THE HISTORY OF UNIVERSAL ALGEBRAIC GEOMETRY

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Abstract. In this paper we give a sketch of the evolution of Universal Algebraic Geometry and Logical Geometry. These topics were systematically developed in the works by B.Plotkin and his followers starting from 1996. A similar theory was constructed by V.Remeslennikov, A.Miasniakov and others. We provide the reader with the examples of problems typical to universal algebraic geometry and discuss thoroughly the open problem on isomorphism of isotypic finitely generated groups.

1. Boris Plotkin recalls

1.1. **Universal Algebraic Geometry.** The idea of universal algebraic geometry arose in the middle of the 1990s. I attended the algebraic seminar in Jerusalem (now this seminar is named after Shimshon Amitsur). The talk of Zlil Sela was devoted to geometric ideas and methods related to the solution of the Tarski problem.

This model-theoretic problem remained open since the end of the 1940s. Its meaning is whether it is possible to distinguish between two free non-abelian groups by means of their elementary theories.

During the talk Sela claimed that the way to a solution of the Tarskii problem goes through the theory of solutions of systems of polynomial equations over a free group. In other words, the problem has a geometrical nature, since a geometrical object (solution of a system of equations) comes to play. By that time such a theory was built in Moscow by V.Makanin and A.Razborov, and E.Rips told me about that. He was sure that this theory yields solutions of the various model theoretic problems and accelerates a new model theory related to free groups.

In his talk Zlil Sela mentioned the words that touched me deeply. I mean the notions of ideal, radical and the Nullstellensatz with respect to a free group.

After the talk I spoke to Eliyahu Rips about the topic of the lecture and gradually the feeling that under an appropriate technique all these concepts can be defined for arbitrary groups and even for arbitrary algebras captured me. The bus ride from Jerusalem to Tel Aviv takes about an hour. During this trip thoughts that were unclear and fuzzy at the beginning turned into the very well-determined concepts. On the way back I stopped at Bar Ilan University and told Zhenya and other algebraists about them. This was the initial point of the work on the new topic which subsequently received the name Universal Algebraic Geometry.

It became clear pretty quickly that studying the theory of solutions of systems of equations over arbitrary algebras gives an opportunity to gather under one roof three related sciences: algebra, geometry, model theory. This combination looked to me extremely promising and very attractive. Hilbert's theorem, known as the Nullstellensatz, served as a testing example. Essentially, it describes all consequences that can be derived from a given system of equations. In other words, given a system of equations, one can consider all its solutions and then take all equations having the same solutions as the initial system. Then, it immediately became clear that this situation lies within the frame of the classical abstract Galois correspondence. Moreover, we proceed from the system of equations over algebra. We are interested in a geometrical object, namely in the set of solutions of this equation. Evidently, it is an algebraic set lying in the appropriate affine space, that is a Galois-closed object. On the other hand, another closed object is the set of all consequences derived from the given system of equations, i.e., formally, a logical object having a syntactical nature.

The classical Hilbert theorem states that in the case when we consider equations in the class of commutative associative algebras with unit over a good field, the Galois closure has the far more clear algebraic characterization. E.Rips explained to me that something similar happens for the far more complicated case of equations over a free group. This brought me the idea to consider Nullstellensatz problem in the most general Galois correspondence of closed syntactical and semantical objects.

Another important motivating factor was my experience with algebraic logic. This science provides a complete algebraization of first-order methods and theorems. Algebraic logic appeared in the works of Henkin, Monk and Tarski under the name of cylindric algebras [8]. Almost simultaneously, Paul Halmos came to another algebraization of first-order calculus and defined the so-called polyadic algebras [7]. In my papers [18], [20] I introduced a multi-sorted version of algebraic logic. These algebras got the name Halmos algebras. I felt that multi-sorted algebraic logic is a necessary ingredient for constructing a general harmonious semantic-syntactic Galois correspondence (see Section 2 for the details).

