Algebraic logic and logical geometry

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Part I

Preliminaries

Chapter 1

Basics of Universal Algebra and Category Theory

1.1 Universal Algebra

1.1.1 Sets

We shall start with some notation. As usual, $a \in A$ means that a is an element of a set A, and $A \subset B$ indicates that A is a subset of B. The empty set is denoted by $\varnothing$. Given two sets A and B, we use notation $f\colon A \to B$ and $A \xrightarrow{f} B$ for a map f of A to B. The image b of the element $a \in A$ under the map f is denoted by $b = f(a)$. Sometimes we use also "right-hand" notation: $b = a^f$.

As usual, $\cap$ and $\cup$ denote the intersection and union of sets. For the complement of a set A we use $\bar{A}$.

The *Cartesian product* $A = A_1 \times \cdots \times A_n$ consists of n-tuples $(a_1, \ldots, a_n)$, where $a_i \in A_i$, $i = 1, \ldots, n$. For any integer $n \geq 0$ the Cartesian power A^n of the set A is $A^n = A \times \cdots \times A$ where the product is taken n times. For $n = 0$ the set A^0 is assumed to be a one-element set.

Every sequence $(a_1, \ldots, a_n)$ can be treated as the function that takes every $i = 1, \ldots, n$ to the element a_i of A_i. Now assume that I is any set, and let a set A_α be associated to every $\alpha \in I$. The Cartesian product $A = \prod_\alpha A_\alpha$ is the set of functions a defined on I and selecting, for any $\alpha \in I$, an element $a(\alpha) = a_\alpha$ in A_α. If all sets A_α coincide with some fixed A, then $A^I = \prod_\alpha A$ is the set of all functions $a\colon I \to A$.

A *binary relation* ρ between elements of sets A and B is a subset of the Cartesian product $A \times B$. The subset consists of the pairs (a, b) such that a stands in the given relation ρ to b. We use the notation $a\rho b$ or $(a, b) \in \rho$. With every map $f\colon A \to B$ one can associate a binary relation which consists of all pairs of the form $(a, f(a))$. Therefore, a map can be viewed as a binary relation of a special kind.

Consider binary relations between elements of the same set. We select the following properties of such relations.

1. *Reflexivity*: $a\rho a$ for every $a \in A$.
2. *Symmetry*: $a\rho b$ implies $b\rho a$, for all a and b in A.

$2'$. *Antisymmetry*: $a\rho b$ and $b\rho a$ implies $a = b$.

3. *Transitivity*: $a\rho b$ and $b\rho c$ implies $a\rho c$.

Definition 1.1.1. *A reflexive, symmetric and transitive relation is called an equivalence on A. A reflexive, and transitive relation is a preorder relation. A reflexive, antisymmetric and transitive relation is an order relation. An order relation is total order if $a\rho b$ or $b\rho a$ for all a and b in A.*

Let A be a partially ordered set, i.e., a set with an order relation $\leq$ on it. If B is a subset of A and $a \in A$, then a is an *upper bound* of B if every element of B is comparable with a and does not exceed a. The upper bound that is less than any other upper bound is the *least upper bound* of B. *Lower bounds* and the *greatest lower bound* of a subset are defined in a similar way. The least upper bound and the greatest lower bound may not exist. An element a of A is called *maximal* if for every $a' \in A$, $a \leq a'$ implies $a' = a$.

Lemma 1.1.2 (Zorn)**.** *Suppose that every totally ordered non-empty subset of an ordered set A has an upper bound in A. Then the set A has a maximal element.*

Zorn's Lemma is equivalent to the axiom of choice and plays a principal role in many considerations.

Definition 1.1.3. *A directed set is a non-empty set A together with a preorder relation $\leq$, subject to condition: for any a and b in A there exists c in A such that $a \leq c$ and $b \leq c$.*

Given an equivalence ρ on A, denote by $[a]$, $a \in A$, the class of all $a' \in A$ satisfying the condition $a\rho a'$. This class of equivalent elements is called a coset with respect to the equivalence ρ with the representative a. To emphasize the relation ρ, the notation $[a]_\rho$ is used. Every element of the coset $[a]$ may be chosen to be its representative: $[a] = [a']$ if $a' \in [a]$. All mutually distinct cosets with respect to a given equivalence ρ make up a partition of A into disjoint classes, and, moreover, partitions and equivalences are in one-to-one correspondence to each other.

Let A be a set, and let ρ be an equivalence on A. We denote by A/ρ the set of all cosets of the form $[a]_\rho$, $a \in A$. This set is called the *quotient set* of A modulo ρ. We also obtain the associated canonical surjection $\tau : A \to A/\rho$ determined by transition from a to $[a]$. This surjection is called the *natural map* of A onto the quotient set A/ρ.

Denote by $\mathbf{2}$ the set consisting of two elements: $\mathbf{2} = \{0, 1\}$. Given a set M, we define the *power set* $\mathcal{P}(M)$ of M as the set of all subsets of M. Let A be a subset of M. Assign to A a two-valued function $\chi_A : M \to \mathbf{2}$ defined by $\chi_A(a) = 1$ if $a \in A$ and $\chi_A(a) = 0$ otherwise. This function is called *the characteristic* function of A. There is a bijection between the set of characteristic functions $Fun(M, \mathbf{2})$ and $\mathcal{P}(M)$.

1.1.2 One-sorted algebras

Definitions and examples

Let H be a set. A map $\omega : H^n \to H$ is called an n-ary *algebraic operation* on H:

$$\omega : \underbrace{H \times H \times \cdots \times H}_{n} \to H.$$

In particular, a nullary algebraic operation takes any element of H into a distinguished element of H and can be identified with this element. These distinguished elements of H are called *constants.* The notation $a_1 \dots a_n\omega$ or $\omega(a_1, \dots, a_n)$ is used for the result of application of an n-ary operation ω to the argument $(a_1, \dots a_n)$.

Definition 1.1.4. *A set Ω of symbols of operations, where each symbol $\omega \in \Omega$ is equipped with an integer $n(\omega) \geq 0$ is called a signature of operations.*

Definition 1.1.5. *An algebra H of signature Ω is a triple (H, Ω, f) where H is the underlying set, Ω is the set of symbols of operations, and the function f realizes every symbol of operation ω as the operation $f(\omega)$ of arity $n(\omega)$ acting on H.*

In fact, we usually omit the reference to the realization f and write simply $a_1 \cdots a_n\omega$ instead of $a_1 \cdots a_n f(\omega)$. Furthermore, if the set Ω is already fixed, we speak merely about an algebra H. We use also the term Ω-algebra in order to emphasize the role of the signature Ω.

Every n-ary operation can be treated as an $(n+1)$-ary relation: if ω is an n-ary operation and ω' is the corresponding relation, then $(a_1, \dots, a_n, a_{n+1}) \in \omega'$ whenever $a_1 \cdots a_n\omega = a_{n+1}$.

For a fixed Ω, we have the class of Ω-algebras. In this large class various subclasses may be distinguished by selecting sets of axioms. For the commonly known classes of algebras we use for symbols of operations the standard notation: $+$, $\times$, $\cdot$, $\otimes$, $\vee$, ..., etc. We shall list briefly some classes of algebras.

Recall that *a semigroup* is an algebra S with one operation $\cdot$ subject to the associativity condition $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Usually the sign $\cdot$ is omitted in the notation. A semigroup S is said to be a *monoid* if it has an identity element e such that $xe = ex = x$ for every $x \in S$. The element e is easily seen to be unique. It is often denoted as 1. The signature of a monoid consists of one binary operation and one nullary operation that are subject to three axioms.

A group G is a monoid in which all elements are invertible. Thus, it is an algebra with three operations: the binary multiplication, the unary inverse element operation, and the nullary operation that distinguishes the identity element. These operations satisfy the axioms:

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
2. $1 \cdot x = x \cdot 1 = x$;
3. $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

From now on we will omit $''\cdot''$ in the notation. A group G is called *abelian or commutative* if $xy = yx$ for every $x, y \in G$. For abelian groups the additive notation is common. In this notation the binary operation is called addition and denoted by $+$. Then the commutatitivity law looks as $x + y = y + x$, the inverse of a is denoted by $-a$ and is said to be the opposite of a, and the zero element 0 stands for the identity in those groups.

