra). All structure constants $N_{\alpha\beta ij}$ for F_4 and G_2 are tabulated in [VaPl].

• For A_1 one needs another relation

$$w_\alpha(u)x_\alpha(r)w_\alpha(u)^{-1} = x_{-\alpha}(-u^{-2}r) \quad \text{for } \alpha\in\Phi,\ u\in R^\times,\ r\in R,$$

where

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u). \tag{1}$$

Remark. If Φ does not have irreducible components $\cong \mathsf{A}_1$ this extra relation follows from additivity and the Chevalley commutator formula.

1.3. **Arithmetic case.** Let F be a global field and S be a finite non-empty set of places of F containing all archimedean places when F is a number field. Following [BMS] we will say that

$$R = \{x\in F \mid v(x)\geq 0\ \forall v\notin S\}$$

is *the Dedekind ring of arithmetic type* defined by the set S. Obviously, R is indeed a Dedekind domain, and one can canonically identify the maximal ideals of R with the places outside S.

The following result proven by Hideya Matsumoto in [Mat, Théorème 12.7] explains why we usually prefer to work with simply connected groups.

Lemma 1.1. *Let R be a Dedekind ring of arithmetic type and Φ a reduced irreducible root system of rank at least 2. Then*

$$\mathrm{E}_{\mathrm{sc}}(\Phi,\, R) = \mathrm{G}_{\mathrm{sc}}(\Phi,\, R).$$

In fact, for $\Phi = \mathsf{A}_l$, C_l this result was established already by Hyman Bass, John Milnor and Jean-Pierre Serre in [BMS]. Recently Anastasia Stavrova generalised it to isotropic reductive groups and to polynomial rings over R, see [Sta, Corollary 1.2].

2. Uniform bounded generation of Chevalley groups: proof of Theorem A

For a group G and a subset $X \subseteq G$ such that $X^{-1} = X$, we say that G is boundedly generated by X if there exists a constant C such that any element $g \in G$ is a product of at most C elements of X.

In this section we collect and complement several known results on bounded elementary generation of Chevalley groups $\mathrm{G}(\Phi, R)$ over Dedekind rings of arithmetic type R that together amount to Theorem A. In fact, we establish *uniform* bounded generation, with the bound that depends on the root system Φ alone, and does not depend on R.

Recall that the results of [Mor, MRS, Tr22, KMR] completely solve the problem of the uniform bounded elementary generation for the special linear groups $\mathrm{SL}(n, R)$, $n \geq 3$, — and when $R^\times$ is infinite, even for $\mathrm{SL}(2, R)$.

An unexpected observation is that the methods of our previous paper [KPV] completely reduce the proof of similar result for almost all other Chevalley groups, including even the *symplectic* groups $\mathrm{Sp}(2l, R)$, $l \geq 3$, to the case of $\Phi = \mathsf{A}_2$.

Thus, *the only* case that does not follow rightaway by combining results of the above papers, is that of $\mathrm{Sp}(4, R)$. The analysis of that case is longer and far too technical, its inclusion would tilt the balance of the present paper. Below we give the proof for all other cases (somewhat more details and [better] explicit bounds are contained in [KPV2]).

2.1. Tavgen rank reduction theorem.

In most cases the reduction to A_1 or A_2 is based on the following cunning observation, whose idea goes back to the work of Oleg Tavgen [Tav]. His trick was then generalised in [VSS] and [SSV]. The following final form is proven in our previous paper [KPV, Theorem 3.2].

Lemma 2.1. *Let Φ be a reduced irreducible root system of rank $l \geq 2$, and R be a commutative ring. Let $\Delta_1, \ldots, \Delta_t$ be some subsystems of Φ, whose union contains all fundamental roots of Φ. Suppose that for all Δ_i the elementary Chevalley group $\mathrm{E}(\Delta_i, R)$ admits a unitriangular factorisation*

$$\mathrm{E}(\Delta_i, R) = U^+(\Delta_i, R)\, U^-(\Delta_i, R)\, U^+(\Delta_i, R) \ldots U^\pm(\Delta_i, R)$$

of length N (not depending on i). Then the elementary group $\mathrm{E}(\Phi, R)$ itself admits unitriangular factorisation

$$\mathrm{E}(\Phi, R) = U^+(\Phi, R)\, U^-(\Phi, R)\, U^+(\Phi, R) \ldots U^\pm(\Phi, R)$$

of the same length N.

Below, we essentially apply it to two cases, when all Δ_i's are A_1, and when all of them are A_2.

2.2. **The case when $R^\times$ is infinite.** The case where R has infinitely many units and uts field of fractions is a number field is *completely* solved, with very small *absolute* constant. We cannot describe the whole chain of events here, and mention all contributors. After the initial breakthrough by Maxim Vsemirnov [Vse], which was a first unconditional result of this sort, not depending on the GRH, Aleksander Morgan, Andrei Rapinchuk and Sury [MRS] succeded in solving the number case, with the bound $L = 9$. This bound was then improved to $L = 8$ by Bruce Jordan and Yevgeny Zaytman [JoZa] (and can be further improved in the presence of finite or real valuations in S).

Lemma 2.2. *For any Dedekind ring of arithmetic type R in a number field with the infinite multiplicative group $R^\times$ any element in $\mathrm{SL}(2, R)$ is a product of at most 8 elementary transvections.*

In the paper presently under way the first author, Dave Morris and Andrei Rapinchuk [KMR] improve the bound to $L = 7$ in the number case (which we believe is the best possible and cannot be further improved, in general). Also, they obtain a similar result in the function case, with the bound $L = 8$ (which, we believe, can be further improved to $L = 7$).