In this language Galois-closed objects acquired a natural look of algebraic sets of the corresponding closed systems of equations. I realized that the Galois correspondence may be related to the given variety, while the identities of the variety play the role of the data type, and the free algebras of the variety can be viewed as atomic algebras containing atomic formulas. Moreover, I got the feeling that we need not be restricted by equations and may consider arbitrary systems of formulas.

The entire structure of consistent model-theoretic concepts of geometrical and algebraic nature was developed within a few months. As a result, the paper "Varieties of algebras and algebraic varieties" [23] appeared in 1996 in the Israel Mathematical Journal. In the following year these notions acquired an almost modern form in the paper "Varieties of algebras and algebraic varieties. Categories of algebraic varieties" [24] published in the Siberian Mathematical Journal. An important digression must be made here. Classical algebraic geometry, especially over algebraically closed fields, deals with the description of the structure of solutions of systems of algebraic equations. In other words, we are focused on the question how various algebraic manifolds can look like and what are the properties they can possess. Step by step dimension theory and the problem of classification of such manifolds came to stage.

It became clear pretty soon that the classification of algebraic manifolds for the case of an arbitrary variety of algebras is extremely complicated, and, in a sense, a meaningless problem. Only in rare cases of special simple systems of equations and of rather good varieties of groups and Lie algebras one can hope to manage a description of solutions. Thus, it became of utmost importance to look for new problems and new approaches that match the goals and general viewpoints of universal algebraic geometry. The notion of *geometric equivalence of algebras* became a cornerstone of the future theory.

I was looking for the notion that expresses the idea that algebras possess the same possibilities with respect to solution of systems of polynomial equations. For example, let us take two algebras: a field and its extension. It is clear that the possibilities of these algebras in relation to solving equations are different. Hence, given two algebras one needs to formulate a condition that guarantees the coincidence of solutions of any equation over both algebras. Such a condition was successfully formulated in a very natural way which fits well with the constructed Galois correspondence between pairs of elements in a free algebra and points in an affine space.

The only catch was a representation of an affine space as a Cartesian product H^n. I replaced it with the space of homomorphisms $Hom(W(X), H)$, where $W(X)$ is a free finitely generated algebra, H is the algebra under consideration, and everything fell into the right place. From that moment the theory began to develop according to laws dictated by the internal logic of concepts.

Two algebras H_1 and H_2 are called *geometrically equivalent* if for any system of equations their radicals calculated with respect to the first and the second algebra coincide. If one considers these radicals syntactically, then it means basically that the quasi-identities of the algebras H_1 and H_2 coincide. In its turn, semantically such a definition implies coincidence of quasi-varieties generated by H_1 and H_2. This is a rather effective fact falling under a Birkhoff-type theorem. But one subtlety immediately became visible. In usual algebraic geometry we do not encounter infinite systems of equations since any ideal in a polynomial algebra is finitely generated. The same situation occurs for equations over a free group, thanks to the Guba-Briant theorem. But what can be said about infinite systems of equations in the general case of an arbitrary variety of algebras?

This obstruction was quite natural, and this is the reason to introduce the notion of geometrically noetherian algebras. An algebra is called *geometrically noetherian* if any system of equations over this algebra is equivalent to a finite subsystem. One note to make here. In parallel to my ideas, a similar theory began to be developed by V.Remeslennikov and his collaborators A.Miasnikov, E.Daniyarova and others (see [4], [15] and references therein). Amazingly our thoughts resonated and some notions arose almost simultaneously. In the future this cooperation turned out to be extremely fruitful and important. The difference was in the nature of emergence of concepts but in the end everything came together for the best.

Returning to geometrical equivalence of algebras, it became clear that there are a lot of non-geometrically noetherian algebras. Hence, two algebras will be geometrically noetherian if their infinitary quasi-identities coincide. Since such formulas are not included in the elementary theory of an algebra, geometrical equivalence of algebras does not follow from classical elementary equivalence of algebras.