A *ring* is a set R endowed with two binary operations called addition ($+$) and multiplication ($\cdot$) that are subject to the following conditions:

1. R is an abelian group with respect to addition;
2. R is a semigroup with respect to multiplication;
3. addition and multiplication are related by the distributive laws:

$$x(y + z) = xy + xz, \quad (x + y)z = xz + yz.$$

A ring R is commutative if the multiplication is commutative. A *field* is a commutative ring with the unit element 1 in which any non-zero element is invertible with respect to multiplication.

In order to define a vector space, we should specify a field of scalars K. A *vector space* A over a field K is an abelian group A with respect to addition and the corresponding 1-ary and 0-ary operations, on which a multiplication of the elements of A by elements of K is defined: for any $a \in A$ and $\alpha \in K$, we have $\alpha a \in A$. Moreover, the following axioms should be satisfied:

1. $\alpha(x + y) = \alpha x + \alpha y$;
2. $1 \cdot x = x$;
3. $(\alpha + \beta)x = \alpha x + \beta x$;
4. $(\alpha\beta)x = \alpha(\beta x)$.

Here 1 denotes the unit element of K, $x, y \in A$, and $\alpha, \beta \in K$. In this definition, we regard any element α of K as a 1-ary operation that assigns to an element a of the underlying set A the element αa. We include all the elements of K in the signature of operations Ω.

If in the definition of a vector space we replace a field K of scalars by a commutative ring R, we obtain the notion of a *module over a ring*. We can regard any additive abelian group A as a module over the ring of integers $\mathbb{Z}$ by letting $na = a + \cdots + a$ and $(-n)a = -na$.

Let H be a R-module over a commutative ring R. Suppose also that H is a multiplicative semigroup.

The module H is said to be an *associative algebra* over R if it satisfies the following conditions:

1. H is a ring with respect to addition and multiplication;
2. $\lambda(xy) = (\lambda \cdot x) \cdot y = x \cdot (\lambda y), \quad \lambda \in R,\ x, y \in H$.

The non-commutative polynomials $f(x_1, \ldots, x_n)$ with coefficients in a given field K constitute an associative algebra. Another example is provided by the algebra of square matrices of order n whose entries are elements of a commutative ring R. We denote this algebra by $M(n, R)$. The group of invertible elements of $M(n, R)$ is denoted by $GL(n, R)$ and called the *general* linear group.

A *Lie algebra* is a module L over a commutative ring R together with a bilinear product $[\ ,\ ] : L \times L \to L$ subject to conditions:

1. $[x, y] = -[y, x]$,
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,

for all $x, y, z \in L$.

If A is an Ω-algebra, then a subset B of A is called a *subalgebra* of A if B is closed with respect to all operations $\omega \in \Omega$. This means that for any n-ary operation $\omega \in \Omega$ and any $b_1, \ldots, b_n \in B$, the element $b_1 \cdots b_n\omega$ belongs to B. In particular, it should be closed with respect to all nullary operations, that is B contains all constants.

Let B be a subset of A. Denote by $\langle B \rangle$ the minimal subalgebra containing B, that is the intersection of all subalgebras of A containing B. We say that the subalgebra $\langle B \rangle$ *is generated* by the set B.

Homomorphisms of algebras

Let A and B be Ω-algebras of the same signature, and let $\mu \colon A \to B$ be a map. This map is said to be *compatible* with an n-ary operation $\omega \in \Omega$ if

$$(a_1 \cdots a_n\omega)^\mu = a_1^\mu \cdots a_n^\mu \omega,$$

where $(a_1, \ldots, a_n)$ is an arbitrary n-tuple in A^n.

Definition 1.1.6. *A map $\mu \colon A \to B$ compatible with all operations $\omega \in \Omega$ is called a homomorphism from A to B.*

If $\mu \colon A \to B$ is a bijective homomorphism, then $\mu^{-1} \colon B \to A$ is a homomorphism, too. Moreover, then μ is an *isomorphism* and μ^{-1} is the inverse isomorphism. Algebras A and B are isomorphic if there is an isomorphism $\mu \colon A \to B$.

A property of an algebra is called *abstract* if it respects isomorphic images. Surjective and injective homomorphisms are also called *epimorphism* and *monomorphism*, respectively. The set of all endomorphisms is denoted by $End(A)$. If $A = B$, then homomorphisms $A \to A$ are called *endomorphisms*. Bijective endomorphisms are called *automorphisms* of A. All endomorphisms of a given algebra A constitute a monoid with respect to composition of maps. Analogously, all automorphisms of A form its *group of automorphisms* denoted by $Aut(A)$.

Let A be an Ω-algebra, and let ρ be an equivalence on A, i.e., a reflexive, symmetric, and transitive binary relation. If ω is an n-ary operation in Ω, then the relation ρ and the operation ω are said to be *compatible* if, for any n-tuples $(a_1, \ldots, a_n)$ and $(a'_1, \ldots, a'_n)$ the condition $a_i \rho a'_i$, $i = 1, \ldots, n$, implies

$$(a_1 \cdots a_n\omega)\rho(a'_1 \cdots a'_n\omega).$$

Definition 1.1.7. *An equivalence ρ is called a congruence of an Ω-algebra A, if ρ is compatible with every operation from Ω.*

If ρ is a congruence then any n-ary operation $\omega \in \Omega$ induces an operation on the quotient set A/ρ by the rule $[a_1 \cdots a_n]\omega = [a_1 \cdots a_n\omega]$, for every n-tuple of cosets from A/ρ. Here the element $[a_1] \cdots [a_n]\omega$ does not depend on the choice of representatives of the cosets because of compatibility of ρ and ω.

Thus, we have an Ω-algebra A/ρ which is called *the quotient algebra* of A modulo ρ. The natural map $\tau\colon A \to A/\rho$ becomes *a natural homomorphism.* In fact, it is an epimorphism of algebras.

If $\mu : A \to B$ is a homomorphism of algebras then *the kernel* $\rho = Ker\mu$ is defined to be the binary relation on A determined by the rule: $a_1 \rho a_2$ if and only if $a_1^\mu = a_2^\mu$, $a_1, a_2 \in A$. This means that a_1 and a_2 are ρ-equivalent if their images under μ coincide. The equivalence $\rho = Ker\mu$ is always a congruence. Thus, the quotient algebra $A/Ker\mu$ is defined. The commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\mu"] \arrow[dr, "\tau"'] & & B \\ & A/Ker\mu \arrow[ur, "\varphi"'] & \end{tikzcd}$$

states that τ is an epimorphism and the quotient algebra $A/Ker\mu$ is naturally isomorphic to the image of A.

For some classes of algebras the kernel equivalence, i.e., the elements of $A/Ker\mu$, can be specified by distinguishing a single equivalence class. For example, if $\mu : G_1 \to G_2$ is a homomorphism of groups, then $Ker\mu = \mu^{-1}(1_{G_2})$ is the set of elements in G_1 equivalent to 1_{G_1}. This set is always a *normal subgroup.* Recall that a subgroup N of a group G is called normal if $gNg^{-1} = N$ for every $g \in G$.

A homomorphism of rings $\mu : R_1 \to R_2$ preserves addition, multiplication and the zero element of the ring. The set $H = Ker\mu = \mu^{-1}(0_{R_2})$ of elements in R_1 equivalent to 0_{R_1} determines the kernel congruence. The set H is *a two-sided ideal of the ring*, that is

1. H is a subgroup of the additive group $(R_1, +)$.
2. For every $r \in R_1$ and $h \in H$ we have $rh \in H$,
3. For every $r \in R_1$ and $h \in H$ we have $hr \in H$.

If H is a two-sided ideal of a ring R then the relation ρ defined by $a\rho b$ if and only if $a - b \in H$ is a congruence on R and each congruence can be obtained in such a way by taking an appropriate $Ker\mu$.

Note that even in the case when the ring R_1 is a monoid with respect to multiplication, the set $\mu^{-1}(1_{R_2})$ does not determine the whole kernel congruence since $(R_1, \cdot)$ is not a group.

1.1.3 Boolean algebras and lattices

Boolean algebras and lattices play a special role in this book. We will meet them once again in Section 3.2 devoted to algebraization of the propositional logic.

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Definition of a Boolean algebra.

