WORD MAPS, WORD MAPS WITH CONSTANTS AND REPRESENTATION VARIETIES OF ONE-RELATOR GROUPS

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To Efim Zelmanov on the occasion of his 60th birthday

INTRODUCTION

Word maps. Let F_m be the free group of rank m. Fix its generators $x_1, \ldots, x_m$. Then for any word $w = w(x_1, \ldots, x_m) \in F_m$ and any group G one can define the *word map*

$$\widetilde{w} \colon G^m \to G$$

by evaluation. Namely, $\widetilde{w}(g_1, \ldots, g_m)$ is obtained by substituting g_i in place of x_i and g_i^{-1} in place of x_i^{-1} followed by computing the resulting value $w(g_1, \ldots, g_m)$.

Word maps have been intensely studied over at least two past decades in various contexts (see, e.g., [**?**], [**?**], [**?**], [**?**] for surveys). In this paper, we consider the case where $G = \mathcal{G}(K)$ is the group of K-points of a simple linear algebraic group $\mathcal{G}$ defined over an algebraically closed field K. We are mainly interested in studying the image of $\widetilde{w}$. Borel's theorem [**?**] says that $\widetilde{w}$ is dominant, i.e., its image contains a Zariski dense open subset of G. However, $\widetilde{w}$ may not be surjective: this may happen in the case of power maps on groups with non-trivial centre (say, squaring map on $\mathrm{SL}(2, \mathbb{C})$) and, if $\mathcal{G}$ is not of type A, even on adjoint groups, see [**?**], [**?**], [**?**]. For the adjoint groups of type A, the surjectivity problem is wide open, even in the case of groups of rank 1, and even for words in two variables.

The goal of the present paper is two-fold. First, we extend our viewpoint on the dominance and surjectivity problems from genuine word maps to word maps with constants and establish a partial, "generic" analogue of Borel's dominance theorem. Another extension concerns a continuation of the word map $\widetilde{w} \colon \mathrm{GL}_n(K)^m \to \mathrm{GL}_n(K)$ to the map $\widetilde{w}^* \colon \mathrm{M}_n(K)^m \to \mathrm{M}_n(K)$. Being interesting in its own right, this method yields, as a by-product, a new proof of some results of Bandman and Zarhin [**?**], who proved the surjectivity of $\widetilde{w}$ for $G = \mathrm{PGL}_2(K)$ in the case where K is an algebraically closed field of characteristic zero, $m = 2$, and $w \in F_m \setminus F_m^2$, where $F_m^1 = [F_m, F_m], \ldots, F_m^i = [F_m^{i-1}, F_m^{i-1}], \ldots$. Our second goal consists in studying the geometric structure of the representation variety of the one-relator group $\Gamma_w := F_m / \langle w \rangle$ with an eye towards applying the data on its irreducible components to searching unipotent elements in the image of the word map. This often allows one to prove the surjectivity of the word map on $\mathrm{PGL}_2(K)$. We give a non-trivial example of such a $w \in F_2$, in the spirit of [**?**] but avoiding their computer calculations.

Word maps with constants. Let G be a group, let

$$\Sigma = \{\sigma_1, \ldots, \sigma_r \mid \sigma_i \in G \setminus Z(G) \text{ for every } i = 1, \ldots, r\},$$

and let $w_1, \ldots, w_{r+1} \in F_m$ be reduced words. The expression

$$w_\Sigma = w_1 \sigma_1 w_2 \sigma_2 \cdots w_r \sigma_r w_{r+1}$$

is called *a word with constants* (or a generalized monomial) if the sequence $w_2, \ldots, w_r$ does not contain the identity word. We will view a word $w \in F_m$ as a word with constants w_Σ with $\Sigma = \emptyset$ and $w = w_1$.

A word with constants also induces a map

$$\widetilde{w}_\Sigma \colon G^m \to G.$$

Let now $G = \mathcal{G}(K)$ where $\mathcal{G}$ is a simple algebraic group defined over an algebraically closed field K. In general, the image $\mathrm{Im}_{\widetilde{w}_\Sigma}$ is not dense in G as in Borel's Theorem. However, there are examples when this image is dense. For instance, the problem of the density of $\mathrm{Im}_{\widetilde{w}_\Sigma}$ is related to the definition of *covering numbers* for products of conjugacy classes (see [?]). Namely, let $r = m$ and let

$$w_\Sigma = x_1 \sigma_1 x_1^{-1} x_2 \sigma_2 x_2^{-1} \cdots x_m \sigma_m x_m^{-1}.$$

Then $\mathrm{Im}_{\widetilde{w}_\Sigma} = C_1 C_2 \cdots C_m$ where C_i is the conjugacy class of σ_i. Thus,

$$\overline{\mathrm{Im}_{\widetilde{w}_\Sigma}} = \overline{C_1 C_2 \cdots C_m}$$

where $\overline{X}$ is the Zariski closure of $X \subset G$. In [?] it has been proved that $\overline{\mathrm{Im}_{\widetilde{w}_\Sigma}} = G$ if $\mid \Sigma \mid > 2\,\mathrm{rank}\,G + 1$.

In the present paper, we prove (Theorem **??**) that for a "general" word with constants w_Σ the induced map $\widetilde{w}_\Sigma$ turns out to be dominant.

Note that words with constants are also related to other problems in the theory of algebraic groups (see, e.g., [?], [?], [?], [?]).

Word maps and representation varieties.

For a simple algebraic group $\mathcal{G}$ defined over an algebraically closed field K one can define the quotient map $\pi \colon \mathcal{G} \to \mathcal{G} /\!/ \mathcal{G} \approx T/W$ where $\mathcal{G} /\!/ \mathcal{G}$ and T/W are categorical quotients with respect to the action of $\mathcal{G}$ on $\mathcal{G}$ by conjugation and the natural action of the Weyl group W on a maximal torus T, respectively, see [?] for details. Then we have the map

$$\pi \circ \widetilde{w} \colon \mathcal{G}^m \to T/W.$$

We denote by $\widetilde{w}_K \colon \mathcal{G}(K)^m \to \mathcal{G}(K)$ the induced word map on the group $G := \mathcal{G}(K)$. Borel's theorem implies that $\mathrm{Im}(\pi \circ \widetilde{w}_K)$ is dense in T/W. However, we do not know when $\mathrm{Im}(\pi \circ \widetilde{w}_K) = T(K)/W$. Moreover, *we have no example* when $\mathrm{Im}(\pi \circ \widetilde{w}_K) \neq T(K)/W$. On the other hand, we have not so many examples when $\mathrm{Im}(\pi \circ \widetilde{w}_K) = T(K)/W$. The latter equality holds, say, for the Engel words $w = [\cdots [[x, y], y], \cdots y]$ (see [?]) and for the power words $w = x^k$. Bandman and Zarhin proved in [?] that $\mathrm{Im}(\pi \circ \widetilde{w}_K) = T(K)/W$ when $G = \mathrm{SL}_2(K)$.

Note that the equality $\mathrm{Im}(\pi \circ \widetilde{w}_K) = T(K)/W$ implies that for every semisimple element $s \in T(K)$ one can find an element of the form su in the image of $\widetilde{w}_K$ where u is a unipotent element of G which commutes with s, see [?]. This implies, in its turn, that

$$\mathrm{Im}(\pi \circ \widetilde{w}_K) = T(K)/W \Rightarrow \text{ all regular semisimple elements of } G \text{ belong to } \mathrm{Im}\,\widetilde{w}_K.$$

Thus, if $\mathcal{G} = \mathrm{SL}_2$ then every element of G belongs to $\mathrm{Im}\,\widetilde{w}_K$ except, possibly, $-1, \pm u$ where u is a unipotent element. Hence, if $\mathcal{G} = \mathrm{PGL}_2$ then $\mathrm{Im}\,\widetilde{w}_K \supseteq G \setminus \{u\}$. Then we have only one obstacle to proving the surjectivity of $\widetilde{w}_K$ for $\mathrm{PGL}_2(K)$, namely, we have to prove the existence of a non-trivial unipotent element in the image. The existence of unipotent elements in $\mathrm{Im}\,\widetilde{w}_K$ is somehow related to the structure of the representation variety. Namely, denote

$$\mathcal{W}_w = \widetilde{w}^{-1}(1), \quad \mathcal{T}_w = (\pi \circ \widetilde{w})^{-1}(\pi(1))$$

(here we denote by 1 the identity of G). Then $\mathcal{W}_w$ is the representation variety $R(\Gamma, G)$ where $\Gamma = \Gamma_w$ is the m-generated group with one defining relation w (see, e.g., [**?**]). The set $\mathcal{T}_w \subset G^m$ is the set of m-tuples $(g_1, \ldots, g_m) \in G^m$ such that

$$w(g_1, \ldots, g_n) = u \text{ is a unipotent element of } G,$$

that is, $\mathcal{T}_w$ is the preimage of the unipotent subvariety of G. We have an inclusion $\mathcal{W}_w \subseteq \mathcal{T}_w$ of affine subsets of G^m, and the inequality $\mathcal{W}_w \neq \mathcal{T}_w$ is a sufficient condition for the existence of a non-trivial unipotent element in $\mathrm{Im}\,\widetilde{w}$. We calculate several examples of words in F_2 for the group $\mathcal{G} = \mathrm{SL}_2$. In all these examples $\mathcal{W}_w \neq \mathcal{T}_w$ but there are cases when some irreducible components of $\mathcal{W}_w$ coincide with a irreducible component of $\mathcal{T}_w$. Possibly, the investigation of structure properties of the representation varieties $\mathcal{W}_w$ would give an answer to the question on the existence a non-trivial unipotent element in $\mathrm{SL}_2(\mathbb{C})$ lying in the image of $\widetilde{w}$.

Some results of this paper were announced in [**?**].

Notation and conventions.

Below, if not stated otherwise, K is an algebraically closed field and $\mathcal{G}$ is an algebraic group defined over K, so we *identify the group $\mathcal{G}$ with $G := \mathcal{G}(K)$*.

We denote the identity element of G by 1;
$N_G(H)$ denotes the normalizer of H in G;
$R * Q$ denotes the free product of groups R and Q;
for a group Δ and $x, y \in \Delta$, we use the symbol $x \smile y$ if x is conjugate to y in Δ;
$\mathbb{G}_a, \mathrm{GL}_n, \mathrm{SL}_n$ denote the additive, general linear, special linear groups;
$\mathrm{M}_n(K)$ is the set of $n \times n$-matrices over K;
$I_n \in \mathrm{M}_n(K)$ is the identity matrix;
for $A \in \mathrm{M}_n(K)$ by A^* we denote the adjugate matrix, i.e., the matrix such that $AA^* = A^*A = \det(A)I_n$ (note that for a generic matrix $M = (x_{ij})$ the entries of the matrix M^* are homogeneous polynomials in $\{x_{ij}\}$ of degree $n - 1$);
for a map $f \colon X \to Y$ and a subset $S \subset X$ we denote by $\mathrm{Res}_S f$ the restriction of f to S.

Let X be an algebraic variety defined over K, and let $\{X_i\}$ be a countable set of proper closed subsets $X_i \subsetneqq X$. Then we call the set $X \setminus \cup_i X_i$ a *quasi-open subset of* X. (In topology and real analysis such sets are often called "G_δ-sets".)

If X is an algebraic variety and $Y \subset X$, then $\overline{Y}$ is the Zariski closure of Y in X.

1. Word maps with constants

Let $\widetilde{w}_\Sigma \colon G^m \to G$ be a word map with constants of a simple algebraic group G. Note that there are words with constants w_Σ such that $\operatorname{Im} \widetilde{w}_\Sigma = 1$ (so-called identities with constants, see, e.g., [?]). Such identities exist if and only if the root system of G contains roots of different length. However, even in the cases when all roots are of the same length we cannot expect the analogue of the Borel Theorem $\overline{\operatorname{Im} \widetilde{w}_\Sigma} = G$. It would be interesting to understand the influence of the properties of the set Σ on the dimension of $\overline{\operatorname{Im} w_\Sigma}$. In such a generality, this seems to be a difficult question, and we start with considering some particular situations.

Theorem 1.1. *Let $w_1, \dots, w_{r+1} \in F_m$ be words where $w_2, \dots, w_r \neq 1$. There exists an open set $\mathcal{U} \subset G^r$ such that for every $\Sigma = (\sigma_1, \dots, \sigma_r) \in \mathcal{U}$ and for the word with constants $w_\Sigma = w_1 \sigma_1 w_2 \sigma_2 \cdots w_r \sigma_r w_{r+1}$, the dimension $\dim \overline{\operatorname{Im} \widetilde{w}_\Sigma}$ is a fixed number $\mathfrak{d} = \mathfrak{d}(w_1, \dots, w_{r+1})$ depending only on $w_1, \dots, w_{r+1} \in F_m$. Moreover,*

$$\dim \overline{\operatorname{Im} \widetilde{w}_{\Sigma'}} \leq \mathfrak{d}$$

for every $\Sigma' = (\sigma_1', \dots, \sigma_r') \in G^r$ (here we admit the possibility $\sigma_i' \in Z(G)$).