The corresponding example for varieties of groups was brilliantly built by Remeslennikov and Miasnikov. The question of geometric noetherianity of algebras turned out to be rather hard. For instance, the problem of geometric noetherianity of a free Lie algebra is still open and there are no visible ways of its solution!

Despite the clearly successful definition of geometric equivalence of algebras, for me there was an important flaw in it. I mean that for the whole constructed theory the key idea was the concept of a point not as a set of elements of a fixed algebra, but as a special homomorphism from a free algebra into a specific algebra from a variety, that "computes" the value of each specific coordinate. In its turn, this means that the theory should be constructed not on consideration of each object separately but rather on the base of interactions of the objects. Therefore I decided to attract categorical notions and modify the definitions in such a way that they take into account dynamics of the transition from one algebra to another.

All categories arose naturally. First, equations were built over some free algebra $W(X)$, where $X = \{x_1, \dots, x_n\}$, needless to say, should vary. This is how the category Θ^0 of finitely generated free algebras appeared. A category of affine spaces $Hom(W(X), H)$ where H runs the whole fixed variety of algebras Θ corresponds to Θ^0. The next two necessary categories are the dual categories of closed congruences and algebraic sets, that is, precisely those categories that are subject of the classical algebraic geometry. Moreover, for any variety of algebras Θ rational mappings as morphisms are defined naturally for algebraic sets thus determining the corresponding category. By the Galois correspondence categories of coordinate algebras are defined immediately for an arbitrary variety of algebras. This is how a set of categories that stimulated further theory emerged. In addition to the categories I have defined several functors that play an important role in the entire theory. The main one was the closure functor C_H that associates a set of H-closed congruences to each free algebra $W(X)$. Here H is an arbitrary algebra from the variety Θ.

My desire was to incorporate the definition of geometrical equivalence into some setting taking into account the categorical nature of the objects. A lot of papers were published on this topic, theorems were proved, but in fact search for the categorical definition of geometrical equivalence lasts till now.

Very soon, in the beginning of the 2000s, it became clear that objects which had not been considered yet in algebra play a special role in the whole picture. I mean the group of automorphisms of a category and the notion of inner automorphism of a category. An automorphism φ of the category C is called inner if it is isomorphic to the identity isomorphism of the category C. Expanding this definition using the concept of isomorphism of functors, its similarity to the definition of inner automorphism of a group becomes obvious. Anyway, all inner automorphisms form a subgroup $Inn(C)$ of the group of all automorphisms $Aut(C)$ of the category C.

It turned out that the group $Aut(\Theta^0)$ is very important to the entire theory [31], [32], [33], [40], [41], [42]. Remind that Θ^0 is the category of all free finitely generated algebras of the basic variety Θ.

Let us give a generalization of geometrical equivalence of algebras. Let us call two algebras H_1 and H_2 *geometrically similar* if the categories of algebraic sets $AG(H_1)$ and $AG(H_2)$ over these algebras are isomorphic. It is clear that if two algebras are geometrically equivalent, then they are geometrically similar. When the converse is true?

The situation is as follows. Let us imagine an algebra possessing some geometrical properties and try to deform it preserving geometry. Deform in this case means to define a derived structure so that the geometry of the new algebra coincides with the geometry of the old one. It happens that such a nontrivial deformation is possible if the group $Aut(\Theta^0)$ has outer automorphisms! If all automorphisms of $Aut(\Theta^0)$ are inner then the notions of geometrical equivalence and geometrical similarity coincide.

At the same time there were defined geometrical automorphic equivalence of algebras and many other important notions which dictated the path of development of universal algebraic geometry. It is very interesting to find out how the arising notions look like for specific varieties of algebras. But with no doubt the most tempting was the feeling that besides geometrical theory there exists a parallel logical theory of general geometrical character. This idea was the one that was realized.

1.2. **Logical geometry.** At some point I realized that I can construct a Galois correspondence also for a general case of systems of First Order formulas. It was a real event since it meant that I can investigate solutions of systems of formulas in such a manner like solutions of systems of equations. This is offering us a universal ability to apply algebraic geometry methods to a logical event and spread all ideas of universal algebraic geometry to the absolutely new situation of model theory.