Definition 1.1.8. *A Boolean algebra is a set A viewed together with two binary operations $+, \cdot$ and one unary operation $^-$. These operations are subject to the following axioms.*

1. $a + a = a; \quad a \cdot a = a.$
2. $a + b = b + a; \quad a \cdot b = b \cdot a.$
3. $(a + b) + c = a + (b + c); \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$
4. $a \cdot (b + c) = a \cdot b + a \cdot c; \quad a + b \cdot c = (a + b) \cdot (a + c).$
5. $\overline{(\overline{a})} = a.$
6. $\overline{a + b} = \overline{a} \cdot \overline{b}; \quad \overline{a \cdot b} = \overline{a} + \overline{b}.$
7. $(a + \overline{a}) \cdot b = b; \quad a \cdot \overline{a} + b = b.$

It follows from the axioms that the identities $a \cdot \overline{a} = b \cdot \overline{b}$ and $a + \overline{a} = b + \overline{b}$ always hold. Therefore, one can single out the elements $0 = a \cdot \overline{a}$ and $1 = a + \overline{a}$, and then

$$a + 0 = a, \quad a \cdot 1 = a,$$

$$a + 1 = 1, \quad a \cdot 0 = 0.$$

The zero and identity elements, regarded as nullary operations, could be included into the signature of Boolean algebras. One can check that in this signature the system of axioms defining a Boolean algebra is equivalent to the following one.

1. $a + b = b + a; \quad a \cdot b = b \cdot a.$
2. $(a + b) + c = a + (b + c); \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$
3. $a(b + c) = a \cdot b + a \cdot c; \quad a + b \cdot c = (a + b) \cdot (a + c).$
4. $a \cdot (a + b) = a + a \cdot b = a.$
5. $a + 0 = a, \quad a \cdot 1 = a,$
   $a + 1 = 1, \quad a \cdot 0 = 0.$

6. $a \cdot \overline{a} = 0, \quad a + \overline{a} = 1.$

We will freely use the notation $\vee$ and $\wedge$ instead of $+$ and $\cdot$, and also replace $^-$ by $\neg$. This is especially the case when we want to underline relations between Boolean algebras and propositional calculi or two distinguish between the operations in a Boolean algebra and in some other algebra. Another system of axioms equivalent to above defined is given in Section 3.2.

Homomorphisms of Boolean algebras play a special role in further considerations. A map μ of Boolean algebras is a homomorphism if it preserves all operations of the signature. Thus,

$$\mu(a+b) = \mu(a) + \mu(b),$$

$$\mu(ab) = \mu(a)\mu(b),$$

$$\mu(\overline{a}) = \overline{(\mu(a))},$$

$$\mu(1) = 1 \text{ and } \mu(0) = 0.$$

Let us give some examples of Boolean algebras.

Example 1.1.9. The simplest example of a non-trivial Boolean algebra is the algebra $\mathbf{2}$. It is two-element set $\{0,1\}$ with the operations defined in terms of usual arithmetic operations as follows:

$$b_1 \vee b_2 = max(b_1, b_2),$$

$$b_1 \wedge b_2 = b_1 \cdot b_2,$$

$$\neg b_1 = 1 - b_1,$$

where $b_1, b_2 \in \{0,1\}$.

Example 1.1.10. Let A be an arbitrary set and B a Boolean algebra with operations $\vee$, $\wedge$, $\neg$. Then the set $Fun(A,B) = B^A$ of all functions from A to B is also a Boolean algebra with operations $\vee$, $\wedge$, $\neg$ defined by the rules:

$$(f_1 \vee f_2)(a) = f_1(a) \vee f_2(a),$$

$$(f_1 \wedge f_2)(a) = f_1(a) \wedge f_2(a),$$

$$(\neg f_1)(a) = \neg(f_1(a)),$$

where $a \in A$, $f_1, f_2 \in B^A$.

Example 1.1.11. Let S be a set. Denote by $\mathcal{P}(S)$ the power set of S, that is the set of all subsets of S. The set $\mathcal{P}(S)$ with the set-theoretic operations union $\cup$, intersection $\cap$ and complementation $\neg$ forms a Boolean algebra, which is called the power set algebra (Boolean). Note, $\mathcal{P}(S) \cong Fun(S, \mathbf{2}) = \mathbf{2}^S$.

The well-known Stone's theorem states that:

Theorem 1.1.12. *Every Boolean algebra is isomorphic to a subalgebra of the Boolean power set algebra of some set.* □

The set **2** is also a ring, via the addition and multiplication modulo 2. $\mathbf{2}^M$ is a ring as well, so here we have simultaneously a ring and a Boolean algebra. There are certain links between ring operations and Boolean operations.

Boolean rings. Now we consider definition and some properties of Boolean rings.

Definition 1.1.13. *An associative ring R with unity is called a Boolean ring, if the identity $x^2 = x$ holds in R.*

An element a of a ring R is called *idempotent* if $a^2 = a$. Hence, a Boolean ring consists of idempotent elements.

Moreover, in a Boolean ring R

$$(a+b) = (a+b)^2 = a + ab + ba + b,$$

which implies $ab + ba = 0$. Replacing $a = b$ we get $a = -a$ for any $a \in R$. Now $ab + ba$ can be rewritten as $ab - ba = 0$, and each Boolean ring is commutative. Thus:

Proposition 1.1.14. *Every Boolean ring is commutative and satisfies the identity $x = -x$.* □

There is a natural connection between Boolean algebras and Boolean rings.

Proposition 1.1.15. *Let R be a Boolean ring. We define the operations $\vee$, $\wedge$, $^-$ on R by setting*

$$a \vee b = a + b + ab, \quad a \wedge b = ab, \quad \overline{a} = a + 1.$$

Relatively to these operations, R is a Boolean algebra denoted by $A = R^*$.

Conversely, if A is a Boolean algebra with respect to operations $\vee$, $\wedge$, $^-$, we define addition and multiplication on A by the rules

$$a + b = (a \wedge \overline{b}) \vee (\overline{a} \wedge b), \quad ab = a \wedge b.$$

Then A becomes a Boolean ring denoted by $R = A^*$.

Moreover, $R^{**} = R$ and $A^{**} = A$. The elements 0 and 1 are the zero and unit elements both of the ring and the Boolean algebra.
□

Lattices. In any Boolean algebra, an order relation is introduced in a natural way: $a \leq b$ if $ab = a$ or, equivalently, if $a + b = b$. It is easy to check that $\leq$ satisfies the order axioms.

Definition 1.1.16. *A partially ordered set A is called a lattice if every pair of elements a, b of A has the least upper bound $sup_A(a, b)$ and the greatest lower bound $inf_A(a, b)$.*

It is easy to see that $a+b$ is the least upper bound of the elements a and b of a Boolean algebra, and that ab is the greatest lower bound. Therefore, every Boolean algebra is a lattice.

Now assume that A is a lattice. For arbitrary $a, b \in A$, we denote by $a + b$ and ab their least upper bound and greatest lower bound, respectively. A simple checking shows that the following axioms hold:

1. $a + a = a; \quad aa = a.$
2. $a + b = b + a; \quad ab = ba.$
3. $a + (b + c) = (a + b) + c; \quad a(bc) = (ab)c.$
4. $a(a + b) = a; \quad a + ab = a.$

These axioms occurred in the definition of a Boolean algebra. They are called *idempotent*, *commutative*, *associative* and *absorption laws*, respectively.

Therefore, to any lattice we have assigned an algebra with two binary operations, addition and multiplication, which satisfy conditions 1 – 4.

Proposition 1.1.17. *Define an order relation on an algebra A satisfying axioms 1 – 4 by the rule: $a \leq b$ if $ab = a$. Then A becomes a lattice, and the algebra corresponding to it coincides with the original algebra A.* $\square$

The proposition shows that a lattice can be defined as an algebra of a specific kind.

Definition 1.1.18. *A lattice is said to be distributive, if it satisfies the identity $a(b + c) = ab + ac$.*

Proposition 1.1.19. *A lattice A is distributive if it satisfies one of the equivalent conditions*

1. $a(b + c) = ab + ac,$

2. $a + bc = (a + b)(a + c),$

3. $a(b+c) \leq ab+ac$,

4. $a+b=a+c$ *and* $ab=ac$ *imply* $b=c$. □

A lattice is said to be *bounded* if it has elements 0 and 1 such that $0 \leq a \leq 1$ for every $a \in A$. If A is a bounded lattice and $a \in A$, then b is a *complement* of a if $a+b=1$ and $ab=0$. If A is a bounded distributive lattice, then every element $a \in A$ has at most one complement. Indeed, let $a+b_1=a+b_2$ and $ab_1=ab_2$. Then $b_1=b_1+ab_1=b_1+ab_2=(b_1+a)(b_1+b_2)=(b_2+a)(b_2+b_1)=b_2+ab_1=b_2+ab_2=b_2$. A bounded lattice A is called *complemented* if every element of A has a complement. Hence in a complemented distributive lattice each element a has a unique complement denoted by $\overline{a}$. This means that $^-$ is an unary operation on a complemented distributive lattice.