Proof. Define the word

$$w^Y = w_1 y_1 w_2 y_2 \cdots w_r y_r w_{r+1} \in F_{m+r} = \langle x_1, \dots, x_m, y_1, \dots, y_r \rangle.$$

We have dominant maps

$$\widetilde{w}^Y \colon G^{m+r} \to G \quad \text{and} \quad p_Y \colon G^{m+r} \to G^r$$

where p_Y is the projection onto the components $m+1, \dots, m+r$. Consider the map

$$\Phi \colon G^{m+r} \overset{(\widetilde{w}^Y, p_Y)}{\to} G \times G^r.$$

Let $X = \overline{\operatorname{Im} \Phi} \subset G \times G^r$, and let $p_Y' \colon X \to G^r$ be the projection onto G^r. Then $p_Y'(X) = G^r$ because for every r-tuple $\Sigma = (\sigma_1, \dots, \sigma_r) \in G^r$ there is a non-empty set

$$Z_\Sigma = \{(\widetilde{w}_\Sigma(g_1, \dots, g_m), \sigma_1, \dots, \sigma_r) \mid (g_1, \dots, g_m) \in G^m\} \subset X.$$

One can show that there exists an open subset $\mathcal{V}$ of X such that:

   (a) $\mathcal{V} \subset \operatorname{Im} \Phi$,

   (b) for every $v \in \mathcal{V}$ the dimension of every irreducible component of the preimage $\Phi^{-1}(v)$ is a fixed number $\mathfrak{f}$,

   (c) for every $u \in \operatorname{Im} \Phi$ the dimension of every irreducible component of the preimage $\Phi^{-1}(u)$ is greater than or equal to $\mathfrak{f}$

(cf., e.g., [?, Chapter I, n. 6, Th. 7]). Let now $\mathcal{U} \subset G^r$ be an open subset contained in $p_Y'(\mathcal{V})$, and let $\Sigma = (\sigma_1, \dots, \sigma_r) \in \mathcal{U}$. Let $v \in \mathcal{V}$ be such that $p_Y'(v) = \Sigma$. Then $v = (\widetilde{w}_\Sigma(g_1, \dots, g_m), \sigma_1, \dots, \sigma_r)$ for some $(g_1, \dots, g_m) \in G^m$, and the dimension of every irreducible component of the preimage $\Phi^{-1}(v)$ is equal to $\mathfrak{f}$, see (b). Further, the Zariski closure $\overline{Z}_\Sigma$ is an irreducible closed subset of X. Indeed, Z_Σ is the image of an irreducible variety under the morphism $\Phi_\Sigma \colon G^m \to G \times G^r$ given by the formula $\Phi_\Sigma(x_1, \dots, x_m) = (w_\Sigma(x_1, \dots, x_m), \sigma_1, \dots, \sigma_r)$ (in other words, Φ_Σ is the restriction of Φ to $G^m \times (\sigma_1, \dots, \sigma_r) \subset G^m \times G^r$). Note that $v \in Z_\Sigma \cap \mathcal{V}$. Hence there is an open subset $\mathcal{W}$ of $\overline{Z}_\Sigma$ such that $v \in \mathcal{W} \subset \mathcal{V}$. Since $\mathcal{W} \subset \mathcal{V}$, the dimension of every irreducible

component of $\Phi^{-1}(v')$ for every point $v' \in \mathcal{W}$ is equal to $\mathfrak{f}$, see (b). Also, for every $v' \in \mathcal{W}$ the closed subset $\Phi^{-1}(v') \subset G^m \times G^r$ is isomorphic (as an affine variety) to the closed subset $\Phi_\Sigma^{-1}(v') \subset G^m$. Hence the dimension of the general fibre of the morphism $\Phi_\Sigma \colon G^m \to G \times G^r$ is equal to $\mathfrak{f}$, and therefore

$$\dim \overline{\operatorname{Im} \Phi_\Sigma} = m \dim G - \mathfrak{f}.$$

The construction of Φ_Σ shows that $\overline{\operatorname{Im} \Phi_\Sigma}$ is isomorphic to $\overline{\operatorname{Im} \widetilde{w}_\Sigma}$ (the projection of $G \times G^r$ onto the first component gives this isomorphism). Hence $\dim \overline{\operatorname{Im} \widetilde{w}_\Sigma} = m \dim G - \mathfrak{f}$ for every $\Sigma \in \mathcal{U}$.

Let $\Sigma' = (\sigma'_1, \ldots, \sigma'_r) \in G^r$ (possibly, $\sigma'_i \in Z(G)$ for some i). Note that over the points of $G \times \Sigma \subset G \times G^r$ the maps $\widetilde{w}_{\Sigma'} \colon G^m \to G$ and $\Phi_{\Sigma'} \colon G^m \to G \times G^r$ have the same fibres. Moreover, these fibres are also fibres of the map $\Phi \colon G^m \times G^r \to G \times G^r$ which correspond to points of the form $(\widetilde{w}_{\Sigma'}(g_1, \ldots, g_m), \sigma'_1, \ldots, \sigma'_r)$. Since the dimension of every fibre of Φ is at least $\mathfrak{f}$ (see (c)), the dimension $\mathfrak{f}'$ of the general fibre of $\widetilde{w}_{\Sigma'}$ is at least $\mathfrak{f}$. Hence

$$\dim \overline{\operatorname{Im} \widetilde{w}_{\Sigma'}} = m \dim G - \mathfrak{f}' \leq \mathfrak{d} = m \dim G - \mathfrak{f}.$$

$\square$

Definition 1.2. Given an $(r+1)$-tuple of words $\Omega_r = (w_1, \ldots, w_{r+1})$ of F_m where $w_2, \ldots, w_r \neq 1$, we say that $\Sigma = (\sigma_1, \ldots, \sigma_r)$ is *regular* for Ω_r if $\Sigma \in \mathcal{U}$ where $\mathcal{U}$ is an open subset satisfying the conditions of Theorem **??**.

Example 1.3. Let $m = 1$ and $\Omega_2 = \{w_1 = x, w_2 = x^{-1}\}$. Then for every $\Sigma = \{\sigma\}$ the image of $\widetilde{w}_\Sigma$ is the conjugacy class of σ, and therefore for a regular Σ the dimension of $\operatorname{Im} \widetilde{w}_\Sigma$ is equal to the dimension of the conjugacy class of a regular element σ of G, that is, $\dim G - \operatorname{rank} G$.

Corollary 1.4. *Let $\Omega_r = (w_1, \ldots, w_{r+1})$ be such that $w_2, \ldots, w_r \neq 1$. Suppose $\prod_{i=1}^{r+1} w_i \neq 1$. Then if $\Sigma = (\sigma_1, \ldots, \sigma_r)$ is regular for Ω_r, the map $\widetilde{w}_\Sigma \colon G^m \to G$ is dominant.*

Proof. Indeed, for $\Sigma = (1, 1, \ldots, 1)$ we have $w_\Sigma = w = \prod_{i=1}^{r+1} w_i \neq 1$, and therefore $\widetilde{w}_\Sigma$ is dominant according to Borel's Theorem. Now the statement immediately follows from Theorem **??**. $\square$

Remark 1.5. Since the point $(1, 1, \ldots, 1)$ is the most peculiar case, presumably, the condition $w = \prod_{i=1}^{r+1} w_i \neq 1$ is sufficient for every $\widetilde{w}_\Sigma$ to be dominant.

Below we consider one important example where the word with constants is obtained by substitution of an element of G instead of one variable and where the condition $\prod_{i=1}^{r+1} w_i \neq 1$ of Corollary **??** may not hold (such words with constants are used in [**?**]).
Let $w(x_1, \ldots, x_m, y) \in F_{m+1}$,

$$w = w_1 y^{k_1} w_2 y^{k_2} w_3 \cdots w_r y^{k_r} w_{r+1},$$

where $w_1, \ldots, w_{r+1} \in F_m$ and $w_2, \ldots, w_r \neq 1$. Then for every $\sigma \in G$ we have the word with constants

$$w_\sigma = w_1 \sigma^{k_1} w_2 \sigma^{k_2} w_3 \cdots w_r \sigma^{k_r} w_{r+1}$$

(actually, according to the definition, we have to exclude the cases where $\sigma^{k_i} \in Z(G)$ for some i).

Theorem 1.6. *Suppose $\sum_{j=1}^m k_j = 0$. Then there exists an open subset $\mathcal{V} \subset G$ such that for any $\sigma \in \mathcal{V}$ the map*

$$\pi \circ \widetilde{w}_\sigma \colon G^m \to T/W$$

is dominant.

Proof. Choose a sequence $G_1 < G_2 < \cdots < G_\ell = G$ of simple algebraic subgroups of G and a sequence $T_1 \leq \cdots \leq T_\ell = T$ of their maximal tori so that $\operatorname{rank} G_i = i$ and all the G_i are T-invariant.

This can be done as follows. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be an irreducible root system corresponding to G. We may assume that the simple roots α_i are numbered so that for every $i \leq \ell$ the set $\{\alpha_1, \ldots, \alpha_i\}$ is an irreducible root system. Let G_i be the corresponding subgroup of G (generated by the root subgroups of the root system $\{\alpha_1, \ldots, \alpha_i\}$). Then we have $G_1 < G_2 < \cdots < G_\ell = G$. Further, let $T_i = \{h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) \cdots h_{\alpha_i}(t_i) \mid t_1, \ldots, t_i \in K^*\}$ be the corresponding maximal torus of G_i (here $\{h_{\alpha_i}(t) \mid t \in K^*\}$ is the corresponding one-dimensional torus in the simple algebraic group generated by the root subgroups $X_{\pm\alpha_i}$), and let $T = T_\ell$. Then we have $T_1 < T_2 < \cdots < T_\ell = T$. Obviously, T normalizes every root subgroup X_β, $\beta \in \langle \alpha_1, \ldots, \alpha_\ell \rangle$, and therefore it normalizes every G_i. Then in the reductive group $G_i T$ we have $T = T_i H_i$ where H_i is the subtorus of T which centralizes G_i. Let $\pi_i \colon G_i \to T_i/W_i$ denote the quotient morphisms.

The embedding $\iota \colon T_{i-1} \to T_i$ induces a morphism of varieties

$$\iota^* \colon T_{i-1}/W_{i-1} \to T_i/W_i.$$

Obviously, the morphism ι^* is quasi-finite (every non-empty fibre is finite). Then

$$\dim \overline{\iota^*(T_{i-1}/W_{i-1})} = \dim T_{i-1} = i - 1.$$

The set $\overline{\iota^*(T_{i-1}/W_{i-1})}$ is an irreducible codimension one subset of T_i/W_i.

Now suppose that $\sigma \in T$. Recall that for every i we have $T = T_i H_i$ where H_i is the centralizer of G_i in T. Hence $\sigma = \sigma_i h_i$ for some $\sigma_i \in T_i, h_i \in H_i$. Then for every m-tuple $(g_1, \ldots, g_m) \in G_i^m$ we have

$$w_\sigma(g_1, \ldots, g_m) = w(g_1, \ldots, g_m, \sigma_i) h_i^{\sum_j k_j} = w(g_1, \ldots, g_m, \sigma_i) \in G_i \qquad (1.1)$$

(recall that $\sum_{j=1}^r k_j = 0$). Hence the restriction of $\widetilde{w}_\sigma$ to G_i^m gives the map

$$\widetilde{w}_\sigma^i \colon G_i^m \to G_i.$$

Lemma 1.7. *For every $1 < i \leq \ell$ there is an element $\sigma \in T$ such that*

$$\overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)} \not\subseteq \overline{\iota^*(T_{i-1}/W_{i-1})}.$$

Proof. Assume to the contrary that $\overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)} \subseteq \overline{\iota^*(T_{i-1}/W_{i-1})}$ for every $\sigma \in T$. Then for every m-tuple $(g_1, \ldots, g_m) \in G_i^m$ and every $\sigma_i \in T_i$ we have

$$\pi_i \circ \widetilde{w}(g_1, \ldots, g_m, \sigma_i) \subset \overline{\iota^*(T_{i-1}/W_{i-1})}.$$

Note that any $(m+1)$-tuple of semisimple elements $(\gamma_1, \ldots, \gamma_m, \gamma)$ is conjugate by an element of G_i to an $(m+1)$-tuple of the form $(g_1, \ldots, g_m, \sigma_i)$ where $\sigma_i \in T$. Since the set of $(m+1)$-tuples $(\gamma_1, \ldots, \gamma_m, \gamma)$ of semisimple elements is dense in G_i^{m+1} and the map $\pi_i \circ \widetilde{w}$

is invariant under conjugation by elements of G_i, we have $\pi_i \circ \widetilde{w}(G_i^{m+1}) \subset \overline{\iota^*(T_{i-1}/W_{i-1})}$, and therefore

$$\dim \overline{\pi_i \circ \widetilde{w}(G_i^{m+1})} \leq i - 1.$$

This implies that $\overline{\pi_i \circ \widetilde{w}(G_i^m)} \neq \overline{\pi_i(G_i)}$, which is a contradiction with the Borel Theorem [**?**]. $\square$

Lemma 1.8. *For every $1 < i \leq \ell$ there is a non-empty open subset $S_i \subset T$ such that*

$$\overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)} \nsubseteq \overline{\iota^*(T_{i-1}/W_{i-1})}$$

for every $\sigma \in S_i$.