I decided that first of all it is necessary to describe in detail the philosophy of universal algebraic geometry and having this soil in one's pocket to publish a long paper on the foundations of logical geometry. That is what was done.

First of all, the one hundred pages paper "Algebraic logic, varieties of algebras and algebraic varieties" [21] appeared in the volume dedicated to the anniversary of E.S.Liapin. Then the consistent presentation of the theory "Algebras with the same algebraic geometry" [25] was published in the MIAN volume. Finally, the entire theory was accumulated in the preprint "Seven lectures on universal algebraic geometry" [22] published by the Hebrew University. This series of papers gave rise to a detailed exposition of the basics of Universal Algebraic Geometry. It also allowed one to concentrate on the logical geometry case. This work resulted in the principal paper "Algebraic geometry in first-order logic" (2006) [26] and in the draft of the book "Algebraic logic and logical geometry" (2013) [19]. After all universal algebraic geometry and logical geometry began to represent a single organism and went together all the time.

How to make logic and model theory geometry? Thinking about this question I came to the conclusion that the necessary connecting link is algebra. We would like to view systems of formulas as an algebraic object like systems of equations are just congruences over the free algebra of a variety. Therefore we need to find an algebraic place where systems of first-order formulas live in a comfortable way. Algebraic logic gave us a hint.

As I already mentioned, I had a lot of expertise with the methods of algebraic logic. It was the middle of the 1980s when I became interested in the works of Henkin, Monk, Tarski, Halmos, Pigocci, Nemeti and others on algebraic logic. As a result I came up with the notion of multi-sorted algebraization of first order logic which entered mathematics under the name of Halmos algebras. This construction turned out to be very successful not only for logic itself but also for solution of some old algebraic problems. In particular, it was applied for the solution of a classical problem of quasigroups theory by A.Gvaramia [6]. I summarised all the accumulated information on Halmos algebras in a large monograph "Universal algebra, algebraic logic and databases" [18].

Syntactical Halmos algebra is that very object in which first order formulas live in the most natural way. Fortunately, but rather due to the matter of things, Boolean algebra of all subsets of an affine space perceives the action of quantifiers and can be christened semantical Halmos algebra. Then the procedure of calculation of truth value of a system of formulas has acquired the form of a homomorphism of a Halmos algebra into another one. At this point everything was ready for constructing a logical Galois correspondence by algebraic means.

In any Galois correspondence the main role is played by Galois-closed objects. In geometrical Galois correspondence closed objects "from below" are algebraic sets, while "from above" they are closed congruences. This gives coincidence of closed Galois-objects with the closed objects in Zariski topology. In logical Galois correspondence closed objects "from below" are definable sets, while the closed "from above" systems of formulas are just closed filters in a syntactical Halmos algebra. This correspondence gave a universal way to transfer ideas of algebraic geometry to the logical case, and therefore to the problems of model theory. This was exactly what I was looking for.

Now everything was ready to define the notion of a logical kernel of a point. A few words need to be said about it. Let a point in an affine space be given. Recall that a point is a homomorphism from $W(X)$ into an algebra H. Like any homomorphism, it has a kernel. By definition, this kernel is a set of equations satisfied by a given point. Since we have now a homomorphism of Halmos algebras, we can similarly define a logical kernel $LKer(\mu)$ of the point μ as a set of elements of a syntactical Halmos algebra, i.e., first order formulas which get the value "true" on this point. That is, the logical kernel of a point is the set of formulas satisfiable on this point, i.e., it is an ultrafilter of the Halmos algebra, i.e., it is exactly the type of a point in model theoretic terms. This gave a bridge linking logical geometry and classical model theory and allowing one to view model theory from the perspectives of geometry and algebra.