Combining Lemmas 2.1 and 2.2 we immediately get the following stronger form of Theorem A in this special case, with *explicit* bounds.

Theorem D. *For any Dedekind ring of arithmetic type R in a number field with the infinite multiplicative group $R^\times$ any element in $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$ is a product of at most $L = 8|\Phi^+|$ elementary unipotents.*

It is indeed shocking that one does not have to do essentially any extra work to solve the general case with not the best possible but still rather plausible bound (asymptotically L cannot be smaller than something like $3N$ to $4N$ anyway).

Thus, if we are not interested in actual bounds, but just in uniform boundedness, the rest of this section is dedicated to the Dedekind rings of arithmetic type with *finite* multiplicative groups in the number field case, i.e. to $\mathbb{Z}$ and the rings of integers in imaginary quadratic number fields.

As discovered by Alexander Trost [Tr22], in the function case for ranks ≥ 2 we do not have to distinguish between rings with finite and infinite multiplicative group. Below we provide a uniform proof that covers the cases D_l, $l \geq 4$, E_6, E_7, E_8 and F_4 (with worse bounds than the bound in Theorem D). And then, by two different methods, the case by case proofs that cover B_l, $l \geq 3$ and G_2, and C_l, $l \geq 3$ (with better bounds than the bound in Theorem D).

2.3. **The simply laced case and $\Phi = \mathsf{F}_4$.** By a theorem of Carter–Keller–Paige (redeveloped by Dave Morris [Mor]), bounded generation for groups of type A_l, $l \geq 2$, holds for Dedekind rings R in number fields K, with a bound depending on l and also on the degree d of K. But since for all degrees $d \geq 3$ the existence of uniform bound already follows from Theorem D, we only need to take maximum of that, and the universal bound for $d = 1, 2$.

Combining this result with the subsequent work of Trost [Tr22] on the function field case, one obtains the following result, see [Tr22, Theorem 4.1].

Lemma 2.3. *For each $l \geq 2$, there exists a constant $L = L(l) \in \mathbb{N}$ such that for any Dedekind ring of arithmetic type R, any element in $\mathrm{G}_{\mathrm{sc}}(\mathsf{A}_l, R)$ is a product of at most L elementary root unipotents.*

In fact, in the sequel we only need the special case of the above result pertaining to $\mathrm{SL}(3, R)$, which corresponds to $l = 2$. In the function case Trost [Tr22] gave the estimate $L(2) \leq 65$. No such explicit estimate is known in the number case.

Since the fundamental systems of the simply laced systems and F_4 are covered by copies of A_2, combining Lemma 2.1 with Lemma 2.3 one gets another stronger form of Theorem A, now without the assumption that $R^\times$ is infinite, but only in the special case of simply laced systems of rank ≥ 2 and F_4. This is the only part of Theorem A on which Theorem B relies.

Theorem E. *Let Φ be simply laced of rank ≥ 2 or $\Phi = \mathsf{F}_4$ and R be any Dedekind ring of arithmetic type. Then $\mathrm{G}_{\mathrm{sc}}(\Phi, R)$ is a product of at most $L(2) \cdot |\Phi^+|$ elementary unipotents.*

The bound here is very rough, since $L(2)$ is the number of *elementary* factors, the number of unitriangular ones can be much smaller. Also, the use of stability allows to get much better bounds, of the type $L = L(2) + 4|\Phi^+|$, where some multiple of $|\Phi^+|$ occurs as a summand, not as a factor. Below we only do it for $\Phi = \mathsf{B}_l$, $l \geq 3$, and $\Phi = \mathsf{G}_2$, but it could be applied also in all cases covered by Theorem E, only that such an approach would require much more work.

2.4. **The use of stability.** Here we prove Theorem A for the Chevalley groups of types $\Phi = \mathsf{B}_l$, $l \geq 3$, and $\Phi = \mathsf{G}_2$.

The following result is essentially [KPV], Proposition 4.3. In particular it implies the universal bound $L(2) + 20$ for the elementary generation of $\mathrm{E}_{\mathrm{sc}}(\mathsf{G}_2, R)$ over all Dedekind rings of arithmetic type.

Lemma 2.4. *Let R be a Dedekind ring and assume that any element of $\mathrm{G}(\mathsf{A}_2, R)$ is a product of at most L elementary root unipotents. Then any element of $\mathrm{G}(\mathsf{G}_2, R)$ is a product of at most $L + 20$ elementary root unipotents.*

A similar result for $\Phi = \mathsf{B}_3$ is also essentially proven in [KPV, Section 6.2]. The situation is only marginally more complicated in view of the fact that the vector representation used there furnishes not the simply connected group but the adjoint one. Of course, the case $\Phi = \mathsf{B}_l$, $l \geq 3$, is reduced to $\Phi = \mathsf{B}_3$ either by Lemma 2.1 in conjunction with Lemma 2.3, or again by stability (see Corollary 2.6 below).

Lemma 2.5. *Let R be a Dedekind ring such that of $\mathrm{G}_{\mathrm{sc}}(\mathsf{A}_2, R)$ is a product of L elementary root unipotents. Then any element of $\mathrm{E}_{\mathrm{ad}}(\mathsf{B}_3, R)$ is a product of at most $L+31$ elementary root unipotents.*

Proof. First, observe that [KPV, Lemma 6.3] and [KPV, Lemma 5.1] are valid for any Dedekind ring R (although they are formally stated under the assumption $R = \mathbb{F}_q[X]$).