Proof. Let $\omega^i \colon G_i^m \times T \to G_i$ be given by the formula

$$\omega_i(g_1, \ldots, g_m, \sigma) = \widetilde{w}_\sigma^i(g_1, \ldots, g_m).$$

Consider the composite map

$$\pi_i \circ \omega^i \colon G_i^m \times T \xrightarrow{\omega^i} G_i \xrightarrow{\pi_i} T_i/W_i.$$

Let $X = (\pi_i \circ \omega^i)^{-1}(\overline{\iota^*(T_{i-1}/W_{i-1})})$. Then X is a proper closed subset in $G_i^m \times T$ by Lemma **??**. Let $X = \cup_q X_q$ be the decomposition of X into the union of irreducible closed subsets. Further, let $i_T \colon X \to T$ be the map induced by the projection $G_i^m \times T \to T$. Suppose $\overline{i_T(X_q)} \neq T$ for some q. Put $S_{iq} = T \setminus \overline{i_T(X_q)}$. Now, if $\sigma \in S_{iq}$ then $i_T^{-1}(\sigma) \cap X_q = \emptyset$. Suppose $\overline{i_T(X_q)} = T$ for some q. Since $\dim X_q < \dim G_i^m \times T$, there is a non-empty open subset $S_{iq} \subset T$ such that $\dim i_T^{-1}(\sigma) \cap X_q < \dim G_i^m$ for every $\sigma \in S_{iq}$. Thus, if $\sigma \in S_i = \cap_q S_{iq}$, the set $G_i^m \times \{\sigma\}$ is not contained in X and therefore $\overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)} \nsubseteq \overline{\iota^*(T_{i-1}/W_{i-1})}$. $\square$

Lemma 1.9. *There is a non-empty open subset $S \subset T$ such that for every $1 < i \leq \ell$ and every $\sigma \in S$ we have*

$$\overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)} \nsubseteq \overline{\iota^*(T_{i-1}/W_{i-1})}.$$

Proof. Take $S = \cap_i S_i$ where S_i is an open set from Lemma **??**. $\square$

We can now prove the theorem. Choose S as in Lemma **??**. Let $\sigma \in S$. Suppose

$$\overline{\pi_{i-1}(\operatorname{Im} \widetilde{w}_\sigma^{i-1})} = T_{i-1}/W_{i-1}. \tag{1.2}$$

Note that for $i = 1$ we have $T_0 = \{1\}$, $W_0 = \{1\}$ and therefore for $i = 1$ assumption (**??**) holds. Since for every i the map $\widetilde{w}_\sigma^i$ is the restriction of $\widetilde{w}_\sigma$ to G_i^m (see, (**??**)), we have

$$\pi_i(\operatorname{Im} \widetilde{w}_\sigma^{i-1}) \subset \pi_i(\operatorname{Im} \widetilde{w}_\sigma^i).$$

Since $\pi_i(\operatorname{Im} \widetilde{w}_\sigma^{i-1}) \subset T_i/W_i$ is the image of $\pi_{i-1}(\operatorname{Im} \widetilde{w}_\sigma^{i-1}) \subset T_{i-1}/W_{i-1}$ with respect to the map $\iota^* \colon T_{i-1}/W_{i-1} \to T_i/W_i$, we obtain from (**??**) and the assumption on σ that

$$\overline{\iota^*(T_{i-1}/W_{i-1})} \subsetneq \overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)}.$$

Since $\overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)}$ and $\overline{\iota^*(T_{i-1}/W_{i-1})}$ are closed irreducible subsets of T_i/W_i and $\dim \overline{\iota^*(T_{i-1}/W_{i-1})} = i - 1$, we have

$$\overline{\pi_i(\operatorname{Im} \widetilde{w}_\sigma^i)} = T_i/W_i.$$

The induction step now finishes the proof. $\square$

Remark 1.10. Presumably, the assumption $\sum_{j=1}^{r} k_j = 0$ can be replaced with a weaker condition: the word $w(x_1, \ldots, x_m, y) \in F_{m+1}$ is not of the form $\omega y^l \omega^{-1}$ for some $\omega = \omega(x_1, \ldots, x_m, y) \in F_{m+1}$. However, the example $w(x, \sigma) = x\sigma x^{-1}$ shows that we cannot avoid some restrictions on the word w.

Remark 1.11. It would be interesting to extend our considerations to words with constants and automorphisms (anti-automorphisms) of G where we have variables x_i and variables $x_i^{\varphi_i}$ marked by automorphisms. Such a word also gives rise to a map $G^m \to G$. Note that given a word with constants w_Σ and a collection $\Phi = \{\varphi_i\}$ ($i = 1, \ldots, m$) of automorphisms of G, there are (at least) two natural ways to produce such a map: first replace each appearance of $x_i^{a_i}$ in w_Σ with $(x_i^{\varphi_i})^{a_i}$, then either

   (1) replace each such expression with $\varphi_i(g_i)^{a_i}$, or
   (2) do this only in the cases where the exponent a_i is positive,

and compute the resulting value in G. Although the second option might seem a little artificial, it would include (at least) two important cases: 1) twisted conjugacy classes, see, e.g., [?], [?] and the references therein for a survey of various aspects of this theory, and 2) twisted commutators, see, e.g., [?], [?], [?], [?].

2. Word maps and central functions

General definitions. Let $G = \mathrm{SL}_n(K)$. Then we extend a word map with constants $\widetilde{w}_\Sigma \colon G^m \to G$ to a map

$$\widetilde{w}_\Sigma^* \colon \mathrm{M}_n^m(K) \to \mathrm{M}_n(K)$$

in the following way. Let

$$w_\Sigma = w_1 \sigma_1 w_2 \sigma_2 \cdots w_r \sigma_r w_{r+1} = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_{s_1}}^{a_{s_1}} \sigma_1 x_{i_{s_1+1}}^{a_{s_1+1}} \cdots x_{i_{s_2}}^{a_{s_2}} \sigma_2 x_{i_{s_2+1}}^{a_{s_2+1}} \cdots \tag{2.1}$$

be a word map with constants where $\Sigma = \{\sigma_1, \ldots, \sigma_r\} \subset \mathrm{SL}_n(K)$. For any m-tuple of matrices $(\mu_1, \ldots, \mu_m) \in \mathrm{M}_n^m(K)$ we define

$$\widetilde{w}_\Sigma^*(\mu_1, \ldots, \mu_m) \in \mathrm{M}_n(K)$$

by putting in formula (**??**) instead of each expression $x_{i_k}^{a_k}$ the matrix $\bar{\mu}_{i_k}^{|a_k|}$ where

$$\bar{\mu}_{i_k} = \begin{cases} \mu_{i_k} & \text{if } a_k > 0 \\ \mu_{i_k}^* & \text{if } a_k < 0 \end{cases}.$$

In other words, we substitute in formula (**??**) matrices instead of variables replacing each negative exponent with its module and in each such case replacing the given matrix with the adjugate matrix.

Remark 2.1. The word map with constants $\widetilde{w}_\Sigma \colon \mathrm{SL}_n^m(K) \to \mathrm{SL}_n(K)$ admits a natural extension to the group $\mathrm{GL}_n(K)$. Below we also denote by $\widetilde{w}_\Sigma$ the corresponding map $\mathrm{GL}_n^m(K) \to \mathrm{GL}_n(K)$.

Note that the map $\widetilde{w}_\Sigma^*$ can be represented by an $n \times n$-matrix whose entries are polynomial functions in the entries of the matrices μ_i. More precisely:

Proposition 2.2. (i) *Let $A = K[\{y_{pq}^r\}_{r=1}^m]$ be the ring of polynomial functions on M_n^m where y_{pq}^r corresponds to the $(p,q)^{th}$ entry of the rth component of M_n^m (here $1 \le p, q \le n, \ 1 \le r \le m$). Further, let*

$$a = \sum_{a_k > 0} a_k, \quad b = \sum_{a_k < 0} \mid a_k \mid.$$

Then

$$\widetilde{w}_\Sigma^* = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & & & \\ w_{n1} & w_{2n} & \cdots & w_{nn} \end{pmatrix} \in \mathrm{M}_n(A)$$

where each w_{ij} is a homogeneous polynomial in $\{y_{pq}^r\}$ of degree $a + (n-1)b$.

(ii) *Let Δ_r be the homogeneous polynomial of degree n in the variables y_{pq}^r given by the determinant of the rth component of M_n, and let*

$$\Delta = \prod_{r=1}^m \Delta_r^{b_r} \ \ where \ b_r = \sum_{a_k < 0, i_k = r} \mid a_k \mid.$$

Then, restricting to GL_n and SL_n, we obtain, respectively

$$\mathrm{Res}_{\mathrm{GL}_n} \widetilde{w}_\Sigma^* = \Delta \widetilde{w}_\Sigma \ \ and \ \ \mathrm{Res}_{\mathrm{SL}_n} \widetilde{w}_\Sigma^* = \widetilde{w}_\Sigma.$$

Proof. Straightforward from the definition. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 2.3. Let $m = 1, n = 2$,

$$\Sigma = \left\{ \sigma_1 = \sigma = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \sigma_2 = \sigma^{-1} = \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} \ \mid \ s \neq 1 \right\},$$

$w_\Sigma = \sigma x \sigma^{-1} x^{-1}$. Then $a = 1, b = 1$, and $\widetilde{w}_\Sigma^*$ is a function on M_2 which can be expressed by the formula

$$\widetilde{w}_\Sigma^* = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{pmatrix} =$$

$$= \begin{pmatrix} y_{11}y_{22} - s^2 y_{12}y_{21} & y_{11}y_{12}(s^2 - 1) \\ y_{21}y_{22}(1 - s^{-2}) & y_{11}y_{22} - s^{-2}y_{12}y_{21} \end{pmatrix}.$$

Central functions on $\mathrm{Im}_{\widetilde{w}_\Sigma^*}$. Let $X \in \mathrm{M}_n(K)$, and let

$$\chi(X) = \lambda^n + \chi_1(X)\lambda^{n-1} + \cdots + \chi_n(X)$$

be the characteristic polynomial of X. Then the map $\chi_i \colon \mathrm{M}_n(K) \to K, \ X \mapsto \chi_i(X)$, is given by a homogeneous polynomial of degree i in the entries of X.

The following fact should be compared with [**?**, Lemma 2.1].

Theorem 2.4. *Let*

$$\widetilde{w}_\Sigma \colon \mathrm{SL}_n^m(K) \to \mathrm{SL}_n(K).$$

Then $\chi_i \circ \widetilde{w}_\Sigma \colon \mathrm{SL}_n^m(K) \to K$ is either a constant function, or takes every value.

Proof. We start with the following lemma.

Lemma 2.5. *Let F be a polynomial function on $\mathrm{M}_n^m(K)$ which is homogeneous on each component of $\mathrm{M}_n^m(K)$, that is, there exist natural d_i, $i = 1, \ldots, m$, such that*

$$F(g_1, \ldots, g_{i-1}, cg_i, g_{i+1}, \ldots, g_m) = c^{d_i} F(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_m)$$

for every $(g_1, \ldots, g_m) \in \mathrm{M}_n^m(K)$ and every $c \in K$. Then the restriction of F to $\mathrm{SL}_n^m(K)$ is either a constant function, or takes every value.

Proof. Suppose that the restriction of F to $\mathrm{SL}_n^m(K)$ is not a constant function. Let $\alpha \in K$, and denote by X_α the hypersurface in the affine space $\mathrm{M}_n^m(K)$ defined by the equation

$$F^n - \alpha \prod_{r=1}^m \Delta_r^{d_r} = 0.$$

(Here $A = K[\{y_{pq}^r\}_{r=1}^m]$, Δ_r, Δ are defined as in Proposition **??**.) Denote by X_0 the hypersurface of $\mathrm{M}_n^m(K)$ given by the equation $\prod_{r=1}^m \Delta_r = 0$. If $X_\alpha \subset X_0$, then

$$F^n - \alpha \prod_{r=1}^m \Delta_r^{d_r} = \beta \Delta_{i_1}^{e_{i_1}} \Delta_{i_2}^{e_{i_2}} \cdots \Delta_{i_s}^{e_{i_s}}$$

for some $1 \le i_1 \le i_2 \le \cdots \le i_s \le m$, $\beta \in K^*$ and $e_{i_j} \in \mathbb{N}$ (note that each Δ_r is an irreducible polynomial in a polynomial ring A), and therefore $F^n(g_1, \ldots, g_m) = \alpha + \beta$ for every $(g_1, \ldots, g_m) \in \mathrm{SL}_n^m(K)$, which contradicts our assumption. Hence, there is $(g_1, \ldots, g_m) \in \mathrm{GL}_n^m(K)$ such that

$$F^n(g_1, \ldots, g_m) = \alpha \prod_{r=1}^m \Delta_r^{d_r}(g_r).$$

On the other hand, $g_r = \sqrt[n]{\Delta_r(g_r)} g_r'$ for some $g_r' \in \mathrm{SL}_n(K)$. Now, using the condition on F we obtain

$$F^n(g_1, \ldots, g_m) = F^n(g_1', \ldots, g_m') \prod_{r=1}^m \Delta_r^{d_r}(g_r).$$

Hence $F^n(g_1', \ldots, g_m') = \alpha$. Thus, the function F^n takes every value on $\mathrm{SL}_n^m(K)$. The same is of course true for the function F. $\qquad\square$

Let now define $F = \chi_i \circ \widetilde{w}_\Sigma^* \colon \mathrm{M}_n^m(K) \to K$. Then F satisfies the assumptions of Lemma **??** (see Proposition **??**). Hence $\mathrm{Res}_{\mathrm{SL}_n^m} F$ is either a constant function, or takes every value. We have

$$\mathrm{Res}_{\mathrm{SL}_n^m} F = \mathrm{Res}_{\mathrm{SL}_n^m} \chi_i \circ \widetilde{w}_\Sigma^* = \chi_i \circ \mathrm{Res}_{\mathrm{SL}_n^m} \widetilde{w}_\Sigma^* = \chi_i \circ \widetilde{w}_\Sigma,$$

which proves the theorem. $\qquad\square$

Corollary 2.6. *If*

$$\chi_1 \circ \widetilde{w}_\Sigma \colon \mathrm{SL}_2^m(K) \to \mathrm{SL}_2(K)$$

is not a constant function, then every non-central semisimple conjugacy class of $\mathrm{SL}_2(K)$ intersects $\mathrm{Im}\,\widetilde{w}_\Sigma$.