Definition 1.1.20. *A lattice A is Boolean if it is distributive, has 0 and 1, and every element of A has a complement.*

Proposition 1.1.21. *Let A be a Boolean lattice. Then A is a Boolean algebra with respect to the operations $+$, $\cdot$, and $^-$ defined on it. Denote this algebra by $B=A^*$. Conversely, if B is a Boolean algebra, then, by setting $a \leq b$ if $ab=a$, we obtain a Boolean lattice $A=B^*$. Moreover, $A^{**}=A$, and $B^{**}=B$.* □

The proposition means that, when dealing with Boolean algebras, we may treat them as Boolean lattices, and vice versa.

Definition 1.1.22. *A lattice is called modular if it satisfies the identity $a(ab+c)=ab+ac$.*

This property is equivalent to the following one: if $a \leq c$, then $(a+b)c=a+bc$. Every distributive lattice is modular since the modular identity is a particular case of the distributive law.

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Ideals and filters.

Definition 1.1.23. *A subset U of a Boolean algebra A is called an ideal of A if U is closed with respect to addition and $ab \in U$ whenever $a \in U$ and $b \in A$.*

In fact, a subset U of A closed with respect to addition is an ideal if and only if $a \in U$ implies $b \in U$ for any $b \leq a$. In particular, an ideal always contains the zero element of the algebra. Every Boolean algebra H has the trivial ideal consisting only of the zero element.

The minimal ideal containing $a \in A$ is denoted by $\langle a\rangle_{id}$. For an arbitrary $a \in A$, the ideal $\langle a\rangle_{id}$ consists of all elements b, such that

$b \leq a$. Any ideal of such kind is called *principal.* An ideal U of a Boolean algebra B is proper if $U \neq B$. Clearly, an ideal is proper if and only if it does not contain 1.

Proposition 1.1.24. *A subset U of a Boolean algebra A is an ideal if only if U is an ideal of the corresponding Boolean ring. Every ideal U of A determines a congruence ρ of the Boolean algebra as follows: $a\rho b$ if and only if $\overline{a}b + a\overline{b} \in U$. Every congruence can be obtained in such a way.* $\square$

If $\mu : A \to B$ is a homomorphism of Boolean algebras then the equivalence classes with respect to $Ker\mu$ are determined by the set $\mu^{-1}(0_B)$. This is an ideal of A and each ideal of A can be represented in such a way. However, there is another approach to characterize these classes using the inverse image of 1_B. It leads to the notion of filter which is dual to the notion of ideal.

Definition 1.1.25. *A subset F of a Boolean algebra A is a filter of A if it is closed with respect to multiplication, and $a + b \in U$ whenever $a \in U$ and $b \in A$.*

A statement dual to one for ideals states that F is a filter of A if and only if U is closed with respect to multiplication and $a \in F$ implies $b \in F$ for any $b > a$. In particular, a filter always contains the unit element of A. A filter is called trivial if it coincides with 1.

For any $a \in A$, the set $F = \langle a \rangle_{fil}$ consisting of all the elements $b > a$ is a filter, called the *principal filter* generated by a.A filter F of a Boolean algebra B is proper if $F \neq B$. Clearly, F is proper if and only if it does not contain 0.

The duality between ideals and filters in Boolean algebras is given via the operation $^-$. If U is an ideal, then the set $F = \overline{U}$ consisting of all $\overline{u}$, such that $u \in U$, is a filter. Conversely, the ideal $U = \overline{F}$ corresponds to the filter F, and $\overline{\overline{U}} = U$, $\overline{\overline{F}} = F$. Duality also implies

Proposition 1.1.26. *Every filter F of A determines a congruence ρ of the Boolean algebra as follows: $a\rho b$ if and only if*

$$(\bar{a} + b)(\bar{b} + a) \in F.$$

The same congruence is determined by the ideal $U = \bar{F}$. Every congruence can be obtained in such a way. $\square$

We denote the quotient algebra A/ρ by A/U or A/F as well. We will see that the notion of a filter is tightly related to derivability of formulas in propositional calculus. This is a reason to write $a \to b$ for $\overline{a} + b$ and regard $\to$ as a derived binary operation on a Boolean algebra.Then

Proposition 1.1.27. *A subset F of a Boolean algebra A is a filter if and only if the following two conditions hold:*
1. $1 \in F$,
2. *if* $a \in F$ *and* $a \to b \in F$, *then* $b \in F$.

Proof. See Proposition 3.2.10. □

Corollary 1.1.28. *A subset U of a Boolean algebra A is an ideal if and only if the following two conditions hold:*
1. $0 \in U$,
2. *if* $a \in U$ *and* $b \setminus a = b\bar{a} \in U$, *then* $b \in U$.

A subset C of a Boolean algebra B has the finite intersection property if for every finite set of elements $\{c_1, \dots, c_n\}$ in C their product $c_1 \cdots c_n$ is not 0.

Proposition 1.1.29. *Any subset C with finite intersection property lies in a minimal proper filter. This is the filter generated by C.*

Indeed, one can add to C all finite products of elements of C and extend the obtained set with all bigger elements. The obtained filter is proper since it does not contain zero.

Now we postpone further consideration of Boolean algebras till Section 3.2.2

1.1.4 Multi-sorted algebras

Our next aim is to define multi-sorted algebras. There are many reasons to deal with algebras of such kind. For instance, we will need a multi-sorted variant of Halmos algebras (see Part 2, Chapter **??**
!!! Ssylka ???) in order to work with finite dimensional affine spaces and construct a geometry related to first-order calculus in an arbitrary variety Θ.

Basic definitions

Let Γ be an arbitrary set, which is treated as a set of sorts. There are no restrictions on Γ, this set can be finite or infinite. Consider a *multi-sorted set* $D = (D_i,\ i \in \Gamma)$, where Γ is a set of sorts, and D_i is a set called a *domain of the sort* i. Now we shall make D a multi-sorted algebra.

Every operation ω on D has a specific type $\tau = \tau(\omega)$, which is an $(n+1)$–tuple of the form $(i_1, \dots, i_n; j)$, $i_k,\ j \in \Gamma$. This notion generalizes the notion of arity of an operation defined on a one-sorted set.

Definition 1.1.30. *A map*

$$\omega : D_{i_1} \times \cdots \times D_{i_n} \to D_j$$

is called an operation on D of the type $(i_1, \dots, i_n; j)$.

Denote by Ω a set of symbols of operations.

Definition 1.1.31. *A set Ω of symbols of operations, such that each symbol $\omega \in \Omega$ is equipped with an $(n+1)$-tuple $\tau(\omega) = (i_1, \dots, i_n; j)$, where $i_k, j \in \Gamma$, is called a signature of multi-sorted operations. A tuple (j) corresponds to a symbol of nullary operation, $j \in \Gamma$.*

Definition 1.1.32. *A multi-sorted algebra $D = (D_i,\ i \in \Gamma)$ of the signature Ω is a 4–tuple (D, Γ, Ω, f), where D is a multi-sorted set with set of sorts Γ, Ω is the signature of multi-sorted operations, and the function f realizes every symbol of operation ω of type $\tau = (i_1, \dots, i_n; j)$ as the operation*

$$f(\omega) : D_{i_1} \times \cdots \times D_{i_n} \to D_j.$$

In the sequel we will not use the function f in the notation for operations on D and simply write $D = (D_i,\ i \in \Gamma)$. Now we shall define binary relations on a multi-sorted set $D = (D_i,\ i \in \Gamma)$. Each binary relation ρ has a type τ, which is an n-tuple $(i_1, \dots, i_n)$, $i_k \in \Gamma$.

A binary relation ρ of type τ on $D = (D_i,\ i \in \Gamma)$ is a collection $(\rho_{i_1}, \dots, \rho_{i_n})$, where ρ_{i_k} is a binary relation on D_{i_k}. A relation $\rho = (\rho_{i_1}, \dots, \rho_{i_n})$ is called *an equivalence* if each ρ_{i_k} is an equivalence.