By [KPV, Lemma 6.3], each element $x \in \mathrm{E}_{\mathrm{ad}}(\mathsf{B}_3, R)$ is a product of an image of $y \in \mathrm{G}_{\mathrm{ad}}(\mathsf{B}_2, R)$ and at most 21 elementary root unipotents. However, since the image of y in $\mathrm{G}_{\mathrm{ad}}(\mathsf{B}_3, R)$ is elementary and, in particular, lies in the kernel of the spinor norm, we conclude that y itself lies in the kernel of the spinor norm [Bas, Proposition 3.4.1], and therefore y is the image of some $z \in \mathrm{G}_{\mathrm{sc}}(\mathsf{B}_2, R)$ [Bas, (3.3.4)].

Next, by [KPV, Lemma 5.1], z is equal to a product of the image of some $w \in \mathrm{G}_{\mathrm{sc}}(\mathsf{A}_1, R)$ and at most 10 elementary root unipotents (where $\mathsf{A}_1 \subset \mathsf{B}_2$ is the inclusion on long roots). Therefore x is the product of the image of w in $\mathrm{G}_{\mathrm{ad}}(\mathsf{B}_3, R)$ and at most 31 elementary root unipotents.

However, since the inclusion $\mathsf{A}_1 \subset \mathsf{B}_3$ factors through A_2, we conclude that x is a product of the image of some element from $\mathrm{G}_{\mathrm{sc}}(\mathsf{A}_2, R)$ and at most 31 elementary root unipotents. The claim follows. $\square$

Corollary 2.6. *In the notation of Lemma 2.3, for any Dedekind ring of arithmetic type R, any element of $\mathrm{G}_{\mathrm{sc}}(\mathsf{B}_3, R)$ is a product of at most $L(2)+41$ elementary root unipotents.*

Proof. Any element of $\mathrm{G}_{\mathrm{sc}}(\mathsf{B}_3, R)$ is elementary by Lemma 1.1, and therefore its image in $\mathrm{G}_{\mathrm{ad}}(\mathsf{B}_3, R)$ is a product of at most $L(2)+31$ elementary root unipotents by Proposition 2.5. However, $\mathrm{G}_{\mathrm{sc}}(\mathsf{B}_3, R)$ is a central extension of $\mathrm{G}_{\mathrm{ad}}(\mathsf{B}_3, R)$ with the kernel cyclic of order 2. The generator of the kernel comes from $\mathrm{G}_{\mathrm{sc}}(\mathsf{D}_3, R)$ [Bas, (3.4)], where it can be expressed as a product of at most 10 elementary root unipotents by [HO'M, Theorem 7.2.12]. $\square$

2.5. **The case of $\mathrm{Sp}(2l, R)$, $l \geq 3$.** Thus, we are left with the analysis of the symplectic groups $\mathrm{Sp}(2l, R)$, $l \geq 3$. Quite amazingly, the results of [KPV] and [Tr22] allow to reduce $\mathrm{Sp}(6, R)$ to $\mathrm{SL}(3, R)$ as well. Actually, in the special case $R = \mathbb{Z}$ the idea of such a reduction was contained already in Zakiryanov's thesis, see [Zak], but we have not realised this fact before rediscovering the same idea in the general case in March 2023[1]. Of course, as above, the case $\Phi = \mathsf{C}_l$, $l \geq 3$, is immediately reduced to $\Phi = \mathsf{C}_3$ either by Lemmas 2.1 and 2.3, or again by stability as in Section 2.4.

[1]It is wrongly claimed in [Zak] that $\mathrm{Sp}(4, \mathbb{Z})$ is not boundedly generated. As a result this work has not been given the credit it deserves. In particular, we should have cited it in the historical survey of [KPV].

Lemma 2.7. *Let R be a Dedekind ring such that of* $\mathrm{G}_{\mathrm{sc}}(\mathsf{A}_2,\,R)$ *is a product of L elementary root unipotents. Then any element of* $\mathrm{E}_{\mathrm{sc}}(\mathsf{C}_3,\,R)$ *is a product of at most $L+40$ elementary root unipotents.*

Proof. As in the case of $\Phi = \mathsf{B}_3$ we first invoke [KPV, Lemma 6.1], to reduce a matrix from $\mathrm{Sp}(6,\,R)$ to a matrix from $\mathrm{Sp}(4,\,R)$ by 16 elementary transformations. Then we invoke [KPV, Lemma 5.1], to reduce a matrix from $\mathrm{Sp}(4,\,R)$ to a matrix from $\mathrm{Sp}(2,\,R) = \mathrm{SL}(2,\,R)$ in *long* root position by 10 elementary transformations. After that we invoke [CaKe, Lemma 3] to get a square in the non-diagonal position by 4 elementary transformations in the number case or following [Tr22, Proof of Lemma 3.1] do the same in the function case by 3 elementary transformations. We remark that the case of characteristic 2 did not get special consideration in [Tr22, Lemma 3.1]; however, it can be settled as in the proof of [KPV, Lemma 5.4].

Now, we can invoke [KPV, Lemma 5.15], to move such a matrix in the long root fundamental position to a matrix in the short root fundamental position by 10 elementary transformations. At this stage we can apply Lemma 2.3 to the *short* root $\widetilde{\mathsf{A}}_2 \leq \mathsf{C}_3$, which gives us $\leq 16 + 10 + 4 + 10 + L$ elementary moves in all cases. □

Remark 2.8. As we have seen above, the analysis of bounded elementary generation for *all* cases apart from $\mathrm{Sp}(4, R)$ readily reduces to $\mathrm{SL}(3, R)$. The case $\mathrm{Sp}(4, R)$ does not, but it can be done by the same methods as in [Mor, Tr21, Tr22], using our Lemmas from [KPV]. The details, and the resulting explicit bounds in the function case, are reproduced in [KPV2].