Proof. Let C be a non-central semisimple conjugacy class of $\mathrm{SL}_2(K)$, and let $\alpha = \chi_1(g) = \mathrm{tr}\,g$ for $g \in C$. Further, let $(g_1, \ldots, g_m) \in \mathrm{SL}_2^m(K)$ be such that $\mathrm{tr}(\widetilde{w}_\Sigma(g_1, \ldots, g_m)) = \alpha$. Then $\widetilde{w}_\Sigma(g_1, \ldots, g_m) \in C$. $\qquad\square$

This result gives rise to an alternative proof of the following theorem by Bandman and Zarhin.

Corollary 2.7 (cf. [**?**]). *The image of a non-trivial word map $\widetilde{w} \colon \mathrm{SL}_2^m(K) \to \mathrm{SL}_2(K)$ contains every semisimple element except, possibly, -1.*

Proof. We may view the word w as w_Σ with $\Sigma = \emptyset$. Also, the identity element 1 is always in $\mathrm{Im}\, w$. $\square$

Here is another corollary.

Theorem 2.8. *Let G be a simple algebraic group. Suppose that G is not of type $A_r, r > 1$, D_{2k+1}, $k > 1$, or E_6, and let $\widetilde{w} \colon G^m \to G$ be a non-trivial word map. Then every regular semisimple element of G is contained in $\mathrm{Im}\,\widetilde{w}$. Moreover, for every semisimple $g \in G$ there exists $g_0 \in G$ of order ≤ 2 such that $gg_0 \in \mathrm{Im}\,\widetilde{w}$.*

Proof. Let R be an irreducible root system of rank r. Let us check the property: there exists a root subsystem $R' \subset R$ such that

$$R' = R_1 \cup R_2 \cup \cdots \cup R_r \subset R \ \text{ where } \ R_i = A_1. \tag{$*$}$$

Property (*) implies that

$$G_1 \times G_2 \times \cdots \times G_r \leq G_R \ \text{ where } \ G_i = SL_2(K), PGL_2(K) \tag{$**$}$$

The following fact is apparently well known (one can extract it, e.g., from [**?**, Table 5 on page 234]). We present a proof for the sake of completeness. We use the notation of [**?**] throughout.

Lemma 2.9. *Property (*) holds for all irreducible root systems except for A_r, $r > 1$, D_{2k+1}, $k > 1$, and E_6.*

Proof. **Case A_r.** Obviously, (**) does not hold ($\mathrm{SL}_2^r \not\leq \mathrm{SL}_{r+1}$). Then (*) does not hold for $r > 1$.

Case B_r. We have $R = \langle \mathfrak{e}_1 - \mathfrak{e}_2, \ldots, \mathfrak{e}_{r-1} - \mathfrak{e}_r, e_r \rangle$.

Subcase $r = 2m + 1$. We have $R_1 = \langle \mathfrak{e}_1 - \mathfrak{e}_2 \rangle$, $R_2 = \langle \mathfrak{e}_1 + \mathfrak{e}_2 \rangle$, $R_3 = \langle \mathfrak{e}_3 - \mathfrak{e}_4 \rangle$, $R_4 = \langle \mathfrak{e}_3 + \mathfrak{e}_4 \rangle, \ldots, R_{2m} = \langle \mathfrak{e}_{2m-1} + \mathfrak{e}_{2m} \rangle$, $R_{2m+1} = \langle \mathfrak{e}_{2m+1} \rangle$.

Subcase $r = 2m$. We have $R_1 = \langle \mathfrak{e}_1 - \mathfrak{e}_2 \rangle$, $R_2 = \langle \mathfrak{e}_1 + \mathfrak{e}_2 \rangle$, $R_3 = \langle \mathfrak{e}_3 - \mathfrak{e}_4 \rangle$, $R_4 = \langle \mathfrak{e}_3 + \mathfrak{e}_4 \rangle, \ldots, R_{2m} = \langle \mathfrak{e}_{2m-1} + \mathfrak{e}_{2m} \rangle$.

Case C_r. We have $R_i = \langle 2\mathfrak{e}_i \rangle$ (long roots).

Case D_{2m}. The same as B_{2m}.

Case D_{2m+1}. Consider the standard representation of D_{2m+1} with weights $\pm\mathfrak{e}_1, \ldots, \pm\mathfrak{e}_{2m+1}$. For a root $\alpha = \mathfrak{e}_i \pm \mathfrak{e}_j$, the semisimple root subgroup $h_\alpha(t)$ acts non-trivially exactly on the four weight vectors corresponding to $\pm\mathfrak{e}_i, \pm\mathfrak{e}_j$. Then if (*) holds, we can divide the dimension of the representation by 4. But this dimension equals $2(2m + 1)$, so (*) does not hold for $r = 2m + 1$.

Case E_6. In E_6 there are no six mutually orthogonal positive roots. Indeed, since all roots lie in the same W-orbit, we may start with

$$\alpha = \frac{1}{2}(\mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3 + \mathfrak{e}_4 + \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 + \mathfrak{e}_8).$$

There are altogether fifteen positive roots orthogonal to α: ten roots of the form

$$\beta = -\mathfrak{e}_i + \mathfrak{e}_j, \ 1 \le i < j \le 5,$$

and five roots of the form

$$\gamma = \frac{1}{2}(\pm\mathfrak{e}_1 \pm \mathfrak{e}_2 \pm \mathfrak{e}_3 \pm \mathfrak{e}_4 \pm \mathfrak{e}_5 - \mathfrak{e}_6 - \mathfrak{e}_7 + \mathfrak{e}_8)$$

(where the number of minus signs is equal to 4). The set of roots of the form β contains orthogonal subsets of size at most two, and the set of roots of the form γ contains no orthogonal subsets. Hence one can find at most three mutually orthogonal positive roots, and thus (*) does not hold.

Case E_7. Here $D_6 \cup \langle \mathfrak{e}_7 - \mathfrak{e}_8 \rangle \subset R$, and therefore (*) holds.

Case E_8. Here $D_8 \subset E_8$, and therefore (*) holds.

Case F_4. Here $D_4 \subset F_4$, and therefore (*) holds.

Case G_2. Here $\langle \mathfrak{e}_1 - \mathfrak{e}_2 \rangle \cup \langle -2\mathfrak{e}_3 + \mathfrak{e}_1 + \mathfrak{e}_2 \rangle \subset G_2$, and therefore (*) holds. $\square$

We can now prove the theorem. Let $\Gamma_i = \mathrm{SL}_2(K)$, and let

$$\Psi \colon \prod_{i=1}^{r} \Gamma_i \to G$$

be the natural homomorphism induced by the inclusion (**).

Denote by $\widetilde{w}_i$ the word map $\Gamma_i^m \to \Gamma_i$ defined by the same word w. Then $\mathrm{Im}\,\widetilde{w}_i$ contains every semisimple element of Γ_i except, possibly, -1.

Let T_i be a maximal torus of Γ_i.

We have $\Psi(\prod_{i=1}^{r} T_i) = T$. (Indeed, $\Psi(\prod_{i=1}^{r} T_i) \subset T$, and $\Psi(\prod_{i=1}^{r} T_i)$ is a torus of dimension $r = \dim T$.) Let $t \in T$. Then

$$t = \Psi(t_1, t_2, \ldots, t_r) \ \text{for some} \ t_i \in T_i.$$

First suppose that $t_i \ne -1$ for every i. Then

$$t_i = \widetilde{w}_i(g_{i1}, \ldots, g_{im}) \ \text{for some} \ (g_{i1}, \ldots, g_{im}) \in \Gamma_i^m,$$

and therefore

$$t = \widetilde{w}(\gamma_1, \ldots, \gamma_m) \ \text{where} \ \gamma_j = \Psi(g_{1j}, g_{2j}, \ldots, g_{rj}).$$

Note that the condition $t_i \ne -1$ holds for all regular semisimple elements t (indeed, if $t_i = -1$, then $\Psi(1, \ldots, 1, u_i, 1 \ldots 1) \ne 1$ commutes with t for a non-trivial unipotent element $u_i \in \Gamma_i \approx \mathrm{SL}_2(K)$). Then all such elements lie in $\mathrm{Im}\,\widetilde{w}$.

If $t_i = -1$ for some i, then we may take

$$t_0 = (t_1', t_2', \ldots, t_r') \text{ where } t_i' = \begin{cases} -1, \text{ if } t_i = -1 \\ 1 \text{ if } t_i \neq -1 \end{cases}.$$

Then the order of $\Psi(t_0)$ is at most two, and $\Psi(tt_0) \in \mathrm{Im}\, \widetilde{w}$. $\qquad\square$

Remark 2.10. Presumably, Theorem **??** can be extended in the following way. Instead of one function $\chi \circ \widetilde{w}$ one could consider the map $\pi \circ \widetilde{w}$ where $\pi = (\chi_1, \ldots, \chi_{n-1})\colon \mathrm{SL}_n \to T/W = \mathbb{A}^{n-1}$ is the quotient map. Say, for a tuple $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in T/W$ one can consider the system of equations

$$[\chi_i \circ \widetilde{w}^*(Y)]^n - \alpha_i \prod_{r=1}^m \Delta_r^{d_{ir}}(Y) = 0$$

where d_{ir} is the homogeneous degree of $\chi_i \circ \widetilde{w}^*(Y)$ with respect to the variables y_{ij}^r (see Proposition **??**, Lemma **??**). Obviously, this system has solutions in $\mathrm{M}_n^m(K)$. If the variety of these solutions is not contained in the variety $\prod_{r=1}^m \Delta_r = 0$, we can find a solution in $\mathrm{GL}_n^m(K)$ of the system $\chi_i \circ \widetilde{w}(Y) - \sqrt[n]{\alpha_i} = 0$ for some $Y \in \mathrm{SL}_n^m(K)$.

Remark 2.11. Note that if we could prove that $\pi \circ \widetilde{w}$ is a surjective map, we would have every regular semisimple element of any simple group G in $\mathrm{Im}\, \widetilde{w}$. Indeed, in every irreducible root system of rank r there is a subsystem of rank r which is a union of systems A_i (see, e.g., [**?**]).

3. Representation varieties and generic groups

General constructions (see, e.g., [**?**], [**?**], [**?**], [**?**]).
Let $\Gamma = \langle g_1, \ldots, g_m \rangle$ be a finitely generated group, and let $\mathfrak{R}_\Gamma \subset F_m$ be the set of all relations of Γ.
Put

$$R(\Gamma, G) = \{(x_1, \ldots, x_m) \in G^m \mid \omega(x_1, \ldots, x_m) = 1 \text{ for every } \omega \in \mathfrak{R}_\Gamma\}. \qquad (3.1)$$

Obviously, $R(\Gamma, G)$ is a Zariski closed subset of G^m (which is defined over K), and for every $(x_1, \ldots, x_m) \in R(\Gamma, G)$ the subgroup $\langle x_1, \ldots, x_m \rangle \leq G$ is a quotient of Γ. One can identify the sets

$$R(\Gamma, G) = \mathrm{Hom}(\Gamma, G)$$

using the one-to-one correspondence

$$(x_1, \ldots, x_m) \in R(\Gamma, G) \leftrightarrow \rho \in \mathrm{Hom}(\Gamma, G)$$

given by

$$(\rho(g_1), \ldots, \rho(g_m)) = (x_1, \ldots, x_m).$$

The set $R(\Gamma, \mathrm{GL}_n(K))$ is called *the variety of n-dimensional representations of* Γ.

The "variety" $R(\Gamma, G)$ may be non-reduced and reducible, so the scheme language is most appropriate, see, e.g., [**?**]. However, we will freely use the abusive term "variety" in what follows.
Let $R(\Gamma, G) = \cup_i R(\Gamma, G)^i$ be the decomposition into the union of irreducible closed subsets. Then we have the following property for the components.