Let now $D = (D_i, i \in \Gamma)$ be an Ω-algebra, ω an operation from Ω of type $\tau(i_1, \dots, i_n; j)$, and ρ an equivalence of type τ. Compatibility of ρ and ω means that if $(a_{i_1}, \dots, a_{i_n})$ and $(a'_{i_1}, \dots, a'_{i_n})$ are elements of $D_{i_1} \times \cdots \times D_{i_n}$, and $a_{i_s} \rho_{i_s} a_{i_s}$ for all $s = 1, \dots, n$, then

$$(a_{i_1} \cdots a_{i_n} \omega) \rho_j (a'_{i_1} \cdots a'_{i_n} \omega).$$

An equivalence ρ is called *a congruence* if it is compatible with all operations $\omega \in \Omega$.

Homomorphisms of multi-sorted algebras act componentwise and are of the form $\mu = (\mu_i, i \in \Gamma) : D \to D'$, where $\mu_i : D_i \to D'_i$ are homomorphisms of algebras and, besides that, every μ respects each operation ω of type $\tau = (i_1, \dots, i_n; j)$:

$$(a_1 \cdots a_n \omega)^{\mu_j} = a_1^{\mu_1} \cdots a_n^{\mu_n} \omega, \quad a_s \in D_{i_s},\ s = 1, \dots, n.$$

The kernel of a homomorphism $\mu : D = (D_i,\ i \in \Gamma) \to D' = (D'_i,\ i \in \Gamma)$ is of the form $\rho = (\rho_i,\ i \in \Gamma)$ where each ρ_i is the kernel congruence of the homomorphism μ_i. If ρ is a congruence, then *the quotient algebra* D/ρ is defined as follows:

$$D/\rho = (D_i/\rho_i,\ i \in \Gamma).$$

Subalgebras and Cartesian products of multi-sorted algebras are defined in a usual way. For example, if $D^\alpha = (D_i^\alpha, i \in \Gamma)$, $\alpha \in I$, are Ω-algebras, then the Cartesian product is the algebra

$$\prod_\alpha D^\alpha = (\prod_\alpha D_i^\alpha,\ i \in \Gamma),$$

and if ω is an operation of type $\tau(\omega) = (i_1, \ldots, i_n; j)$, then

$$(a_1 a_2 \cdots a_n \omega)(\alpha) = a_1(\alpha) a_2(\alpha) \cdots a_n(\alpha)\omega,$$

where $a_1 \in \prod_\alpha D_{i_1}^\alpha, \ldots, a_n \in \prod_\alpha D_{i_n}^\alpha$.

The following general fact, known as Remak's theorem, remains true for multi-sorted algebras.

Theorem 1.1.33. *Let $D = (D_i,\ i \in \Gamma)$ be an Ω-algebra, and let a collection of congruences $\rho_\alpha, \alpha \in I$, be given; we set $\rho = \bigcap_\alpha \rho_\alpha$. Then the quotient algebra D/ρ can be embedded as a subalgebra into the Cartesian product of all D/ρ_α.* $\square$

Examples

Let us consider some examples of multi-sorted algebras.

• A *semigroup representation* is a two-sorted algebra (V, S), where V is a set, S is a semigroup acting on V by the operation $\circ\colon V \times S \to V$ subject to condition

$$u \circ s_1 s_2 = (u \circ s_1) \circ s_2,$$

where $u \in V,\ s_1, s_2 \in S$. Any semigroup representation (V, S) defines a homomorphism $\nu\colon S \to End(V)$ and vice versa.

• A *linear semigroup representation* is a two-sorted algebra (V, S), where V is a R-module, S is a semigroup acting on V by the operation $\circ\colon V \times S \to V$ subject to conditions

$$u_1 \circ s_1 s_2 = (u_1 \circ s_1) \circ s_2,$$

$$(\alpha_1 u_1 + \alpha_2 u_2) \circ s = \alpha_1(u_1 \circ s) + \alpha_2(u_2 \circ s),$$

where $\alpha_1, \alpha_2 \in R\ u_1, u_2 \in V,\ s_1, s_2 \in S$. Any linear semigroup representation (V, S) defines a homomorphism $\nu\colon S \to End(V)$ and vice versa. In view of this observation one can define:

• A *linear group representation* (V, G) is defined if a group homomorphism $\nu\colon G \to AutV = GL(V)$ is given. Introducing the operation $\circ : V \times G \to V$ by

$$a \circ g = ag^{\nu},$$

we view the group representation as a two-sorted algebra.

• A *pure automaton* $\mathfrak{A} = (A, B, C)$ is a three-sorted algebra with two operations, $\circ\colon A \times B \to A$ and $*\colon A \times B \to C$. The corresponding set of sorts consists of three elements: $\Gamma = \{1, 2, 3\}$; 1 corresponds to the set of states A, 2 corresponds to the set of input signals B, and 3 corresponds to the set of outputs C. The operations $\circ$ and $*$ are of types $(1, 2; 1)$ and $(1, 2; 3)$, respectively. No axioms are assumed here. If either of the sets A, B, C is a semigroup or an R-module, then a bunch of axioms appear.

• A *semigroup automaton* $\mathfrak{A} = (A, B, C)$ is an automaton in which B is a semigroup and the operations $\circ\colon A \times B \to A$ and $*\colon A \times B \to C$ are subject to conditions

1. $a \circ b_1 b_2 = (a \circ b_1) \circ b_2$,
2. $a * b_1 b_2 = (a \circ b_1) * b_2$.

where $a \in A$ and $b_1, b_2 \in B$. Any semigroup representation can be viewed as a particular case $C = 0$ of a semigroup automaton.

• A *linear semigroup automaton* $\mathfrak{A} = (A, B, C)$ is a semigroup automaton where A and C are vector spaces over a field or, more generally, modules over a commutative ring R with unit, and the operations $a \to a \circ b$ and $a \to a * b$ are linear maps for any $b \in B$.

Any linear semigroup representation can be viewed as a particular case of a linear semigroup automaton.

According to the general definition, a pure automata *homomorphism* is a map $\mu = (\alpha, \beta, \gamma)\colon (A, B, C) \to (A', B', C')$ which is compatible with the operations. The compatibility conditions are of the form $(a \circ b)^{\alpha} = a^{\alpha} \circ b^{\beta}$ and $(a * b)^{\gamma} = a^{\alpha} * b^{\beta}$.

For semigroup automata, we also assume that the map $\beta\colon B \to B'$ is *a semigroup homomorphism*, while for linear automata we suppose that $\alpha\colon A \to A'$ and $\gamma\colon C \to C'$ are linear maps.

A congruence of an automaton (A, B, C) is a triple of equivalence relations $\rho = (\rho_1, \rho_2, \rho_3)$ compatible with all operations.

1.1.5 Free algebras

One-sorted case

Let us start with the one-sorted case. Assume that a signature of operations Ω and a set X are fixed. The set X will play a role of an alphabet while the operations $\omega \in \Omega$ provide the rules for constructing elements of the absolutely free algebra $\Omega(X)$ over X. We call elements of $\Omega(X)$ words over the given set X. The rules are as follows.

1. All elements of X and all symbols of nullary operations are words in $\Omega(X)$.

2. If ω is an n-ary operation symbol from Ω, $n > 0$, and if $w_1, \ldots, w_n$ are words, then the expression $w_1 \cdots w_n\omega$ is a word in $\Omega(X)$.

3. There are no other rules.

The set $\Omega(X)$ is defined in such a way that all operation symbols from Ω are naturally realized in $\Omega(X)$. Indeed, if ω is an n-ary operation symbol and $w_1, \ldots, w_n$ are words, then, by definition of $\Omega(X)$, the expression $w_1 \cdots w_n\omega$ is a word. So, define the realization $w_{\Omega(X)}$ of ω in $\Omega(X)$ by

$$w_1 \cdots w_n\omega_{\Omega(X)} = w_1 \cdots w_n\omega.$$

A nullary symbol selects the word corresponding to it.

Definition 1.1.34. *The obtained Ω-algebra $\Omega(X)$ is called the absolutely free Ω-algebra over X, or the algebra of words over X.*

This algebra is sometimes called *the term algebra* (see, for example, [DenWis]).

This algebra possesses the following important freeness property in the class of all Ω-algebras.