Remark 2.9. We do not claim that our bounds are sharp in any sense. Of course, in the number case $L(2)$ is not explicit at all. But even for those rings for which there are explicit bounds they can be improved. Even the above reduction to $L(2)$ is far from being optimal and can be *easily* improved for all cases $\Phi \neq \mathsf{A}_l$, see [KPV2] for some results in this spirit.

3. K_2 modeled on Chevalley groups

In this section we collect the classical results on $\mathrm{K}_2(\Phi,\,-)$ which we will use in this paper.

There is a natural map from $\mathrm{St}(\Phi,\,R)$ to $\mathrm{G}_{\mathrm{sc}}(\Phi,\,R)$ sending generators of the Steinberg group $x_\alpha(r)$ to elementary root unipotents $x_\alpha(r)$ of the Chevalley group, $\alpha \in \Phi$, $r \in R$. Following [St78] we denote

$$\mathrm{K}_2(\Phi,\,R) = \mathrm{Ker}\big(\mathrm{St}(\Phi,\,R) \to \mathrm{G}_{\mathrm{sc}}(\Phi,\,R)\big).$$

Let R be a commutative ring. Following Steinberg for $\alpha \in \Phi$, $u \in R^\times$, we define the elements

$$h_\alpha(u) = w_\alpha(u) w_\alpha(-1) \tag{2}$$

of the Steinberg group $\mathrm{St}(\Phi, R)$, where $w_\alpha(u)$ is defined in § 1 (1), see [St71].

Further, for two invertible elements $u,\, v\in R^\times$ we define the **Steinberg symbol**

$$\{u,\, v\}_\alpha = h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}$$

The following fact is well-known, see, for instance, [St73, Proposition 1.3].

Lemma 3.1. *For a ring R and a reduced irreducible root system Φ elements $\{u,\, v\}_\alpha$ for $u,\, v\in R^\times$, $\alpha\in\Phi$, are central in $\mathrm{St}(\Phi,\, R)$ and belong to $\mathrm{K}_2(\Phi,\, R)$.*

The following classical result is due to Matsumoto [Mat, Corollaire 5.11].

Lemma 3.2. *Let k be a field, Φ be a reduced irreducible root system.*
1) *The group $\mathrm{K}_2(\Phi,\, k)$ is generated by $\{u,\, v\}_\alpha$ for any fixed* long *root $\alpha\in\Phi$, and all $u,\, v\in R^\times$.*
2) *Let Φ be a non-symplectic reduced irreducible root system (i.e., $\Phi\neq \mathsf{A}_1,\, \mathsf{B}_2,\, \mathsf{C}_l$). Consider any embedding $\mathsf{A}_2\hookrightarrow\Phi$ on* long *roots. Then the induced map*

$$\mathrm{K}_2(\mathsf{A}_2,\, k)\to\mathrm{K}_2(\Phi,\, k)$$

is in fact an isomorphism.
3) *For a symplectic reduced irreducible root system $\mathsf{C}_1=\mathsf{A}_1$, $\mathsf{C}_2=\mathsf{B}_2$, or C_l, $l\geq 3$, consider any embedding $\mathsf{A}_1\hookrightarrow\mathsf{C}_l$ on* long *roots. Then the induced map*

$$\mathrm{K}_2(\mathsf{A}_1,\, k)\to\mathrm{K}_2(\mathsf{C}_l,\, k)$$

is in fact an isomorphism.

Remark 3.3. In fact, Matsumoto describes $\mathrm{K}_2(\Phi,\, k)$ in terms of generators and relations in [Mat, Corollaire 5.11]. In modern terms, Matsumoto proved that $\mathrm{K}_2(\Phi,\, k)$ coincides with Milnor $\mathrm{K}_2^{\mathrm{M}}(k)$ for non-symplectic Φ and with Milnor–Witt $\mathrm{K}_2^{\mathrm{MW}}(k)$ for symplectic Φ. However, we will not need the explicit description of relations in this paper.

We will also need the following stabilisation results. The next statement is a particular case of the Suslin–Tulenbaev theorem, see [SuTu, Corollary 4.2].

Lemma 3.4. *Let R be a Dedekind domain, then the natural map*

$$\mathrm{K}_2(\mathsf{A}_l,\, R)\to\mathrm{K}_2(\mathsf{A}_{l+1},\, R)$$

is surjective for $l\geq 2$ and injective for $l\geq 3$.

For the root systems other than A_l we only have the surjective stability part, established by Michael Stein [St78, Corollary 3.2, Theorem 4.1].

Lemma 3.5. *Let R be a Dedekind domain. Consider the following embeddings of root systems $\Psi\hookrightarrow\Phi$:*

- *natural embedding $\mathsf{D}_l\hookrightarrow\mathsf{D}_{l+1}$ for $l\geq 3$;*
- *natural embedding $\mathsf{E}_l\hookrightarrow\mathsf{E}_{l+1}$ for $l=6,7$;*
- *the embedding $\mathsf{D}_5=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}\hookrightarrow\mathsf{E}_6$ (with numbering according to Bourbaki* [B, Table V]*).*

Then the induced map

$$\mathrm{K}_2(\Psi,\, R) \to \mathrm{K}_2(\Phi,\, R)$$

is surjective.

On the other hand, for Dedekind rings of arithmetic type with infinite multiplicative groups the bounds in surjective/injective stability can be improved by 1. This was done by Wilberd van der Kallen [Ka81, Theorem 1].

Lemma 3.6. *Let R be a Dedekind ring of arithmetic type with infinitely many units. Then the natural map*

$$\mathrm{K}_2(\mathsf{A}_l,\, R) \to \mathrm{K}_2(\mathsf{A}_{l+1},\, R)$$

is surjective for $l \geq 1$ and injective for $l \geq 2$.