Proposition 3.1. *Let K be an algebraically closed field of infinite transcendence degree over a prime subfield. Then for each i there exists a dense quasi-open subset $U^i \subset R(\Gamma, G)^i$ such that for every $(g_1, \ldots, g_m) \in U^i$ the subgroup $\langle g_1, \ldots, g_m \rangle$ is isomorphic to a fixed quotient of Γ.*

Proof. For any $\omega \in F_m$ the set

$$X_\omega = \{(g_1, \ldots, g_m) \in G^m \mid \omega(g_1, \ldots, g_m) = 1\}$$

is a proper closed subset of G^m. Hence $R(\Gamma, G)^i \cap X_\omega$ is a closed subset of $R(\Gamma, G)^i$. Let now

$$Q_i = \{\omega \in F_m \mid R(\Gamma, G)^i \not\subseteq X_\omega\}.$$

Then

$$\bigcup_{\omega \in Q_i} R(\Gamma, G)^i \cap X_\omega$$

is a countable union of proper closed subsets of $R(\Gamma, G)^i$. Since K is of infinite transcendence degree over a prime subfield, we have

$$U^i = R(\Gamma, G)^i \setminus \bigcup_{\omega \in Q_i} (R(\Gamma, G)^i \cap X_\omega) \neq \emptyset$$

(see [**?**]), and therefore U^i is a dense quasi-open subset of $R(\Gamma, G)^i$. Every group $\langle g_1, \ldots, g_m \rangle$ for $(g_1, \ldots, g_m) \in U^i$ has the same set of relations: $\{\omega \in F_n \mid R(\Gamma, G)^i \subset X_\omega\}$. $\square$

Definition 3.2. A group isomorphic to $\langle g_1, \ldots, g_m \rangle$ for $(g_1, \ldots, g_m) \in U^i$ will be called the *generic group* of the component $R(\Gamma, G)^i$.

For every $\rho \in R(\Gamma, G)$ and every $g \in G$ the map $\rho_g \colon \Gamma \to G$ given by $\rho_g(\gamma) = g\rho(\gamma)g^{-1}$ is also an element of $R(\Gamma, G)$. Hence we have a regular action of the algebraic group G on the affine set $R(\Gamma, G)$. If $G = \mathrm{GL}_n(K)$ or $\mathrm{SL}_n(K)$, orbits correspond to classes of equivalent representations. In these cases (or, more generally, if G is a reductive group) there exists a categorical quotient $R(\Gamma, G) /\!/ G$ which is also a closed affine set. There is a one-to-one correspondence between points of $R(\Gamma, \mathrm{GL}_n(K)) /\!/ \mathrm{GL}_n(K)$ (resp. $R(\Gamma, \mathrm{SL}_n(K)) /\!/ \mathrm{SL}_n(K)$) and classes of completely reducible n-dimensional representations of Γ (see [**?**]).

Let $\mathfrak{B}_\Gamma \subset \mathfrak{R}_\Gamma$ be a minimal set of relations (that is, $\mathfrak{B}_\Gamma$ is a minimal set of generators of the group $\mathfrak{R}_\Gamma$ as a normal subgroup of F_n). Since the equality $\omega(x_1, \ldots, x_m) = 1$, where $x_1, \ldots, x_m \in G, \omega \in F_n$, implies that $\omega(yx_1y^{-1}, \ldots, yx_my^{-1}) = 1$ for every $y \in G$, we may reduce the set of all relations $\mathfrak{R}_\Gamma$ in (**??**) to the set $\mathfrak{B}_\Gamma$. Let us now assume that $\mathfrak{B}_\Gamma = \{\omega_1, \ldots, \omega_k\}$ is a finite set. Then we may consider the map

$$\Phi_{\mathfrak{B}_\Gamma} \colon G^m \to G^k$$

defined by

$$\Phi_{\mathfrak{B}_\Gamma}(x_1, \ldots, x_m) = (\omega_1(x_1, \ldots, x_m), \ldots, \omega_k(x_1, \ldots, x_m)).$$

Then

$$R(\Gamma, G) = \Phi_{\mathfrak{B}_\Gamma}^{-1}(e_k) \tag{3.2}$$

(here $e_k = (1, \ldots, 1)$ is the identity of G^k), see [**?**].

Finitely generated one-relator groups. Let $\mathfrak{B}_\Gamma = \{w\}$. Then we will write Γ_w instead of Γ to emphasize the relation w.

Further, here $k = 1$ and the map $\Phi_{\mathfrak{B}_\Gamma} \colon G^m \to G$ is the word map

$$\widetilde{w} \colon G^m \to G.$$

In this case we denote

$$\widetilde{w}^{-1}(1) = \mathcal{W}_w.$$

From (**??**) we have $\mathcal{W}_w = R(\Gamma_w, G)$. Denote by $\mathcal{W}_w^i$ the irreducible components of $\mathcal{W}_w$ where $i = 0, 1, \ldots$.

In what follows, the image of the identity element 1 of a fixed torus T of G in the quotient variety T/W is also denoted by 1. Define

$$\mathcal{T}_w = (\pi \circ \widetilde{w})^{-1}(1)$$

and denote by $\mathcal{T}_w^j$ the irreducible components of $\mathcal{T}_w$ where $j = 0, 1, \ldots$. Then the following simple statement is of key importance.

Proposition 3.3.

   (a) *For every irreducible component $\mathcal{W}_w^i$ of $\mathcal{W}_w$ there is an irreducible component $\mathcal{T}_w^j$ of $\mathcal{T}_w$ which contains $\mathcal{W}_w^i$.*
   (b) *If $\mathcal{W}_w \neq \mathcal{T}_w$, then $\mathrm{Im}\, \widetilde{w}$ contains a non-trivial unipotent element u.*

Proof.
(a) Obviously, $\mathcal{W}_w \subset \mathcal{T}_w$ and hence every irreducible component of $\mathcal{W}_w$ is contained in an irreducible component of $\mathcal{T}_w$.
(b) Indeed, the set $\mathcal{T}_w \subset G^m$ is exactly the set of m-tuples $(g_1, \ldots, g_m) \in G^m$ such that $w(g_1, \ldots, g_m)$ is a unipotent element. $\qquad\square$

Remark 3.4. Presumably, in (b) one can replace "if" with "if and only if".

If K is a field of infinite transcendence degree over a prime subfield, then for each i there exists a dense quasi-open subset $U^i \subset \mathcal{W}_w^i(K)$ such that for every $(g_1, \ldots, g_m) \in U^i$ the subgroup $\langle g_1, \ldots, g_m \rangle$ is isomorphic to a fixed quotient of Γ_w (Proposition **??**).

The generic group $\langle g_1, \ldots, g_m \rangle$, where $(g_1, \ldots, g_m) \in U^i$, will be denoted by $\widetilde{\Gamma}_w^i$. The question on describing the possibilities for $\widetilde{\Gamma}_w^i$ for a given group Γ_w is interesting in its own right. More specifically, answering it could help in describing words w with the condition $\mathcal{W}_w \neq \mathcal{T}_w$, which guarantees the existence of non-trivial unipotent elements in $\mathrm{Im}\, \widetilde{w}$. Below we consider some examples when $\mathcal{W}_w \neq \mathcal{T}_w$.

4. Examples of $R(\Gamma_w, \mathrm{SL}_2(\mathbb{C}))$

In this section we use the following notation:
$G = \mathrm{SL}_2(\mathbb{C})$;
T, B, B^-, U, U^- are the sets of diagonal, upper and lower triangular, and upper and lower unitriangular matrices;
$\dot{w}_0$ is an element of $N_G(T) \setminus T$;
$w \in F_2$;

$R(\Gamma_w, \mathrm{SL}_2(\mathbb{C})) = \mathcal{W}_w = \bigcup_{i=1}^{l} \mathcal{W}_w^i$ is the decomposition into the union of irreducible components;

$\mathcal{T}_w$ is the hypersurface in $G \times G$ defined by the equation $\mathrm{tr}\, w(x,y) = 2$;

$\mathcal{T}_w = \bigcup_{j=1}^{e} \mathcal{T}_w^j$ is the decomposition into the union of irreducible components.

Obviously, for every j we have

$$\dim \mathcal{T}_w^j = 5. \tag{4.1}$$

Also, for every $i = 1, \ldots, l$ there is $j = 1, \ldots, e$ such that $\mathcal{W}_w^i \subset T_w^j$ and

$$3 \le \dim \mathcal{W}_w^i \le 5. \tag{4.2}$$

(Indeed, the upper inequality follows from (**??**). Since $\mathcal{W}_w$ is defined by three equations $w_{11} = 1, w_{12} = w_{21} = 0$, where

$$\widetilde{w}(x,y) = \begin{pmatrix} w_{11}(x,y) & w_{12}(x,y) \\ w_{21}(x,y) & w_{22}(x,y) \end{pmatrix} \ \mid \ x, y \in G\},$$

the dimension of every component of $\mathcal{W}_w$ is at least $\dim(G \times G) - 3 = 6 - 3 = 3$.)

Note that according to Proposition **??**,

$$\dim \mathcal{W}_w^i \le 4 \Rightarrow \text{there exists a non-trivial unipotent element} \ \ u \in \mathrm{Im}\ \widetilde{w}. \tag{$*$}$$

We start with some simple examples.

Example 4.1. w = [x, y].

In this case $\mathcal{W}_w$ is classically known as the *commuting variety* of G. See, e.g., [**?**] and the references therein for its properties and related problems.

Proposition 4.2. *The set* $\mathcal{W}_w$ *is irreducible,* $\dim \mathcal{W}_w = 4$ *and*

$$\mathcal{W}_w = \overline{\{g(T \times T)g^{-1} \ \mid \ g \in G\}}. \tag{4.3}$$

The set $\mathcal{T}_w$ *is also irreducible,* $\dim \mathcal{T}_w = 5$, *and*

$$\mathcal{T}_w = \overline{\{g(B \times B)g^{-1} \ \mid \ g \in G\}}.$$

Proof. The irreducibility of the commuting variety in $G \times G$ has been proven in [**?**] in a more general case where G is a semisimple simply connected group. A general pair in $\mathcal{W}_w$ is a pair of commuting semisimple elements. Hence we have (**??**). Since the general G-orbit (under conjugation) of a pair (t_1, t_2) is of dimension $\dim G/T = 2$, we have $\dim \mathcal{W}_w = 4$.

Let $(g_1, g_2) \in \mathcal{T}_w^j$ be a general pair. We may assume $\pm 1 \ne g_2 = t \in T$ (since $\dim \mathcal{T}_w^j = 5$, it cannot happen that all pairs (g_1, g_2) consist of $\pm$ unipotent elements). Show that $g_1 \in B \cup B^-$. Assume the contrary. Then either $g_1 = vsu$ for some $1 \ne v \in U^-, 1 \ne u \in U, s \in T$, or $g_1 = \dot{w}_0 u$ for some $\dot{w}_0$ and $u \in U$. In the first case,

$$[vsu, t] = vsutu^{-1}s^{-1}v^{-1}t^{-1} = vu'v' \text{ for some } v^{-1} \ne v' \in U^-, \ 1 \ne u' \in U.$$

Indeed, since both s and t are diagonal matrices in SL_2 and $t \neq \pm 1$, we have $txt^{-1} \neq x$ for every $x \in U$ or $x \in U^-$, hence

$$vsutu^{-1}s^{-1}v^{-1}t^{-1} = v\,s\,\underbrace{u\underbrace{(tu^{-1}t^{-1})}_{\in U, \neq 1}}_{:=u' \in U, \neq 1}s^{-1}\,\underbrace{(tv^{-1}t^{-1})}_{:=v' \in U^-, \neq v^{-1}} = vu'v'.$$

Hence $[vsu, t] \backsim v''u'$ for some $1 \neq v'' \in U^-, 1 \neq u' \in U$. But an element of the form $v''u'$ cannot have the trace equal to 2. This is a contradiction.

In the second case, we have $g_1 = \dot{w}_0 u$. Then $\mathrm{tr}[\dot{w}_0 u, t] = \mathrm{tr}\,t^{-2} \neq 2$. This is also a contradiction.

Thus, we have a pair (g_1, g_2) where $\pm 1 \neq g_1 \in T$ and $g_2 \in B \cup B^-$. Hence $(g_1, g_2) \in \{g(B \times B)g^{-1} \mid g \in G\}$. This implies that

$$\mathcal{T}_w^j \subset \overline{\{g(B \times B)g^{-1} \mid g \in G\}}.$$

The opposite inclusion $\mathcal{T}_w^j \supset \overline{\{g(B \times B)g^{-1} \mid g \in G\}}$ follows from the equality $[B, B] = U$. $\qquad\square$

Example 4.3. $\mathbf{w} = [\mathbf{x^m}, \mathbf{y^n}]$.

Denote

$$\mathfrak{m} = \{k \in \mathbb{N} \mid 2 < k \mid 2m\}, \quad \mathfrak{n} = \{l \in \mathbb{N} \mid 2 < l \mid 2n\}.$$

Let $C_r \subset G$ be the conjugacy class of elements of order r.

Proposition 4.4. *We have*

$$\mathcal{W}_w^0 = \mathcal{W}_{[x,y]} \subset \mathcal{T}_w^0 = \mathcal{T}_{[x,y]}.$$

All other irreducible components are of one of the following forms:

$$\mathcal{W}_w^j = \mathcal{T}_w^j = C_j \times G$$

for $j \in \mathfrak{m}$ or

$$\mathcal{W}_w^j = \mathcal{T}_w^j = G \times C_j$$

for $j \in \mathfrak{n}$.

Proof. The existence of the components $\mathcal{W}_w^0 = \mathcal{W}_{[x,y]} \subset \mathcal{T}_w^0$ is obvious.