Theorem 1.1.35. *Every map $\mu\colon X \to A$, where A is an Ω-algebra, has a unique extension up to a homomorphism $\mu^*\colon \Omega(X) \to A$ of Ω-algebras.*

Proof. First, we shall verify that such an extension exists. Assign to every symbol of a nullary operation in $\Omega(X)$ the corresponding element of A, and then proceed in accordance with the recursive definition of a word. If $w \in X$, then $w^{\mu^*} = w^{\mu}$. If ω is an n-ary operation symbol, $n > 0$, and $w_1, \ldots, w_n$ are words for which the elements $w_1^{\mu^*}, \ldots, w_n^{\mu^*}$ in A are already defined, then we set $(w_1 \cdots w_n\omega)^{\mu^*} = w_1^{\mu^*} \cdots w_n^{\mu^*}\omega$. This rule defines the map μ^* for every element of $\Omega(X)$. By definition, μ^* is a homomorphism extending the map μ. Every homomorphism $\nu : \Omega(X) \to A$ coincides with μ^* since $\nu = \mu^*$ on the set X. $\square$

In the sequel we often denote the map μ and the homomorphism μ^* by the same letter μ.

Theorem 1.1.35 implies that every Ω-algebra A can be represented as a homomorphic image of an absolutely free algebra $\Omega(X)$. Denote by ρ the kernel of the corresponding surjection. Then the quotient algebra $\Omega(X)/\rho$ is isomorphic to A.

It is easy to see that $\langle X\rangle = \Omega(X)$. Hence, X is the *generating set* of $\Omega(X)$.

A class $\mathfrak{X}$ of Ω-algebras is called *abstract* if it contains all algebras isomorphic to a given algebra from $\mathfrak{X}$. Suppose that $\mathfrak{X}$ is an abstract class of Ω-algebras. The property described in Theorem 1.1.35 is a characteristic property and can be taken for the definition of a free in $\mathfrak{X}$ algebra over a set X.

Definition 1.1.36. *Let $W(X)$ be an algebra in $\mathfrak{X}$ with the set of generators X. The algebra $W(X)$ is called free in $\mathfrak{X}$ if for any algebra A in $\mathfrak{X}$ and any map $\nu : X \to A$, there exists a homomorphism $\mu : W(X) \to A$ such that the diagram*

$$\begin{tikzcd} X \arrow[rr, "id"] \arrow[dr, "\nu"'] & & W(X) \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

is commutative. Here id denotes the identity map.

Note that not every abstract class of algebras $\mathfrak{X}$ possesses free algebras. We hold fixed the alphabet X and the algebra $\Omega(X)$ over it. Consider formal expressions (formulas) of the kind $w_1 \equiv w_2$, where w_1 and w_2 are words of $\Omega(X)$.

Definition 1.1.37. *An expression (formula) $w_1 \equiv w_2$, where $w_1, w_2 \in \Omega(X)$, is an identity of an Ω-algebra A if for every homomorphism $\mu\colon \Omega(X) \to A$ we have $w_1^\mu = w_2^\mu$ in A.*

This means that if $w_1 = w_1(x_1, \dots, x_k)$ and $w_2 = w_2(x_1, \dots, x_k)$, then any substitution $x_i \to a_i$, where $a_i \in A$, results into an equality of the corresponding elements of A.

Definition 1.1.38. *A class of algebras satisfying a set of identities is called a variety.*

A formula $w_1 \equiv w_2$ is an *identity of a class* $\mathfrak{X}$ of algebras if it is an identity of every algebra from $\mathfrak{X}$.

In every variety Θ a set X determines a free algebra, which is a quotient algebra of the absolutely free algebra (see Proposition 1.1.42). Given the variety Θ, a set of identities $w_1 \equiv w_2$ in the free in Θ algebra $W(X)$ determines subvarieties of Θ.

The next aim is to give an invariant characteristic of varieties. Let $\mathfrak{X}$ be a class of Ω-algebras. Define

1. $Q(\mathfrak{X})$ is the class of algebras isomorphic to homomorphic images of algebras of $\mathfrak{X}$.

2. $S(\mathfrak{X})$ is the class of algebras isomorphic to subalgebras of algebras from Θ.

3. $C(\mathfrak{X})$ is the class algebras isomorphic to Cartesian products of algebras from $\mathfrak{X}$.

A class $\mathfrak{X}$ *is closed with respect* Q, S, C if $Q(\mathfrak{X}) \subset \mathfrak{X}$, $S(\mathfrak{X}) \subset \mathfrak{X}$, $C(\mathfrak{X}) \subset \mathfrak{X}$, respectively. *The closure of a class* $\mathfrak{X}$ is the minimal closed class of algebras containing $\mathfrak{X}$.

Birkhoff's theorem states:

Theorem 1.1.39. *A class $\mathfrak{X}$ of Ω-algebras is a variety if and only if it is closed with respect to the operators Q, S and C. The closure of $\mathfrak{X}$ is denoted $Var(\mathfrak{X})$ and equals $QSC(\mathfrak{X})$.* $\square$

Hence, the minimal variety containing a given class of algebras $\mathfrak{X}$ is $\Theta = Var(\mathfrak{X}) = QSC(\mathfrak{X})$. This variety Θ is said to be *generated by* $\mathfrak{X}$.

In a variety Θ one can consider free products of algebras.

Definition 1.1.40. *Given two algebras A and B from Θ, the free product $A * B$ is an algebra with homomorphisms $i_A : A \to A * B$ and $i_B : B \to A * B$, such that for any algebra $H \in \Theta$ with homomorphisms $\mu : A \to H$ and $\nu : B \to H$ there is a unique homomorphism*

$$\mu * \nu : A * B \to H$$

extending μ and ν.

Although the free products of algebras exist in any variety, specific constructions realizing free products depend on a particular variety Θ. Free products of algebras can be generalized by the notion of amalgamated products of algebras glued together along a subalgebra.

This construction is most explicit for the case of the amalgamated product of two groups. Let groups G_0, G_1, G_2 and homomorphisms $\varphi_1 : G_0 \to G_1$ and $\varphi_2 : G_0 \to G_2$ be given. Denote by N the normal subgroup of the free product $G_1 * G_2$ generated by the elements $\varphi_1(h)\varphi_2(h)^{-1}$, $h \in G_0$.

Definition 1.1.41. *The quotient group*

$$(G_1 * G_2)/N$$

is called the amalgamated product of the groups G_1 and G_2 over the group G_0.

The free product $G_1 * G_2$ is a particular case of this construction when G_0 is a trivial group. The definition of the amalgamated product for algebras is similar to that for groups. One should replace the normal subgroup N by a congruence generated by $\varphi_1(h)\varphi_2(h)^{-1}$.

Multi-sorted case

The whole setting of free algebras considered above for the one-sorted case can be transferred to the multi-sorted one.

We keep fixed the set Ω of operation symbols, and choose a multi-sorted set $X = (X_i, i \in \Gamma)$ which will play the role of the multi-sorted alphabet. Define the set $\Omega(X)$ of Ω-words as follows.

1. All elements of X_i and all symbols of nullary operations of type (i) are words of sort i, $i \in \Gamma$.
2. If ω is an operation symbol of type $\tau = (i_1, \dots, i_n; j)$ from Ω, and $w_1, \dots, w_n$ are words of sorts $i_1, \dots, i_n$ respectively, then $w_1 \dots w_n \omega$ is a word of sort j.
3. Any word is constructed only by use of these rules.

Similarly to the one-sorted case we come up with the *absolutely free multi-sorted* Ω-algebra $\Omega(X) = (\Omega(X_i),\ i \in \Gamma)$ generated by $X = (X_i,\ i \in \Gamma)$. In particular, every map $X = (X_i,\ i \in \Gamma) \to D = (D_i,\ i \in \Gamma)$, where D is an Ω-algebra, is extended uniquely to a homomorphism $\Omega(X) \to D$.

An expression (or formula) $w_1 \equiv w_2$, where w_1 and w_2 are words of $\Omega(X) = (\Omega(X_i),\ i \in \Gamma)$ of the same sort, is called *an identity of an algebra* $A = (A_i, i \in \Gamma)$ if for any homomorphism

$$\mu = (\mu_i, i \in \Gamma) \colon \Omega(X) = (\Omega(X_i),\ i \in \Gamma) \to A = (A_i, i \in \Gamma)$$

we have $w_1^{\mu_i} = w_2^{\mu_i}$, where i is the sort of w_1 and w_2.