Finally, we will need also another result by Michael Stein claiming that surjective stability implies centrality of K_2 [St71, Theorem 5.1].

Lemma 3.7. *Let Π denote a set of simple roots in a reduced root system Φ. For $\alpha \in \Pi$ let $\Psi \subseteq \Phi$ be a subsystem generated by $\Pi \setminus \alpha$. Then*

$$\mathrm{K}_2(\Phi,\, R) \cap \mathrm{Im}\big(\mathrm{St}(\Psi,\, R) \to \mathrm{St}(\Phi,\, R)\big)$$

is a central subgroup of $\mathrm{St}(\Phi,\, R)$ for any commutative ring R.

4. Stable linear K_2

Recall that

$$\mathrm{K}_2(R) = \lim_{l\to\infty} \mathrm{K}_2(\mathsf{A}_l,\, R)$$

is the usual stable linear K_2-functor for any ring R (cf. [W, Chapter III, Section 5]).

For a place v of a field F let κ_v denote the corresponding residue class field and

$$\partial_v \colon \mathrm{K}_2(F) \to \kappa_v^\times$$

the corresponding residue homomorphism (also called *tame symbol*) sending generators $\{x,\, y\}_\alpha$ of $\mathrm{K}_2(F)$ (see Proposition 3.2) to

$$(-1)^{v(x)v(y)}\,\overline{\left(\frac{y^{\,v(x)}}{x^{\,v(y)}}\right)} \in \kappa_v^\times,$$

see [W, Chapter III, Lemma 6.3]. We will need the following result due to Christophe Soulé, see, for instance, [W, Chapter V, Theorem 6.8].

Lemma 4.1. *Let R be a Dedekind domain whose field of fractions F is a global field. Then there is an exact sequence*

$$0 \to \mathrm{K}_2(R) \to \mathrm{K}_2(F) \xrightarrow{\oplus\partial_{\mathfrak{p}}} \bigoplus_{\mathfrak{p}} (R/\mathfrak{p})^\times \to 0,$$

where the first arrow is induced by the natural inclusion $R \hookrightarrow F$, and the sum is taken over all non-zero prime ideals $\mathfrak{p}$ of R.

Our proof of Theorem B heavily relies on the following classical result. In the function case this is due to Hyman Bass and John Tate [BaTa, Chapter II, Theorem 2.1]. In the number case this was first established by Howard Garland [Gar] by analytic methods (see also [BaTa, Chapter II, Remark after Theorem 2.1]).

Lemma 4.2. *Let F be a global field. Then*

$$\mathrm{H}_2 = \mathrm{Ker}\left(\mathrm{K}_2(F) \xrightarrow{\oplus\partial_v} \bigoplus \kappa_v^\times\right),$$

where the sum is taken over all finite *places of F, is finite.*

Remark 4.3. The case $\mathrm{char}\, F = p > 0$ of Proposition 4.2 was generalised to higher K-theory by Harder [Har, Korollar 3.2.3].

The following corollary is perhaps also well-known, however we do not know the reference and provide an argument for the convenience of the reader.

Corollary 4.4. *Let R be a Dedekind ring of arithmetic type defined by the set of places S. Then $\mathrm{K}_2(R)$ is finite.*

Proof. By Proposition 4.1 we get an exact sequence

$$0 \to \mathrm{K}_2(R) \to \mathrm{K}_2(F) \xrightarrow{\oplus\partial_v} \bigoplus_{v\notin S} \kappa_v^\times \to 0.$$

Therefore we may consider $\mathrm{K}_2(R)$ as a subgroup of $\mathrm{K}_2(F)$, and restricting ∂_v to it we get an exact sequence

$$0 \to \mathrm{H}_2 \to \mathrm{K}_2(R) \xrightarrow{\oplus\partial_v} \bigoplus_{\substack{v\in S\\ \text{finite}}} \kappa_v^\times \to 0,$$

where H_2 is the group from Proposition 4.2. However, S is a finite set, κ_v is a finite field for a finite place v, and H_2 is finite by Proposition 4.2. □

We will denote by $I(k)$ the fundamental ideal of the Witt ring of symmetric bilinear forms $W(k)$ of a field k, see [MH]. Following [Sus, MoRe] we denote $\mathrm{K}_2\mathrm{Sp}(R) = \lim\limits_{l} \mathrm{K}_2(\mathsf{C}_l, R)$. We will need the following result due to Andrei Suslin [Sus, Theorem 6.5].

Lemma 4.5. *For any field k there is an exact sequence*

$$0 \to I^3(k) \to \mathrm{K}_2\mathrm{Sp}(k) \to \mathrm{K}_2(k) \to 0.$$

5. Comparison theorems for K_2: proof of theorem B

5.1. **Simply laced root systems.** The aim of this section is to prove the following result, which, together with Theorem A (or, in fact, already with its special case, Theorem E), implies Theorem B. We believe it is very interesting in its own right, and may have further applications.

Theorem F. *Let R be a Dedekind ring of arithmetic type, and let $\Phi \neq \mathsf{A}_1$ be a simply laced reduced irreducible root system (i.e., $\Phi = \mathsf{A}_l,\ \mathsf{D}_l,\ \mathsf{E}_l,\ l \neq 1$). Assume additionally that*

$$\text{either} \quad \mathrm{rk}(\Phi) \geq 3 \quad \text{or} \quad R \text{ has infinitely many units.}$$

Then $\mathrm{K}_2(\Phi,\ R)$ is a central subgroup of $\mathrm{St}(\Phi,\ R)$, and, moreover,

$$\mathrm{K}_2(\Phi,\ R) = \mathrm{K}_2(R),$$

in particular, $\mathrm{K}_2(\Phi,\ R)$ is finite.