Let now $(x, y) \in \mathcal{W}_w^j$ be such that $x^m \neq \pm 1, y^n \neq \pm 1$. Then the elements x, x^m are either in the same torus of G or in the same unipotent subgroup (modulo the centre). The same is true for y, y^n. Since $[x^m, y^n] = 1$ we then have $[x, y] = 1$ and therefore $(x, y) \in \mathcal{W}_w^0$. Thus, if $(x, y) \in \mathcal{W}_w^j \setminus \mathcal{W}_w^0$ then either $x^m = \pm 1$ or $y^n = \pm 1$ and $\mathcal{W}_w^j$ is one of the components for $j \in \mathfrak{m}$ or $j \in \mathfrak{n}$ which have been pointed out in the statement. Since the components $\mathcal{W}_w^j$, $j > 1$, are isomorphic to the direct product of a conjugacy class C_g for a semisimple element g of order > 2 and the group G, we have $\dim \mathcal{W}_w^j = 5$ and therefore $\mathcal{W}_w^j = \mathcal{T}_w^j$. $\qquad\square$

Definition 4.5. We say that a subgroup $H \leq G$ is a free product modulo the centre if $H/Z(G) \approx R * Q$ for some $R, Q \leq G/Z(G)$. In this case we write

$$H = R * Q \mod Z(G)$$

Now for the integer j put

$$[j] = \begin{cases} j & \text{if } j \text{ is odd,} \\ \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

Proposition 4.6. *For $j > 0$ we have*

$$\widetilde{\Gamma}_w^j = \mathbb{Z}_{[j]} * \mathbb{Z} \mod Z(G) \quad or \quad \widetilde{\Gamma}_w^j = \mathbb{Z} * \mathbb{Z}_{[j]} \mod Z(G)$$

where $\mathbb{Z}$ is the infinite cyclic group and $\mathbb{Z}_{[j]}$ is the cyclic group of order $[j]$.

Proof. For any two non-central conjugacy classes $C_1, C_2 \subset G$, the generic group $\langle g_1, g_2 \mid g_1 \in C_1, g_2 \in C_2 \rangle$ is isomorphic to $\langle g_1 \rangle * \langle g_2 \rangle \mod Z(G)$, see [**?**]. Hence the statement follows from Proposition **??**. $\square$

Example 4.7. $\mathbf{w = [x, y]^2}$.

Let

$$t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G.$$

We have

$$[t, g] = \begin{pmatrix} \alpha\delta - \beta\gamma\lambda^2 & \alpha\beta(-1 + \lambda^2) \\ \gamma\delta(-1 + \lambda^{-2}) & \alpha\delta - \beta\gamma\lambda^{-2} \end{pmatrix}. \tag{4.4}$$

Hence

$$\text{tr}([t, g]) = 2 - \beta\gamma(\lambda - \lambda^{-1})^2.$$

Thus,

$$\text{tr}([t, g]) = a \Leftrightarrow 2 - \beta\gamma(\lambda - \lambda^{-1})^2 = a \Leftrightarrow \beta\gamma = \frac{2 - a}{(\lambda - \lambda^{-1})^2} := p_{a,\lambda}. \tag{4.5}$$

Since $\det g = 1$, we have

$$\beta\gamma = \frac{2 - a}{(\lambda - \lambda^{-1})^2} \Leftrightarrow \alpha\delta = 1 + \frac{2 - a}{(\lambda + \lambda^{-1})^2} = \frac{\lambda^2 + \lambda^{-2} - a}{(\lambda - \lambda^{-1})^2} := q_{a,\lambda}. \tag{4.6}$$

Put

$$T_a := \{t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ \lambda \neq \pm 1, \lambda^2 + \lambda^{-2} - a \neq 0\}. \tag{4.7}$$

. If $t \in T_a$, $\boxed{\mathbf{a \neq 2}}$ then

$$M_t^a = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G \mid \beta\gamma = p_{a,\lambda} \right\} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \beta\gamma = p_{a,\lambda}, \alpha\delta = q_{a,\lambda} \right\} \tag{4.8}$$

is an irreducible closed subset in $G \times G$ and $\dim M_t^a = 2$. The construction of M_t^a implies that

$$M_T^a := \{(t, M_t^a) \mid t \in T_a\} = \{(t, g) \mid t \in T_a, g \in G, \text{tr}([t, g]) = a\}. \tag{4.9}$$

Also, the set M_T^a is an irreducible locally closed subset of $G \times G$, and $\dim M_T^a = 3$. Now let $\Psi \colon M_T^a \times G \to G \times G$ be defined by $\Psi(t, g, y) = (yty^{-1}, ygy^{-1})$. Since M_T^a is an irreducible locally closed subset of $G \times G$ and G is an affine variety, the closure of the image of Ψ is an irreducible closed subset of G. Thus, the set

$$S_a := \overline{\text{Im } \Psi} = \overline{\{gM_T^a g^{-1} \mid g \in G\}} = \overline{\{(gtg^{-1}, gM_t^a g^{-1}) \mid t \in T_a, g \in G\}} \tag{4.10}$$

is an irreducible closed subset of $G \times G$. Further, the projection $p \colon S_a \to G$ onto the first component of $G \times G$ is dominant because the image is invariant under conjugation and contains every $t \in T_a$. The fibre $p^{-1}(t)$ is equal to M_t^a and therefore is of dimension 2. Hence

$$\dim S_a = 5. \tag{4.11}$$

<mark>Note, we also have ?? and the irreducibility for $\mathbf{S_2}$ from Proposition ??</mark> We need the following irreducibility statement, which is probably known to experts. Following the referee's suggestion, we provide a self-contained proof.

Lemma 4.8. *Let $a \in \mathbb{C}$. Then the set $\{(g_1, g_2) \mid \mathrm{tr}([g_1, g_2]) = a\}$ is ireducible and*

$$\{(g_1, g_2) \mid \mathrm{tr}([g_1, g_2]) = a\} = S_a.$$

Proof. The irreducible closed subset S_a is contained in $\{(g_1, g_2) \mid \mathrm{tr}([g_1, g_2]) = a\}$ (see **(??)**, **(??)**). Equality **(??)** implies that S_a is an irreducible component of the set $\{(g_1, g_2) \mid \mathrm{tr}([g_1, g_2]) = a\}$. Suppose that there exists an irreducible component $S_a^1 \neq S_a$ of the set $\{(g_1, g_2) \mid \mathrm{tr}([g_1, g_2]) = a\}$. Since the set $\{(g_1, g_2) \mid \mathrm{tr}([g_1, g_2]) = a\}$ is a hypersurface in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, all its irreducible components are of dimension 5. Thus,

$$\dim S_a^1 = 5. \tag{4.12}$$

Let $p_1 \colon S_a^1 \to G$ be the projection onto the first component of $G \times G$. Since the set S_a^1 is invariant under conjugation by elements of G and it is an irreducible closed subset of $G \times G$, the map p_1 is either dominant or its image is contained in a single conjugacy class C. In the latter case we have $S_a^1 = C \times G$ and $C \neq \pm 1$ (this follows from **(??)**). However, one can find pairs $(g_1, g_2), (g_3, g_4) \in C \times G$ such that $[g_1, g_2] = 1, [g_3, g_4] \neq 1$ and therefore $\mathrm{tr}([g_1, g_2]) = 2, \mathrm{tr}([g_3, g_4]) \neq 2$. Thus, the set S_a^1 cannot be of the form $C \times G$. Hence the map p_1 is dominant and there exists an open subset $T_a^1 \subset T_a$ such that $T_a^1 \subset \mathrm{Im}\, p_1$. Now for every $t \in T_a^1$ we have

$$S_a^1 \supset \{g p_1^{-1}(t) g^{-1} \mid t \in T_a^1, g \in G\} \subset \{g M_T^a g^{-1} \mid g \in G\} \subset S_a.$$

Since T_a^1 is an open subset of the torus T, the set $X = \{g p_1^{-1}(t) g^{-1} \mid t \in T_a^1, g \in G\}$ contains an open subset of the component S_a^1. But X is also a subset of S_a^1. Thus, $S_a = S_a^1$. $\qquad \square$

Now we consider our case $w = [x, y]^2$.
Since the condition $\mathrm{tr}([x, y]^2) = 2$ implies that $[x, y] = \pm u$ where u is a unipotent element, we have $\mathcal{T}_w = S_2 \cup S_{-2}$. The sets $S_{\pm 2}$ are irreducible of dimension 5. Thus, $S_2 = \mathcal{T}_w^0, S_{-2} = \mathcal{T}_w^1$. Note that the set $\mathcal{T}_w^0$ is the variety $\mathcal{T}_{[x,y]}$ considered in Example **??**. Now consider the set $\mathcal{T}_w^1$. The definition T_a (see **(??)**) implies that

$$T_{-2} = \left\{ t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ \lambda \neq \pm 1, \lambda^2 + \lambda^{-2} + 2 \neq 0 \right\} = \left\{ t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ \lambda \neq \pm 1, \pm i \right\}.$$

Then **(??)** implies

$$\mathcal{T}_w^1 = \overline{\{(g t g^{-1}, g M_t^{-2} g^{-1}) \mid t \in T, t^4 \neq 1, g \in G\}}$$

where M_t^{-2} is defined by (**??**). Let $t_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, it is an element of order 4. Then we can also define $M_{t_0}^{-2}$ by formula (**??**) where $p_{a,\lambda} = -1$ and $q_{a,\lambda} = 0$ (see (**??**), (**??**)), namely,

$$M_{t_0}^{-2} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ \mid \ \beta\gamma = -1, \alpha\delta = 0 \right\}.$$

Definitions (**??**), (**??**), (**??**) show that $M_{t_0}^{-2} = \{g \in G \ \mid \ \mathrm{tr}([t,g]) = -2\}$. Hence the set $(t_0, M_{t_0}^{-2})$ is a subset of S_{-2} and therefore we can rewrite the formula for $\mathcal{T}_w^1$:

$$\mathcal{T}_w^1 = \overline{\{gtg^{-1} \times gM_t^{-2}g^{-1} \ \mid \ t \in T, \ t^2 \neq 1, \ g \in G\}}.$$

Proposition 4.9. *Each of the sets $\mathcal{T}_w$ and $\mathcal{W}_w$ has two irreducible components:*

$$\mathcal{W}_w^0 = \mathcal{W}_{[x,y]} \subset \mathcal{T}_w^0 = \mathcal{T}_{[x,y]}, \ \ \mathcal{W}_w^1 \subset \mathcal{T}_w^1,$$

where

$$\mathcal{W}_w^1 = \left\{ g \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix} \right) g^{-1} \ \mid \ \mu \in \mathbb{C}^*, \ g \in G \right\}, \ \ \dim \mathcal{W}_w^1 = 3.$$

The generic group $\widetilde{\Gamma}_w^1$ of $\mathcal{W}_w^1$ is the quaternion group Q_8.

Proof. We have only one component $\mathcal{W}_w^0 = \mathcal{W}_{[x,y]} \subset \mathcal{T}_w^0 = \mathcal{T}_w$ (see Example **??**). Consider a pair $(g_1, g_2) \in \mathcal{T}_w^1$ such that $[g_1, g_2] = -1$. Then $g_1 \neq \pm u$ where u is a unipotent element. Indeed, if $g_1 = \pm u$ we may assume $1 \neq u \in U$ and $g_2 \in \dot{w}_0 U$. Let $g_2 = \dot{w}_0 u', v \in U$. Then

$$[g_1, g_2] = (\pm u)\dot{w}_0 u'(\pm u^{-1})u'^{-1}\dot{w}_0^{-1} = u\underbrace{\dot{w}_0(u'u^{-1}u'^{-1})\dot{w}_0^{-1}}_{=v\in U^-, v\neq 1} = uv \neq \pm 1.$$

Thus, we may assume

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ \ g_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\lambda \neq \pm 1$. Then formula (**??**) shows that the equality $[g_1, g_2] = -1$ is possible if and only if

$$g_1 = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \ g_2 = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}. \tag{4.13}$$

In these cases $\langle g_1, g_2 \rangle = Q_8$. Since every pair (g_1, g_2) with the property $[g_1, g_2] = -1$ is conjugate to a pair of the form (**??**), we have only one irreducible component

$$\mathcal{W}_w^1 = \left\{ g \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix} \right) g^{-1} \ \mid \ g \in G \right\}. \tag{4.14}$$

If C_{t_0} is the conjugacy class of t_0 then $\mathcal{W}_w^1$ is the hypersurface in $C_{t_0} \times C_{t_0}$ given by the equation $[x, y] = -1$. Hence $\dim \mathcal{W}_w^1 = 2 \dim C_{t_0} - 1 = 3$. Also equality (**??**) shows

$$(g_1, g_2) \in W_w^1 \Leftrightarrow \langle g_1, g_2 \rangle = Q_8.$$

$\square$

Example 4.10. $\mathbf{w} = [\mathbf{x}, \mathbf{y}]^{\mathbf{p}}, \ \mathbf{p} \neq \mathbf{2}$, where p is a prime number.