We just follow the internal logic of universal algebraic geometry substituting the notions by their dual ones in the logical geometry. So, the notions of logically equivalent algebras, logically noetherian algebras, logically homogeneous algebras, isotypic algebras and many others come out naturally on this way. In the next section we will dwell on isotypicity of finitely generated groups. This problem is probably the most difficult problem of logical geometry.

2. How do the problems arise? Problem of isotypicity of algebras

It somehow happened that the problem in question was not known in model theory before the advent of universal logical geometry.

The goal of this section is to introduce one more logical invariant describing algebras more rigidly than elementary equivalence. Elementary equivalence of algebras H_1 and H_2 assumes coincidence of all sentences satisfiable on H_1 and H_2. The approach we are going to present here requires coincidence of all types realizable on H_1 and H_2. We call such a situation isotypicity of algebras. Before moving on to results we need to introduce some definitions.

Let an algebra H be given. The set of closed formulas, i.e. sentences satisfiable on every point of any affine space over H, is called its elementary theory $Th(H)$. Elementary theory is an important logical invariant characterizing the given algebra. A classical question going back to the philosophy of Tarski and Malcev is to describe all algebras elementary equivalent to a given one. Characterizations of elementarily equivalent algebraically closed fields, abelian groups [39], nilpotent groups [14], boolean rings [5] and others are well known. In every case of such kind the full characterization of elementarily equivalent algebras is a great success.

Since 1948 till recent years the famous Tarski problem: whether it is possible to distinguish between two non-abelian free groups by the means of elementary theory remained open. It was solved negatively by Kharlampovich-Miasniakov [11] and Sela [35] relatively recently. A similar result is true also for a wide class of hyperbolic groups.

The notion of first order rigidity [1] became especially popular over the last years. It characterizes the case strictly opposite to the one of free groups.

Definition 1. Two algebras H_1 and H_2 are called elementarily equivalent if their elementary theories coincide.

Let us fix a class of algebras C and let $H \in C$. We are interested in all algebras from C elementarily equivalent to H. By the Levenheim-Skolem theorem for a given algebra H in every cardinality there exists an algebra elementarily equivalent to H. Therefore if we intend to describe algebras, elementarily equivalent to a given one, we have to choose the class C so that all algebras of this class be of the same cardinality. For example, this is valid for the class of finite algebras and the class of finitely generated algebras. These considerations hinted at the following definition of rigidity.

Definition 2. A finitely generated algebra H is called first order rigid if any other finitely generated algebra H_1 elementarily equivalent to H is isomorphic to it.

The question is whether there exist some good examples of infinite groups possessing this property, i.e., groups opposite to a free one in some sense. It turned out that such examples can be found first of all in the class of linear algebraic groups.

Theorem 2.1. *(*[3], [34], [38], [1], [2], [10], *etc). An arbitrary Chevalley group* $G(\Phi, O_s)$, $rk(\Phi) > 1$, *over a Dedekind ring of arithmetic type is first order rigid. Any irreducible arithmetic lattice of rank greater than 1 of characteristic zero is first order rigid.*

This theorem indicates the presence of a rich definable subgroup structure of linear groups, which actually gives the desired result. In free groups only the centralizers of points are definable subgroups which makes the situation opposite to rigidity.

It is worth paying attention to the flavour of the notion of elementary equivalence. We assume that formulas that are true in ALL points of the affine space coincide. This means that an algebra (model) can be viewed as a single set of points devoid of individuality. In turn this means that we compare algebras as a whole without caring about individuality of the points. It is clear that such characterisation can rarely be rigid.

Definition 3. Two algebras H_1 and H_2 are called $LG-isotypic$ if for any point $\mu : W(X) \to H_1$ there exists a point $\nu : W(X) \to H_2$ such that $LKer(\mu) = LKer(\nu)$

and vice versa, for any point $\nu : W(X) \to H_2$ there exists a point $\mu : W(X) \to H_1$ such that $LKer(\nu) = LKer(\mu)$.

We can reformulate isotypicity of algebras in more standard logical notions.