Analogously to Definition 1.1.38, a class of multi-sorted algebras satisfying a set of identities is called *a variety* and any $X = (X_i,\ i \in \Gamma)$ with non-empty domains determines a free algebra $W(X) = (W(X_i),\ i \in \Gamma)$ of the variety, which is a quotient algebra of the absolutely free algebra $\Omega(X) = (\Omega(X_i),\ i \in \Gamma)$.

Indeed, given a variety of algebras Θ, consider various homomorphisms $\mu\colon \Omega(X) \to A$ for all algebras A from Θ. Let $\rho = (\rho_i,\, i \in \Gamma)$ be the intersection of the kernels of all these homomorphisms. Then

Proposition 1.1.42. *$W(X) = \Omega(X)/\rho$ is the free algebra over X in Θ.* $\square$

It follows immediately from definitions that a formula $w_1 \equiv w_2$, where w_1, w_2 are words of sort i, is an identity of Θ if and only if $w_1\rho_i w_2$. Here ρ_i is the i-th component of ρ. The relation ρ is called *a verbal congruence* with respect to Θ.

Birkhoff's theorem remains true in the multi-sorted case, that is if $\mathfrak{X}$ is a class of multi-sorted algebras, then $Var(\mathfrak{X}) = QSC(\mathfrak{X})$.

Now we want to characterize congruences in the free algebra $W(X)$ that correspond to varieties.

Definition 1.1.43. *Let $\rho = (\rho_i, i \in \Gamma)$ be a congruence of an algebra $A = (A_i, i \in \Gamma)$, and $\mu = (\mu_i, i \in \Gamma)$ an endomorphism of A. A congruence $\rho = (\rho_i,\, i \in \Gamma)$ is called a fully invariant congruence if $a_1\rho_i a_2$ implies $a_1^{\mu_i}\rho_i a_2^{\mu_i}$ for every $i \in \Gamma$, $a_1, a_2 \in A_i$.*

Proposition 1.1.44. *There is a one-to-one correspondence between fully invariant congruences $\rho = (\rho_i,\ i \in \Gamma)$ of the free algebra $W(X) = (W(X_i),\, i \in \Gamma)$ and subvarieties in Θ.*

1.1.6 Classes of algebras

Varieties, i.e., classes of algebras defined by identities, present one of the most interesting classes of algebras. We shall start with a list of very important varieties:

1. Variety of groups Grp.
2. Variety of semigroups Smg.
3. Variety of associative algebras over a field K, $Ass - K$.
4. Variety of associative and commutative algebras over a field K, $Com - K$.
5. Variety of Lie algebras over a field K, $Lie - K$.
6. Variety of lattices.
7. Variety of Boolean algebras.
8. Variety of representations of groups over a ring R, $Rep - R$.

All varieties of the above list except the last one are varieties of one-sorted algebras. The variety $Rep - R$ is a variety of two-sorted algebras.

Each of these varieties is a universe where the corresponding theories live. In particular, universal algebraic geometry and logic, the subjects of this book, live there. Recall that identities of a variety Θ are simplest formulas of the form $w_1 \equiv w_2$, where w_1, w_2 are words of a free algebra $W(X)$ in Θ. Let us call such formulas atoms. We may and shall consider algebraic structures defined by arbitrary sets of formulas.

Formulas are produced from atoms by means of Boolean connectives and quantifiers. For a precise definition of a formula with respect to a signature of logical operations see Definition 3.1.6. By now, we shall assume that all this is intuitively clear, noting that given $w_i = w_i(x_1, \ldots, x_k)$, the expression

$$\forall x_1 \exists x_3((w_1 \equiv w_3) \vee (w_1 \equiv w_3))$$

is an example of a formula while

$$\forall x_1 \exists x_3((w_1 \equiv w_3) \vee w_1)$$

is not. It will also be explained what does it mean precisely that a formula u is valid on an Ω-algebra H. Meantime we can think that a formula $u = u(x_1, \ldots, x_k)$ is valid on the point $(h_1, \ldots, h_k) \in H^k$ if replacing the variables x_i in u by the elements h_i we obtain a true statement in H. The formula u is valid on H if u holds for any point from H^k.

An *axiomatic*, or *axiomatizable class* of Ω-algebras $\mathfrak{C}$ is a class defined by some collection of formulas S. This means that $\mathfrak{C}$ consists of all Ω-algebras satisfying all formulas from S. If otherwise is not explicitly stated, we consider only first-order formulas, i.e., formulas of the first-order predicate calculus. We emphasize that formulas of S can be multi-sorted, and that axiomatizable classes have certain signature of Ω-algebras.

An axiomatic class $\mathfrak{C}$ is said to be *universal,* or *universally axiomatizable*, if it can be defined by a set S of universal formulas, i.e., formulas which being rewritten in the so-called prenex normal form (see [Mendelson]) do not contain existential quantifiers. The latter means that a formula u is universal if it is equivalent to a formula of the kind $\forall x_1 \forall x_2 \ldots \forall x_s$ (quantifier-free part). The formulas of S are called axioms of the class $\mathfrak{C}$.

Definition 1.1.45. *A formula $u = u(x_1, \ldots, x_k)$ of the form*

$$w_1 \equiv v_1 \wedge \cdots \wedge w_n \equiv v_n \to w \equiv v,$$

where w_i, v_i, w, v belong to $W(X)$, is called a quasi-identity.

A quasi-identity is satisfied in an Ω-algebra H if

$$w_1^\mu = v_1^\mu \wedge \cdots \wedge w_n^\mu = v_n^\mu \to w^\mu = v^\mu$$

is true in H for any homomorphism $\mu\colon W(X) \to H$.

Definition 1.1.46. *A class $\mathfrak{C}$ of Ω-algebras is called a quasivariety if it is defined by a set of quasi-identities.*

Definition 1.1.47. *A formula $u = u(x_1, \ldots, x_k)$ of the form*

$$w_1 \equiv v_1 \vee w_2 \equiv v_2 \vee \cdots \vee w_n \equiv v_n,$$

where w_i, v_i belong to $W(X)$, is called a pseudo-identity.

A pseudo-identity is satisfied in an Ω-algebra H if the statement

$$w_1^\mu = v_1^\mu \vee \cdots \vee w_n^\mu = v_n^\mu$$

is true in H for any homomorphism $\mu\colon W(X) \to H$.

Definition 1.1.48. *A class $\mathfrak{C}$ of Ω-algebras is called a pseudovariety if it is defined by a set of pseudo-identities.*

Remark 1.1.49. *The term "pseudovariety" is often used for classes of algebras closed under homomorphic images, subalgebras, and finite Cartesian products. In this sense all solvable and all nilpotent groups constitute a pseudovariety. In this book the term "pseudovariety" is reserved for classes of algebras in the sense of Definition 1.1.48.*

Remark 1.1.50. *Quasivarieties (hence, varieties) and pseudovarieties are axiomatizable classes of algebras. They can be written in the form*

$$\forall x_1 \ldots \forall x_n (w_1 \equiv v_1 \wedge \cdots \wedge w_n \equiv v_n \to w \equiv v),$$

$$\forall x_1 \ldots \forall x_n (w_1 \equiv v_1 \vee w_2 \equiv v_2 \vee \cdots \vee w_n \equiv v_n),$$

respectively. Note that any identity $w = w'$, where $w = w(x_1, \ldots, x_n)$, $w' = w'(x_1, \ldots, x_n)$ can be represented as the quasi-identity:

$$\forall x_1 \ldots \forall x_n ((x_1 = x_1) \wedge \ldots \wedge (x_n = x_n) \to (w = w')).$$

Thus, a variety of algebras is a quasivariety.

Quasivarieties and pseudovarieties admit invariant Birkhoff-type characterizations that use an important construction of a reduced product. Define, first, filters over sets (cf. filters for Boolean algebras).

Definition 1.1.51. *Let I be a non-empty set. A filter over I is a collection D of non-empty subsets of I subject to conditions:*

1. *The intersection of two subsets from D belongs to D.*
2. *If $J \in D$, then every subset J' of I, including J, belongs to D as well.*
3. *The empty set $\varnothing$ does not belong to D.*

A filter D is said to be an *ultrafilter* if it is not included in a larger filter. This condition is equivalent to another one: for every subset J of I, either J or its complement $\bar{J}$ is an element of D (not both!).