The above theorem was probably never published, but it may be mostly known to experts. In any case, for $\Phi = \mathsf{D}_l$, $l \geq 5$, it follows from a result by Ivan Panin [Pan, Theorem 6.1]. Moreover, in the same paper Panin proves in [Pan, Theorem 9.1] a similar stabilisation result also for higher orthogonal K-theory.

However, to cover also $\Phi = \mathsf{D}_4, \mathsf{E}_6, \mathsf{E}_7$ and E_8 we start with the following result.

Lemma 5.1. *Let R be a Dedekind ring of arithmetic type, and let Φ denote a simply laced reduced irreducible root system of $\mathrm{rk}(\Phi) \geq 3$. Then there exists an embedding $\mathsf{A}_3 \hookrightarrow \Phi$ such that the induced map*

$$\mathrm{K}_2(\mathsf{A}_3,\ R) \to \mathrm{K}_2(\Phi,\ R)$$

is an isomorphism.

Proof. By Proposition 3.5 we conclude that there exists an embedding $\mathsf{A}_3 \hookrightarrow \Phi$ such that the induced map

$$\mathrm{K}_2(\mathsf{A}_3,\ R) \to \mathrm{K}_2(\Phi,\ R)$$

is surjective. Let F be the field of fractions of R. By Proposition 3.2 we conclude that the induced map

$$\mathrm{K}_2(\mathsf{A}_3,\ F) \to \mathrm{K}_2(\Phi,\ F)$$

is an isomorphism. Moreover, by Proposition 3.4 we have $\mathrm{K}_2(\mathsf{A}_3,\ R) = \mathrm{K}_2(R)$, and therefore the natural map

$$\mathrm{K}_2(\mathsf{A}_3,\ R) \to \mathrm{K}_2(\mathsf{A}_3,\ F)$$

is injective by Proposition 4.1. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{K}_2(\mathsf{A}_3,\ R) \arrow[r, tail] \arrow[d, two heads] & \mathrm{K}_2(\mathsf{A}_3,\ F) \arrow[d, "\cong"] \\
\mathrm{K}_2(\Phi,\ R) \arrow[r] & \mathrm{K}_2(\Phi,\ F).
\end{tikzcd}$$

The claim follows by a simple diagram chase. $\square$

Now we are all set to finish the proof of Theorem F.

Proof of Theorem F. For a root system Φ such that $\mathrm{rk}(\Phi) \geq 3$ consider the embedding $\mathsf{A}_3 \hookrightarrow \Phi$ from Lemma 5.1 and use that $\mathrm{K}_2(\mathsf{A}_3, R) = \mathrm{K}_2(R)$ by Proposition 3.4 to get the equality

$$\mathrm{K}_2(\Phi, R) = \mathrm{K}_2(R).$$

To prove centrality use the surjectivity of the map $\mathrm{K}_2(\mathsf{A}_3, R) \to \mathrm{K}_2(\Phi, R)$ for $\mathrm{rk}(\Phi) \geq 4$ or the surjectivity of the map $\mathrm{K}_2(\mathsf{A}_2, R) \to \mathrm{K}_2(\mathsf{A}_3, R)$ from Proposition 3.4 together with Proposition 3.7. The finiteness of $\mathrm{K}_2(R)$ follows from Corollary 4.4.

For $\Phi = \mathsf{A}_2$ the claim follows from Proposition 3.6 (together with Proposition 3.7 and Corollary 4.4). □

Theorem B is a direct consequence of Theorem A and Theorem F.

5.2. **Non-simply laced root systems.** As we see in the next section, for non-simply laced root systems Φ the equality

$$\mathrm{K}_2(\Phi, R) = \mathrm{K}_2(R)$$

may hold for some Dedekind rings R

However, the following counter-example shows that it certainly fails in the symplectic case, in general.

Lemma 5.2. *For $l \geq 3$ one has*

$$\mathrm{K}_2(\mathsf{C}_l, \mathbb{Z}) \neq \mathrm{K}_2(\mathbb{Z}).$$

Proof. Recall that $\mathrm{K}_2(\mathbb{Z}) = \mathbb{Z}/2$ (see, e.g., [M, Corollary 10.2]). However, the map $\mathrm{K}_2(\mathsf{C}_l, \mathbb{Z}) \to \mathrm{K}_2(\mathsf{C}_{l+1}, \mathbb{Z})$ is surjective by [St78, Corollary 3.2], and

$$\lim_{l} \mathrm{K}_2(\mathsf{C}_l, \mathbb{Z}) = \mathrm{K}_2\mathrm{Sp}(\mathbb{Z}) = \mathbb{Z},$$

see, e.g., [Sch, Theorem 2.1]. □

Thus, there is no hope to prove the symplectic analogue of Theorem B along the same lines. This does not mean that bounded generation of $\mathrm{St}(\mathsf{C}_l, R)$, $l \geq 3$, fails. But if it holds, its proof would require some completely different ideas.

6. K_2 for polynomial rings: proof of Theorem C

In this section we consider the polynomial rings $R = \mathbb{F}_q[X]$ and $R = \mathbb{F}_q[X, X^{-1}]$. Bounded generation of the Chevalley groups themselves in these cases is proven in [KPV]. On the other hand, for these rings $\mathrm{K}_2(\Phi, R)$ is generated by the usual Steinberg symbols $\{u, v\}_\alpha$, $u, v \in R^\times$, which allows to explicitly calculate it. Observe that this is rarely the case for more general Dedekind domains, where one needs higher symbols.