Proposition 4.11. *There are $1 + \frac{p-1}{2}$ irreducible components $\mathcal{T}_w^j$ and the same number of components $\mathcal{W}_w^j \subset \mathcal{T}_w^j$. Moreover,*

$$\mathcal{W}_w^0 = \mathcal{W}_{[x,y]} \subset \mathcal{T}_{[x,y]} = \mathcal{T}_w^0$$

and for $j = 1, 2, \ldots, \frac{p-1}{2}$ we have

$$\mathcal{W}_w^j = \mathcal{T}_w^j = \{(x,y) \mid \mathrm{tr}([x,y]) = 2\cos\frac{2j\pi}{p}\}.$$

Proof. Obviously, we have

$$\mathcal{W}_w^0 = \mathcal{W}_{[x,y]} \subset \mathcal{T}_{[x,y]} = \mathcal{T}_w^0.$$

Let now $\mathcal{T}_w^j \neq \mathcal{T}_w^0$. Then

$$(x,y) \in \mathcal{T}_w^j \setminus \mathcal{T}_w^0 \Leftrightarrow \begin{cases} \mathrm{tr}([x,y]) \neq 2 \\ \text{and} \\ [x,y]^p = 1. \end{cases} \Leftrightarrow [x,y] \curvearrowright \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^{-1} \end{pmatrix} \quad \text{where } \epsilon_p = \sqrt[p]{1} \neq 1.$$

The condition $[x,y] \curvearrowright \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^{-1} \end{pmatrix}$ is equivalent to $\mathrm{tr}([x,y]) = 2\cos\frac{2j\pi}{p}$ for some $j = 1, \ldots, \frac{p-1}{2}$. Lemma **??** shows that there are exactly $\frac{p-1}{2}$ irreducible components $\mathcal{T}_w^j = \{(x,y) \mid \mathrm{tr}([x,y]) = 2\cos\frac{2j\pi}{p}\} = S_{2\cos\frac{2j\pi}{p}}$ apart from the component $\mathcal{T}_w^0$. Also, for $j > 0$ we have

$$\mathcal{W}_w^j = \{(x,y) \in \mathcal{T}_w^j \mid [x,y]^p = 1\} = \mathcal{T}_w^j.$$

$\square$

Remark 4.12. The same arguments as in the case $[x,y]^2 = 1$ show that for $j > 0$ we have

$$\mathcal{T}_w^j = \overline{\{gtg^{-1} \times gM_t^{2\cos\frac{2j\pi}{p}}g^{-1} \mid t \in T, t^2 \neq 1, g \in G\}}$$

where

$$M_t^{2\cos\frac{2j\pi}{p}} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G \mid \beta\gamma = \frac{2(1 - \cos\frac{2\pi j}{p})}{(\lambda - \lambda^{-1})^2} \right\}.$$

Proposition 4.13. *For $j > 0$ all $\widetilde{\Gamma}_w^j$ are non-solvable infinite groups isomorphic to each other.*

Proof. A solvable group $\Gamma = \langle g_1, g_2 \rangle$ in $\mathrm{SL}_2(\mathbb{C})$ with the relation $[x,y] \curvearrowright \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^{-1} \end{pmatrix}$ can only be a generalized quaternion group Q_{4p} (this follows from the classification of subgroups of $\mathrm{SL}_2(\mathbb{C})$). However, a finite group may only have a finite number of non-equivalent representation in $\mathrm{SL}_2(\mathbb{C})$. Hence if Γ is the generic group of some irreducible component, then Γ is conjugate in $\mathrm{SL}_2(\mathbb{C})$ to a fixed subgroup $Q_{4p} \leq \mathrm{SL}_2(\mathbb{C})$. It is easy to see that the dimension of such a component is then equal to 3, and therefore such a component is a proper closed subset of some $\mathcal{W}_w^j$. Hence the generic group $\Gamma = \widetilde{\Gamma}_w^j$ of the component $\mathcal{W}_w^j$ cannot be solvable.

Let $F \subset \mathbb{C}$ be a pure transcendental extension of $\mathbb{Q}$ of infinite degree, and let $\overline{F} \subset \mathbb{C}$ be its algebraic closure. There exists $\sigma \in \mathrm{Gal}(\overline{F}/F)$ such that $\sigma(\epsilon_p) = \epsilon_p^2$. Then $\{\sigma^r(\epsilon_p)\}$

where $r = 1, \ldots, p-1$, is the set of all roots $\sqrt[p]{1} \neq 1$. Further, the set $\mathcal{W}_w$ is $\mathbb{Q}$-defined and therefore F-defined. Since $\overline{F}$ is an algebraically closed field of infinite transcendence degree over $\mathbb{Q}$, one can find a pair $(g_1, g_2) \in (\mathrm{SL}_2(\overline{F}) \times \mathrm{SL}_2(\overline{F})) \cap \mathcal{W}_w^1$ such that $\langle g_1, g_2 \rangle = \widetilde{\Gamma}_w^1$ (see Proposition **??**). Further, the set $\mathcal{W}_w$ is $\mathrm{Gal}(\overline{F}/F)$-stable. Thus

$$(\sigma^r(g_1), \sigma^r(g_2)) \in \mathcal{W}_w^j(\overline{F}) = (\mathrm{SL}_2(\overline{F}) \times \mathrm{SL}_2(\overline{F})) \cap \mathcal{W}_w^j \quad \text{for some } j = j(r). \qquad (4.15)$$

Further,

$$[g_1, g_2] \curvearrowright \begin{pmatrix} \epsilon_p & 0 \\ 0 & \epsilon_p^{-1} \end{pmatrix} \Rightarrow [\sigma^r(g_1), \sigma^r(g_2)] \curvearrowright \begin{pmatrix} \epsilon_p^{2r} & 0 \\ 0 & \epsilon_p^{-2r} \end{pmatrix}. \qquad (4.16)$$

Then (**??**) and (**??**) imply that for every $j = 1, \ldots, \frac{p-1}{2}$ there exists r such that $(\sigma^r(g_1), \sigma^r(g_2)) \in \mathcal{W}_w^j(\overline{F})$. Since σ^r is an automorphism of $\mathrm{SL}_2(\overline{F})$, we get

$$\widetilde{\Gamma}_w^j = \langle \sigma^{r(j)}(g_1), \sigma^{r(j)}(g_2) \rangle \approx \widetilde{\Gamma}_w^1.$$

$\square$

Let us now consider a more complicated example.

Example 4.14. $\mathbf{w(x, y)} = \mathbf{[[x, y], x[x, y]x^{-1}]}.$

We have $w \in F_2^2 = [[F_2, F_2], [F_2, F_2]]$.
We remind that by $w_0 \in W$ we denote the non-trivial element of the Weyl group. Note that all elements of the form $\dot{w}_0 = \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \in N_G(T)$ whose image in W is w_0 belong to the same conjugacy class. We denote this class by $C_{\dot{w}_0}$ (and often shorten $\dot{w}_0$ to $\dot{w}$).

Theorem 4.15. *Each of the varieties $\mathcal{T}_w$ and $\mathcal{W}_w$ has three irreducible components:*

$$\mathcal{T}_w^0 = \mathcal{T}_{[x,y]}, \mathcal{T}_w^1 = \overline{\{g(T \times \dot{w}B)g^{-1} \mid g \in G\}}, \mathcal{T}_w^2 = C_{\dot{w}} \times G,$$

$$\mathcal{W}_w^0 = \mathcal{T}_w^0, \mathcal{W}_w^1 = \overline{\{g(T \times \dot{w}T)g^{-1} \mid g \in G\}} \subset \mathcal{T}_w^1, \ \dim \mathcal{W}_w^1 = 4, \mathcal{W}_w^2 = \mathcal{T}_w^2.$$

Proof. If $(g_1, g_2) \in \mathcal{T}_{[x,y]}$, then g_1 and g_2 are in the same Borel subgroup, and therefore $w(g_1, g_2) = 1$. Hence the 5-dimensional irreducible variety $\mathcal{T}_{[x,y]}$ coincides with an irreducible component of $\mathcal{T}_w$ and $\mathcal{W}_w$. Thus we may put

$$\mathcal{W}_w^0 = \mathcal{T}_w^0 = \mathcal{T}_{[x,y]}.$$

We start with the following lemma.

Lemma 4.16. *Let $s \in T, s^4 \neq 1$, and let $h \in \dot{w}B$. Then $w(s, h) = v \in U^-$ and*

$$v = 1 \Leftrightarrow h = \dot{w}.$$

Proof. We have $h = \dot{w}u$ for some $\dot{w}$ and some $u \in U$ and

$$w(s, h) = [[s, \dot{w}u], s[s, \dot{w}u]s^{-1}].$$

Further,

$$[s, \dot{w}u] = s\dot{w}us^{-1}u^{-1}\dot{w}^{-1} = (s\dot{w}s^{-1})\underbrace{(sus^{-1}u^{-1})}_{=u' \in U}\dot{w}^{-1} = \underbrace{(s\dot{w}s^{-1}\dot{w}^{-1})}_{=s^2}\underbrace{\dot{w}u'\dot{w}^{-1}}_{:=v' \in U^-} = s^2 v'$$

and $v' = 1 \Leftrightarrow h = \dot{w}$. Then

$$s[s, \dot{w}u]s^{-1} = s^2 \underbrace{(sv's^{-1})}_{:=v'' \in U^-} = s^2 v''$$

and $v'' = v' \Leftrightarrow v' = 1 \Leftrightarrow h = \dot{w}$. Then

$$w(s, h) = [s^2 v', s^2 v''] = s^2 v' s^2 \underbrace{v'' v'^{-1}}_{:=v_1 \in U^-} s^{-2} v''^{-1} s^{-2} = s^2 v' \underbrace{s^2 v_1 s^{-2}}_{:=v_2 \in U^-} v''^{-1} s^{-2} =$$

$$= s^2 v' v_2 v''^{-1} s^{-2} = s^2 v_2 \underbrace{v' v''^{-1}}_{=v_1^{-1}} s^{-2} = s^2 \underbrace{v_2 v_1^{-1}}_{:=v_3} s^{-2} = s^2 v_3 s^{-2} := v$$

and $v = 1 \Leftrightarrow v_3 = 1 \Leftrightarrow v_2 = v_1 \Leftrightarrow v'' = v' \Leftrightarrow h = \dot{w}$. $\qquad\square$

Lemma **??** implies that $\mathrm{tr}(w(s, h)) = 2$ for every $s \in T$ (if $s^4 = 1$ then $s^2 = \pm 1$ and $w(s, h) = 1$ for every $h \in \dot{w}B$). Then the set

$$\overline{\{g(T \times \dot{w}B)g^{-1} \mid g \in G\}}$$

is contained in an irreducible component of $\mathcal{T}_w$. It is easy to see that

$$\dim \overline{\{g(T \times \dot{w}B)g^{-1} \mid g \in G\}} = \dim T + \dim B + \dim G/T = 1 + 2 + 2 = 5,$$

and therefore the set

$$\overline{\{g(T \times \dot{w}B)g^{-1} \mid g \in G\}} := \mathcal{T}_w^1$$

is an irreducible component of $\mathcal{T}_w$. Further, Lemma **??** implies that a general pair $(g_1, g_2) \in T \times \dot{w}B$ satisfies the condition $w(g_1, g_2) = 1$ if and only if $g = \dot{w}$ for some $\dot{w}$. Hence

$$\mathcal{W}_w^1 := \overline{\{g(T \times \dot{w}T)g^{-1} \mid g \in G\}}$$

is the only irreducible component of $\mathcal{W}_w$ contained in $\mathcal{T}_w^1$, and

$$\dim \mathcal{W}_w^1 = \dim T + \dim T + \dim G/T = 4.$$

Let $(g_1, g_2) \in C_{\dot{w}} \times G$. Since $g_1^2 = -1$, we have

$$g_1[g_1, g_2]g_1^{-1} = g_1(g_1 g_2 g_1^{-1} g_2^{-1})g_1^{-1} = -g_2 g_1^{-1} g_2^{-1} g_1^{-1} = -g_2(-g_1)g_2^{-1}g_1^{-1} = [g_2, g_1] = [g_1, g_2]^{-1},$$

and therefore

$$w(g_1, g_2) = [[g_1, g_2], g_1[g_1, g_2]g_1^{-1}] = [[g_1, g_2], [g_1, g_2]^{-1}] = 1.$$

Hence the 5-dimensional variety $C_{\dot{w}} \times G$ coincides with an irreducible component of $\mathcal{W}_w$ and also with an irreducible component of $\mathcal{T}_w$. Hence we may put

$$\mathcal{W}_w^2 = \mathcal{T}_w^2 = C_{\dot{w}} \times G.$$

To prove that neither $\mathcal{T}_w$, nor $\mathcal{W}_w$ contain additional irreducible components, we need several computational lemmas.