Definition 4. Let $\mathcal{L}$ be a first order language and H and G be $\mathcal{L}$-algebras. If for any n-tuple $\bar{a}$ in H^n there exists an n-tuple $\bar{b}$ in G^n such that $tp^H(\bar{a}) = tp^G(\bar{b})$ and vice versa, then H and G are called isotypic.

Here $tp^H(\bar{a})$ denotes the type of a point a, see [12], [9], [43].

The meaning of Definition 4 is as follows. Two algebras are isotypic if the sets of realizable types over H_1 and H_2 coincide. We can say that these algebras have the same logic of types. Some references to the notion of isotypicity of algebras can be found in [28], [27], [29], [31], [43], [30].

The main property is as follows, see [43].

Theorem 2.2. *Algebras H_1 and H_2 are logically equivalent if and only if they are isotypic.*

This means that we arrived at the notion of isotypicity of algebras following the logic introduced in Universal Algebraic Geometry. This notion is much stronger than elementary equivalence of algebras since it takes into account coincidence of individual logical properties of points.

The next principal conjecture was formulated in [30], see also [16]. In fact, it states that every finitely generated group is rigid with respect to the logic of types. It means that if in addition to coincidence of elementary theories of two finitely generated groups we require coincidence of types of arbitrary points over these groups, then this must exhaust all degrees of freedom leading to differences between groups. The problem of isotypic rigidity after B.Plotkin is as follows:

Problem 1. *Is it true that every two isotypic finitely generated groups are isomorphic?*

Problem 1 has been solved positively for many groups. However the general solution remains unclear.

Starting from the concepts of Universal Algebraic Geometry we call two algebras logically similar if the categories of definable sets over these algebras are isomorphic. We are interested in the question when logical similarity of algebras is reduced to logical equivalence, and, hence, to isotypicity of algebras. It is known that for the geometrical similarity case it is enough to investigate inner automorphisms of the category of finitely generated free algebras Θ^0. Every multi-sorted Halmos algebra can be viewed as the category of Halmos algebras Hal^0_Θ [29]. This category plays a role for logical geometry similar to the one the category Θ^0 plays for Universal Algebraic Geometry.

Hypothesis 2.3. Let Θ^0 be a category of all groups. Then any automorphism of the Halmos category Hal^0_Θ is inner.

To conclude we would like to underline that another important problem is the study of the objects of Universal Logical Geometry with respect to various specific varieties of algebras and determining the precise kind of syntactically-semantic transitions for these categories. Besides standard varieties of groups, associative and Lie algebras, there are a lot of other varieties for which the situation is absolutely unclear. Let us, for example, point out the variety of semirings related to tropical geometry, or the variety of quasigroups.

On the other hand, the varieties of semigroups and of inverse semigroups are thoroughly investigated by G.Zhitomirski. It turns out that for these varieties most of principal problems of Universal Algebraic Geometry and Universal Logical Geometry have quite transparent solutions, see [32], [31], [33].

Let us take a closer look at Universal Geometry over quasigroups. This question is especially interesting since quasigroups have their own well-known geometrical applications. But how do the quasigroups look like from the point of view of Universal Algebraic Geometry and Logical Geometry? This question is totally open and looks challenging. The suggested scheme of investigations of quasigroups from this perspective is as follows:

- Isotypic and isomorphic quasigroups. Foundations of Algebraic and Logical Geometry of quasigroups.
- Geometrical equivalence of quasigroups.
- Automorphisms of the category of free quasigroups.
- Prove that all automorphisms of the category of free quasigroups are inner.
- Theorem. Categories of algebraic sets over two quasigroups are isomorphic if and only if they are geometrically equivalent.
- Geometrical noetherianity for quasigroups.
- When two quasigroups generate one and the same quasi-variety?
- Isotypicity and logical similarity for quasigroups.
- Whether it is true that two finitely generated quasigroups are isotypic if and only if they are isotopic.
- Whether it is true that two quasigroups are logically similar if and only if they are isotypic.
- Elementary equivalence and logical rigidity for quasigroups.

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