Recall that $I(k)$ denotes the fundamental ideal of the Witt ring of symmetric bilinear forms of a field k (see [MH]). We will use the following well-known facts.

The first statement below is the [second] Steinberg theorem, it is proven, e.g., in [S] or in [M, Corollary 9.13]. For the second statement see, e.g., [MH, Chapter IV, Lemma 1.5].

Lemma 6.1. *Let $\mathbb{F}_q$ be a finite field. Then*

1) $\mathrm{K}_2(\mathbb{F}_q) = 0$;
2) $I^2(\mathbb{F}_q) = 0$.

The next result is an immediate corollary of a result by Ulf Rehmann [Reh].

Lemma 6.2. *Let $R = \mathbb{F}_q[X]$ be the polynomial ring over a finite field, Φ any reduced irreducible root system. Then*

$$\mathrm{K}_2(\Phi,\, R) = \mathrm{K}_2(R) = 0.$$

Proof. For any field k the natural embedding induces an isomorphism

$$\mathrm{K}_2(\Phi,\, k) \cong \mathrm{K}_2(\Phi,\, k[t])$$

by [Reh, Korollar zu Satz 1]. It remains to use that $\mathrm{K}_2(\mathbb{F}_q) = 0 = I^3(\mathbb{F}_q)$ by Proposition 6.1, and apply Proposition 4.5. ☐

The key role in the proof of Theorem C is played by the following observation of the second author and Sergei Sinchuk, see [LS20, Lemma 2.2], which in turn relies on deep results of Jürgen Hurrelbrink, Eiichi Abe and Jun Morita.

Lemma 6.3. *For an arbitrary field k and a non-symplectic root system Φ there is an exact sequence of abelian groups*

$$0 \to \mathrm{K}_2(\Phi,\, k) \to \mathrm{K}_2(\Phi,\, k[X,\, X^{-1}]) \to k^\times \to 0$$

split by the map

$$\{X,\, -\}_\alpha \colon k^\times \to \mathrm{K}_2(\Phi,\, k[X,\, X^{-1}])$$

for any fixed long root $\alpha \in \Phi$. In particular, the natural embedding induces an injective map

$$\mathrm{K}_2(\Phi,\, k[X,\, X^{-1}]) \hookrightarrow \mathrm{K}_2(\Phi,\, k(X)).$$

Proof. Since $\mathrm{K}_2(\Phi,\, F) = \mathrm{K}_2(F)$ for any field F by Proposition 3.2, the second statement follows from the first one.

Indeed, the map $\mathrm{K}_2(k) \to \mathrm{K}_2(k(X))$ is injective, e.g., by Milnor's theorem [W, Chapter III, Example 6.1.2, Theorem 7.4], and $k^\times \to \mathrm{K}_2(k(X))$ is injective as a splitting to the residue homomorphism ∂_X corresponding to an order of the zero or the pole at $X = 0$ (see Subsection 4, cf. also [LS20, Proof of Lemma 2.2]).

The first statement is proven for $\Phi \neq \mathsf{G}_2$ in [Hur, Satz 3] (cf. [LS20, Lemma 2.2], Proposition 3.2). For $\Phi = \mathsf{G}_2$ consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{K}_2(\mathsf{A}_2,\, k[X,\, X^{-1}]) \arrow[r, two heads] \arrow[d, tail] & \mathrm{K}_2(\mathsf{G}_2,\, k[X,\, X^{-1}]) \arrow[d] \\
\mathrm{K}_2(\mathsf{A}_2,\, k(X)) \arrow[r, "\cong"] & \mathrm{K}_2(\mathsf{G}_2,\, k(X)),
\end{tikzcd}$$

where the horizontal arrows are induced by the natural embedding $\mathsf{A}_2 \hookrightarrow \mathsf{G}_2$ as a set of long roots.

Since $\mathrm{K}_2(\mathsf{G}_2,\, k[X,\, X^{-1}])$ is generated by $\{u,\, v\}_\alpha$ for $u,\, v \in k[X,\, X^{-1}]^\times$, and $\alpha \in \mathsf{G}_2$ a fixed long root by [AbMo, Corollary 6], we conclude that the top horizontal arrow is surjective. The bottom horizontal arrow is an isomorphism by Proposition 3.2. The injectivity of left horizontal arrow is already discussed above. Therefore, a simple diagram chase shows that the top horizontal arrow is in fact an isomorphism. ☐

Modulo this results we can now summarise the Hurrelbrink, Abe–Morita, and Morita–Rehmann as follows.

Lemma 6.4. *Let* $R = \mathbb{F}_q[X,\, X^{-1}]$ *be the Laurent polynomial ring over a finite field,* Φ *any reduced irreducible root system. Then* $\mathrm{K}_2(\Phi,\, R)$ *is a central subgroup of* $\mathrm{St}(\Phi,\, R)$*, and, moreover,*

$$\mathrm{K}_2(\Phi,\, R) = \mathrm{K}_2(R) = \mathbb{F}_q^\times.$$

Proof. For any field k there is an isomorphism

$$\mathrm{K}_2(\Phi,\, k[X,\, X^{-1}]) \cong \mathrm{K}_2(k) \oplus k^*$$

for $\Phi \neq \mathsf{C}_l$ by Lemma 6.3, and

$$\mathrm{K}_2(\mathsf{C}_l,\, k[X,\, X^{-1}]) \cong \mathrm{K}_2\mathrm{Sp}(k) \oplus \mathrm{P}(k)$$

where $\mathrm{P}(k) = k^\times \oplus I^2(k)$ for $l \geq 1$ by [MoRe, Theorem B]. It remains to observe that $I^2(\mathbb{F}_q) = 0 = \mathrm{K}_2(\mathbb{F}_q)$ by Proposition 6.1, and therefore (using Proposition 4.5) one has

$$\mathrm{K}_2(\Phi,\, R) = \mathrm{K}_2(R) = \mathbb{F}_q^\times.$$

To prove the first statement observe that $\mathrm{K}_2(\Phi,\, k[X,\, X^{-1}])$ is generated by the Steinberg symbols $\{u,\, v\}_\alpha$ for $u,\, v \in k[X,\, X^{-1}]^\times$ by [AbMo, Corollary 6], in particular, it is a central subgroup of $\mathrm{St}(\Phi,\, k[X,\, X^{-1}])$ by Proposition 3.1. ☐