Lemma 4.17. *Let* $g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq \pm 1$, *and let*

$$g_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then

$$[g_1, g_2] \ \text{is a unipotent element if and only if} \ g_2 \in B \cup B^-$$

and

$$\begin{cases} [g_1, g_2] \in B \Leftrightarrow \gamma = 0 \ \text{or} \ \delta = 0 \Leftrightarrow g_2 \in B \ \text{or} \ g_2 \in B\dot{w}, \\ [g_1, g_2] \in B^- \Leftrightarrow \beta = 0 \ \text{or} \ \alpha = 0 \Leftrightarrow g_2 \in B^- \ \text{or} \ g_2 \in \dot{w}B \end{cases} .$$

Proof. We have (see **(??)**)

$$[g_1, g_2] = \begin{pmatrix} \alpha\delta - \beta\gamma\lambda^2 & \alpha\beta(-1 + \lambda^2) \\ \gamma\delta(-1 + \lambda^{-2}) & \alpha\delta - \beta\gamma\lambda^{-2} \end{pmatrix}$$

and $\text{tr}([g_1, g_2]) = 2 - \beta\gamma(\lambda - \lambda^{-1})^2$. Therefore,

$$\text{tr}([g_1, g_2]) = 2 \Leftrightarrow \beta = 0 \ \text{or} \ \gamma = 0 \Leftrightarrow g_2 \in B \ \text{or} \ g_2 \in B^-.$$

Further,

$$[g_1, g_2] \in B \Leftrightarrow \gamma\delta(-1 + \lambda^{-2}) = 0 \Leftrightarrow \gamma = 0 \ \text{or} \ \delta = 0 \Leftrightarrow g_2 \in B \ \text{or} \ g_2 \in B\dot{w}.$$

The case $[g_1, g_2] \in B^-$ can be treated by the same arguments. $\square$

Lemma 4.18. *Let u be a non-trivial unipotent element of G, and let $C_{\pm u}$ be the conjugacy class of $\pm u$. Then $\mathcal{T}_w^j \neq C_{\pm u} \times G$.*

Proof. Obviously, we may consider the case C_u where $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Take

$$g = \begin{pmatrix} i\frac{\sqrt{2}}{2} & -i\frac{\sqrt{2}}{2} \\ -i\sqrt{2} & 0 \end{pmatrix}.$$

Then

$$z = [u, g] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i\frac{\sqrt{2}}{2} & -i\frac{\sqrt{2}}{2} \\ -i\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i\frac{\sqrt{2}}{2} \\ i\sqrt{2} & i\frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} -i\frac{\sqrt{2}}{2} & -i\frac{\sqrt{2}}{2} \\ -i\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} -i\sqrt{2} & 0 \\ i\sqrt{2} & i\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix}.$$

Further,

$$w(u, g) = [z, uzu^{-1}] = (zuzu^{-1})(z^{-1}uz^{-1}u^{-1}) = (zuzu^{-1})((-z)u(-z)u^{-1})$$

$$= (zuzu^{-1})^2 = \left(\begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right)^2 =$$

$$= \left(\begin{pmatrix} 0 & \frac{1}{2} \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 2 \end{pmatrix} \right)^2 = \begin{pmatrix} -1 & 1 \\ 4 & -5 \end{pmatrix}^2.$$

The latter matrix is not unipotent because its trace $\neq 2$. Thus we get a contradiction with our assumption. $\qquad \square$

Lemma 4.19. *Let $\mathcal{T}_w^j \neq \mathcal{T}_w^0, \mathcal{T}_w^1, \mathcal{T}_w^2$ be an irreducible component of $\mathcal{T}_w$, and let $(g_1, g_2) \in \mathcal{T}_w^j$ be a general pair. Then g_1 is a semisimple element of order $\neq 1, 2, 4$, $[g_1, g_2]$ is a semisimple element, and $[g_1, g_2] \neq \pm 1$.*

Proof. Suppose that the projection of $\mathcal{T}_w^j \subset G \times G$ onto the first component is contained in a single conjugacy class C of G. Since $\mathcal{T}_w^j$ is a closed subset invariant under G-conjugation and $\dim T_w^j = 5$, we have $\mathcal{T}_w^j = C \times G$. Lemma **??** implies that C is a semisimple class of order $\neq 1, 2$. Also, it cannot be of order 4 because this would imply $\mathcal{T}_w^j = \mathcal{T}_w^1$. So the order of g_1 is not equal to $1, 2, 4$.

Suppose that $\mathcal{T}_w^j \neq C \times G$.

Let $\mathrm{pr}_1 \colon G \times G \to G$ be the projection onto the first component. Then the set $\mathrm{pr}_1(\mathcal{T}_w^j)$ is not contained in a single conjugacy class. Since $\overline{\mathrm{tr}(\mathrm{pr}_1(\mathcal{T}_w^j))}$ is an irreducible closed subset of K, we have $\overline{\mathrm{tr}(\mathrm{pr}_1(\mathcal{T}_w^j))} = K$, and therefore for a general pair $(g_1, g_2) \in \mathcal{T}_w^j$, g_1 is a semisimple element of infinite order.

We may now assume $g_1 = t \in T$ and $t^4 \neq 1$. Let $(t, g) \in \mathcal{T}_w^j$ be a general pair. Suppose $[t, g]$ is a unipotent element. Then $g \in B$ or $g \in B^-$ by Lemma **??**. Hence $(t, g) \in T \times B$ or $(t, g) \in T \times B^-$, and therefore

$$(t, g) \in \{h(B \times B)h^{-1} \mid h \in G\} \subset \mathcal{T}_{[x,y]}^0.$$

This contradicts the assumption $\mathcal{T}_w^j \neq \mathcal{T}_w^0$. Thus $[g_1, g_2]$ is a semisimple element ==(if $[g_1, g_2] = [t, g] = -u$ where $u \neq 1$ is a unipotent element then elements $-u, -tut^{-1}$ belong to different Borel subgroups by (??),(??) and therefore $(g_1, g_2) = (t, g) \notin \mathcal{T}_w$ by Lemma ??.==

Suppose now $[t, g] = 1$. Then the general pair (t, g) belongs to $T \times T$ and therefore $(t, g) \in \mathcal{W}_{[x,y]}^0 \subset \mathcal{T}_w^0$, once again contradicting the assumption $\mathcal{T}_w^j \neq \mathcal{T}_w^0$.

Suppose $[t, g] = -1$. We have (see (**??**))

$$[t, g] = \begin{pmatrix} \alpha\delta - \beta\gamma\lambda^2 & \alpha\beta(-1 + \lambda^2) \\ \gamma\delta(-1 + \lambda^{-2}) & \alpha\delta - \beta\gamma\lambda^{-2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$$

$$t = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}.$$

This is a contradiction with the choice of t. $\qquad \square$

Lemma 4.20. *Let $g \in G$ be a semisimple element of order different from $1, 2, 4$, let $h \in G$, and suppose that $s = [g, h]$ is a semisimple element $\neq \pm 1$. If s and gsg^{-1} are in the same Borel subgroup, then s and g are also in the same Borel subgroup.*

Proof. We may assume $s \in T$ and $s, gsg^{-1} \in B$. If $g \notin B$, then $g = b_1\dot{w}b_2$ for some $b_1, b_2 \in B$ and

$$gsg^{-1} \in B \Leftrightarrow b_1\dot{w}b_2sb_2^{-1}\dot{w}^{-1}b_1^{-1} \in B \Leftrightarrow \dot{w}b_2sb_2^{-1}\dot{w}^{-1} \in B \Leftrightarrow b_2 \in T.$$

Thus, if $s, gsg^{-1} \in B$ and $g \notin B$ then $g = b\dot{w}$ for some $b \in B$. Then we have

$$[b\dot{w}, h] = s \in T \tag{4.17}$$

for some $b \in B, s \in T$ and $\gamma \in G$. The assumption that $g = b\dot{w}$ is an element of order $\neq 4$ implies that $b \neq 1$. Conjugating both sides of (**??**) with an appropriate element $s' \in T$, we can get

$$\left[\begin{pmatrix} \rho & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \tag{4.18}$$

for some $\rho, \alpha, \beta, \gamma, \delta, \xi \in \mathbb{C}$. A straightforward calculation shows that

$$\left[\begin{pmatrix} \rho & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] = \begin{pmatrix} * & * \\ -\alpha\gamma - \delta\beta + \rho\beta\gamma & * \end{pmatrix} \tag{4.19}$$

Then (**??**) and (**??**) imply that $-\alpha\gamma - \delta\beta + \rho\beta\gamma = 0$. Since $\alpha\delta - \beta\gamma = 1$, we have

$$\begin{cases} -\alpha\gamma - \delta\beta + \rho\beta\gamma = 0 \\ \alpha\delta - \beta\gamma = 1 \end{cases} \Rightarrow \begin{cases} \gamma = -\frac{\beta}{\alpha^2 + \beta^2 - \rho\alpha\beta} \\ \delta = \frac{\alpha - \rho\beta}{\alpha^2 + \beta^2 - \rho\alpha\beta} \end{cases} . \tag{4.20}$$

(We omit the case $\alpha^2 + \beta^2 - \rho\alpha\beta = 0$ because in this case we get $\beta = 0$ and then $\gamma = 0$. But then $h \in T$ is a semisimple element and therefore

$$[b\dot{w}, h] = b \underbrace{\dot{w} h \dot{w}^{-1}}_{=h^{-1}} b^{-1} h^{-1} = b h^{-1} b^{-1} h^{-1} = \underbrace{[b, h^{-1}]}_{\notin T} h^{-2} \notin T.$$

But according to our assumption, $[g, h] \in T$.)
We now substitute (**??**) into (**??**) and obtain

$$\left[\begin{pmatrix} \rho & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\frac{\beta}{\alpha^2 + \beta^2 - \rho\alpha\beta} & \frac{\alpha - \rho\beta}{\alpha^2 + \beta^2 - \rho\alpha\beta} \end{pmatrix}\right] = \begin{pmatrix} * & \rho(1 - (\alpha^2 + \beta^2 - \rho\alpha\beta)) \\ 0 & \alpha^2 + \beta^2 - \rho\alpha\beta \end{pmatrix}. \tag{4.21}$$

Comparing (**??**) and (**??**), we get $\rho(1-(\alpha^2+\beta^2-\rho\alpha\beta)) = 0$. According to our assumption, $\rho \neq 0$. Therefore $\alpha^2 + \beta^2 - \rho\alpha\beta = 1$, and we get as the commutator in (**??**) a matrix of the form

$$s = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $s = 1$. This is a contradiction with our assumption. Thus, $s, g \in B$. The lemma is proved. $\square$

We now continue the proof of the theorem by showing that $\mathcal{T}_w$ has only three irreducible components $\mathcal{T}_w^0, \mathcal{T}_w^1, \mathcal{T}_w^2$. Indeed, assume that there is a component $\mathcal{T}_w^3$. Lemma **??** implies that a general pair $(g_1, g_2) \in \mathcal{T}_w^3$ satisfies the following conditions: $g_1, [g_1, g_2]$ are semisimple elements where the order of g_1 is not 1, 2, or 4 and $[g_1, g_2] \neq \pm 1$. Further, since $\mathrm{tr}(w(g_1, g_2)) = 2$, the element

$$w(g_1, g_2) = [[g_1, g_2], g_1[g_1, g_2]g_1^{-1}]$$

is unipotent, and the elements $[g_1, g_2], g_1[g_1, g_2]g_1^{-1}$ belong to the same Borel subgroup according to Lemma **??**. Then Lemma **??** implies that $[g_1, g_2], g_1$ also belong to the same Borel subgroup. We may assume $g_1 \in T$ and $[g_1, g_2] \in B$. Then applying Lemma **??** once again, we get $g_2 \in B$ or $g_2 \in B\dot{w}$. Thus $(g_1, g_2) \in \mathcal{T}_w^0$ or $(g_1, g_2) \in \mathcal{T}_w^1$ (note that the pair

$(t, b\dot{w})$, where $t \in T$ and $b \in B$, belongs to $\mathcal{T}_w^1$ because $\dot{w}(t, b\dot{w})\dot{w}^{-1} = (t^{-1}, \dot{w}b))$. This contradicts the assumption that (g_1, g_2) is a general pair in $\mathcal{T}_w^3 \neq \mathcal{T}_w^0, \mathcal{T}_w^1$.

Since $\mathcal{W}_w^0 = \mathcal{T}_w^0, \mathcal{W}_w^2 = \mathcal{T}_w^2$ and $\mathcal{W}_w^1$ is the only irreducible component of $\mathcal{W}_w$ contained in $\mathcal{T}_w^1$, we conclude that $\mathcal{W}_w$ also contains only three irreducible components. The theorem is proved. $\qquad\square$

Corollary 4.21. *Let w be a word appearing in any of Examples* **??**, **??**, **??**, **??**, **??**. *Then the induced map $\widetilde{w} \colon \mathrm{PSL}(2, \mathbb{C})^2 \to \mathrm{PSL}(2, \mathbb{C})$ is surjective.*

Proof. By Corollary **??**, the image of $\widetilde{w}$ contains all semisimple elements of $\mathrm{PSL}(2, \mathbb{C})$. The computations of this section provide a 4-dimensional component of $\mathcal{W}_w$ for each w, hence guarantee that all unipotent elements lie in the image of $\widetilde{w}$. As every element of $\mathrm{PSL}(2, \mathbb{C})$ is either semisimple, or unipotent, we are done. $\qquad\square$

Remark 4.22. Note that to prove the corollary, we do not need the complete lists of the irreducible components of the varieties $\mathcal{W}_w$ (whose computation may be technically involved enough, as in Example **??**). It suffices to find a 4-dimensional component, which is much easier.

Remark 4.23. Towards computer-aided search of eventual examples of words w such that $G = \mathrm{SL}(2, \mathbb{C})$ contains unipotents lying outside the image of $\widetilde{w}$, it would be important to make the calculation of irreducible components faster. Towards this end, it makes sense to replace the representation variety $\mathcal{W}_w$ with the character variety $\mathcal{X}_w = \mathcal{W}_w \mathbin{/\!\!/} G$.

Remark 4.24. An approach to proving that all unipotent elements lie in the image of a word map $\widetilde{w}$ where $w \notin F_m^2$ for every simple algebraic group over fields of characteristic zero is based on the Magnus embedding theorem, combined with the Jacobson–Morozov theorem, see [**?**].

Remark 4.25. It is tempting to use representation varieties of associative and Lie algebras (see, e.g., [**?**, Remark 1.5]) to study the images of polynomial maps on such algebras, with an eye towards solving some problems raised in [**?**], [**?**], [**?**].

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