Now Theorem C is a direct consequence of [KPV, Theorem A and Theorem C] and Propositions 6.2, 6.4.

7. Concluding remarks

Here we mention some eventual generalisations of the results of the present paper.

• Let $I \trianglelefteq R$ be an ideal of R. In the present paper we addressed the *absolute* case $I = R$ alone. However, it makes sense to ask similar questions for the *relative* case, in other words for the relative elementary subgroups $E(\Phi, R, I)$ of level $I \trianglelefteq R$. (Unlike the absolute case, $E(\Phi, R, I)$ does not necessarily coincide with the congruence subgroups $G(\Phi, R, I)$ of the same level.)

The expectation is that for $E(\Phi, R, I)$ one can get similar *uniform* bounds in terms of the elementary conjugates

$$x_{-\alpha}(r)x_\alpha(s)x_{-\alpha}(-r), \qquad \alpha \in \Phi, \quad s \in I, \quad r \in R.$$

Otherwise, one could look at the *true* = *unrelativised* elementary subgroup $E(\Phi, I)$ of level I generated by $x_\alpha(s)$, $\alpha \in \Phi$, $s \in I$, and ask a similar question in terms of the elementary generators of level I.

Problem 1. *Establish analogues of Theorem A for the elementary groups $E(\Phi, I)$ and $E(\Phi, R, I)$ of level I, with uniform bounds not depending on either R or I.*

Some partial results in this direction for classical groups are obtained by Sergei Sinchuk and Andrei Smolensky [SiSm] and by Pavel Gvozdevsky [Gv23], but their bounds are not uniform. As a more remote goal one could think of generalisations to birelative subgroups, see [HSVZ] for this context.

• It is known that bounded elementary generation is closely related to many other flavours of bounded generation, including, in particular, finite width in commutators.

Namely, Alexei Stepanov [Ste] has discovered that there exists a *universal* bound $L = L(\Phi)$, depending on Φ alone, such that the commutators $[x, y]$, $x \in G(\Phi, R)$, $y \in E(\Phi, R)$ have elementary width $\leq L$ over an *arbitrary* commutative ring R. (Previously in [SiSt] and [StVa] similar results were proven for finite-dimensional rings, with the bound L depending on Φ and dimension $\dim(R)$).

Thus, for all Chevalley groups bounded elementary generation and bounded commutator width are equivalent! Morally, this says that there are very very few commutators in $x \in G(\Phi, R)$, not much more than elementary generators.

But of course the actual bound for commutator width will be much smaller than the elementary width. So far, using the results of Andrei Smolensky [Smo] we were able to prove that for a Dedekind ring of arithmetic type R with the infinite multiplicative group $R^\times$ every element of $G_{\mathrm{sc}}(\Phi, R)$ is a product of not more than 4,5,6 or 7 commutators, depending on the type Φ and on whether R is a number ring or a function ring, this result is contained in [KPV2].

• Above, Theorem F is stated only for *simply laced* root systems. But there is very strong evidence that suggests that the same is true for all *non-symplectic* root systems. We strongly believe in the following statement and are tempted to call it a conjecture.

Problem 2. *Let R be the Dedekind ring of arithmetic type with infinitely many units and Φ be a reduced irreducible non-symplectic root system* (*i.e.*, $\Phi \neq \mathsf{A}_1$, B_2, C_l)*. Then* $\mathrm{K}_2(\Phi,\ R)$ *is a finite central subgroup of* $\mathrm{St}(\Phi,\ R)$*.*

In particular, $\mathrm{St}(\Phi,\ R)$ *is boundedly generated by the set* $X = \{x_\alpha(r) \mid r \in R,\ \alpha \in \Phi\}$*.*

Remark 7.1. The centrality of $\mathrm{K}_2(\Phi,\ R)$ in fact holds for any commutative ring R and any reduced irreducible root system Φ of rank at least 3. This result was first proven in [Ka77] for $\Phi = \mathsf{A}_l$, and then in [Lav, Sin, LS17, Vor, LSV] for other root systems. However, if Φ has rank 2 then as shown in [Wen] centrality may fail even for some very nice rings.

As Proposition 5.2 shows, one cannot expect an analogue of this to hold in the symplectic case. However, one can still hope that $\mathrm{St}(\mathsf{C}_l, R)$, $l \geq 3$, is boundedly generated and could try to approach it by other means. We state the following problem.

Problem 3. *Let R be the Dedekind ring of arithmetic type with infinitely many units. Is $\mathrm{St}(\mathsf{C}_l, R)$ boundedly elementarily generated?*

• Yet another aspect is that Theorem B is much weaker than Theorem A in that the bound depends on the size of $\mathrm{K}_2(R)$. The natural question arises, whether there is a uniform bound in this case too? However, it seems that an answer to this question is presently out of range, and in any case should involve some hard core arithmetic.

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