Now assume that a collection of Ω-algebras $\mathfrak{A}_\alpha = (A_i^\alpha, i \in \Gamma)$, $\alpha \in I$, is given. We are going to define the reduced product of these algebras relatively to a filter D. We start with the Cartesian product $\mathfrak{A} = \prod_\alpha \mathfrak{A}_\alpha$. Then

$$\mathfrak{A} = (A_i, i \in \Gamma) = (\prod_\alpha A_i^\alpha, i \in \Gamma).$$

Define a relation $\rho = (\rho_i, i \in \Gamma)$ on $\mathfrak{A}$: if a and a' belong to A_i, then $a\rho_i a'$ means that

$$J = \{\alpha\colon\ a(\alpha) = a'(\alpha)\} \in D.$$

This relation is a congruence on $\mathfrak{A}$.

Definition 1.1.52. *The filtered product of the algebras $\mathfrak{A}_\alpha$ relatively to the filter D is the quotient algebra $\mathfrak{A}/\rho$. If D is an ultrafilter, then the corresponding filtered product is called an ultraproduct.*

The reduced product of algebras $\mathfrak{A}_\alpha$ with respect to the filter D is denoted by $\prod_{\alpha\in D} \mathfrak{A}_\alpha$. If all A_α coincide with A and D is an ultrafilter, then $\prod_{\alpha\in D} \mathfrak{A}$ is called the ultrapower of A. Applications of this construction are based on the following theorem.

Theorem 1.1.53. *Every axiomatic class of Ω-algebras is closed with respect to ultraproducts.* $\square$

The following three theorems give us the characterization of universal classes of algebras, quasivarieties and pseudovarieties.

Theorem 1.1.54. *A class of Ω-algebras $\mathfrak{X}$ is universal if and only if the following conditions are fulfilled:*

1. *$\mathfrak{X}$ is abstract,*

2. *$\mathfrak{X}$ is closed with respect to subalgebras,*

3. *$\mathfrak{X}$ is closed with respect to ultraproducts.* □

Theorem 1.1.55. *A class of Ω-algebras $\mathfrak{X}$ is a quasivariety if and only if it satisfies the following conditions:*

1. *$\mathfrak{X}$ is an abstract class containing the trivial algebra,*

2. *$\mathfrak{X}$ is closed with respect to subalgebras,*

3. *$\mathfrak{X}$ is closed with respect to filtered products.* □

Theorem 1.1.56. *A class of Ω-algebras $\mathfrak{X}$ is a pseudovariety if and only if*

1. *$\mathfrak{X}$ is closed with respect to subalgebras,*

2. *$\mathfrak{X}$ is closed with respect to homomorphic images,*

3. *$\mathfrak{X}$ is closed with respect to ultraproducts.* □

Let $\mathfrak{X}$ be an arbitrary class of Ω-algebras. Birkhoff's theorem states that the variety generated by $\mathfrak{X}$ is $QSC(\mathfrak{X})$. Similar characterizations for universal classes of algebras, quasivarieties and pseudovarieties look as follows. Denote by $Uc(\mathfrak{X})$, $qVar(\mathfrak{X})$, and $PsVar(\mathfrak{X})$ the minimal universal class, minimal quasivariety, and minimal pseudovariety of algebras containing $\mathfrak{X}$, respectively. Then the Birkhoff-type theorem ([Mal2],[GrL], [Pl-Datab],[MR]) is as follows:

Theorem 1.1.57. *Let $P_u(\mathfrak{X})$ be the class of algebras isomorphic to ultraproducts of algebras of $\mathfrak{X}$. Then*

1. *$Uc(\mathfrak{X}) = SP_u(\mathfrak{X})$,*

2. *$qVar(\mathfrak{X}) = SCP_u(\mathfrak{X})$,*

3. *$PsVar(\mathfrak{X}) = QSP_u(\mathfrak{X})$.* □

We will need one more class of Ω-algebras.

Definition 1.1.58. *A class $\mathfrak{X}$ is called a prevariety if it is closed under taking subgroups and Cartesian products.*

This class is not necessarily an axiomatizable class of algebras.

Proposition 1.1.59. *Given a class $\mathfrak{X}$ of Ω-algebras, $SC(\mathfrak{X})$ is the prevariety generated by $\mathfrak{X}$.*

1.2 Category Theory

1.2.1 Categories

Definition of a category. Examples

We shall start with basic definitions.

Definition 1.2.1. *A category $\mathcal{K}$ consists of objects and morphisms.*

1. *Let $Ob\mathcal{K}$ denote the class of all objects of category $\mathcal{K}$, and let $Mor\mathcal{K}$ be the class of all its morphisms. A set of morphisms $Mor(A, B)$ is associated with any pair of objects A and B from $Ob\mathcal{K}$. Denote elements of this set by $f\colon A \to B$, or $A \xrightarrow{f} B$, or simply f. We suppose that the class $Mor\mathcal{K}$ is a disjoint union of the sets $Mor(A, B)$. Some of these sets may be empty.*

2. *For any triple of objects A, B and C from $Ob\mathcal{K}$, we are given a map*
$$Mor(A, B) \times Mor(B, C) \to Mor(A, C).$$
This enables us to speak about composite maps, i.e., about products of morphisms. Denote the product of morphisms $f\colon A \to B$ and $\varphi\colon B \to C$ by $f\varphi\colon A \to C$.

   Moreover, we assume that the product of morphisms has the following properties:

   a) $f(\varphi\psi) = (f\varphi)\psi$ for any $f\colon A \to B$, $\varphi\colon B \to C$, and $\psi\colon C \to D$, i.e., the product of morphisms is associative;

   b) for any $A \in Ob\mathcal{K}$ there exists an identity morphism $\varepsilon_A\colon A \to A$ such that $\varepsilon_A f = f$ and $\varphi\varepsilon_A = \varphi$ for any $f\colon A \to B$ and $\varphi\colon C \to A$.

   Such a morphism ε_A is easily seen to be unique for any A.

The set $End(A) = Mor(A, A)$ is a semigroup for any object A. This semigroup is said to be the *endomorphism semigroup* of A. The morphism ε_A is the identity element of $End(A)$. All invertible elements of $End(A)$ form a group. This is the automorphism group of the object A which will be denoted by $Aut(A)$.

Example 1.2.2. Sets and their maps form the category Set. The class of objects of Set is the class of all sets, and $Mor(A, B)$ is $Fun(A, B)$ for any pair A and B. The products of morphisms are defined to be products of maps, the identity map plays the role of ε_A.

One can also consider the category of non-empty sets and the category of finite sets.

Example 1.2.3. The category of Ω-algebras with Ω-algebras as objects and homomorphisms as morphisms.

With every fundamental structure and its homomorphisms one can associate the corresponding category. We obtain therefore the category of groups, the category of semigroups, the category of rings, the category of modules over a given ring, the category of representations, etc.

Example 1.2.4. Let R be a commutative ring. We introduce the category whose objects are all positive integers $1, 2, 3, \ldots$, and morphisms $n \to m$ are $n \times m$-matrices with entries in R. We define the product of morphisms to be the product of matrices. We obtain therefore a category where $\varepsilon_n \colon n \to n$ is the identity matrix of order n.

Example 1.2.5. Let A be a set, and suppose we are given with an order relation on this set. Then A is said to be a *partially ordered* set. We shall construct a category whose objects are the elements of A. For any pair of objects a and b, there is a unique morphism $a \to b$ if $a \leq b$, otherwise $Mor(a, b)$ is empty. Since $a \leq a$, we have the identity morphism $a \to a$, and the product of $a \to b$ and $b \to c$ is $a \to c$ because of transitivity of an order relation. Therefore, any partially ordered set yields a category.

Example 1.2.6. A discrete category is a category with trivial (identical) morphisms. This means that if the objects A and B are distinct, then $Mor(A, B) = \varnothing$. Every class of objects can be equipped with the structure of a discrete category.

Example 1.2.7. The category Bin of binary relations. Its objects are sets, morphisms are presented by binary relations.

It follows from the examples that class $Ob\mathcal{K}$ may be not a set. Also the class $Mor\mathcal{K}$ is not necessarily a set. If $Ob\mathcal{K}$ is a set, then so is $Mor\mathcal{K}$, and the category $\mathcal{K}$ is said to be *small*.

A category $\mathcal{L}$ is said to be a *subcategory* of a category $\mathcal{K}$ if any object of $\mathcal{L}$ belongs to $\mathcal{K}$ and any morphism of $\mathcal{L}$ is a morphism of $\mathcal{K}$. Besides, the product of morphisms in $\mathcal{L}$ coincides with their product in $\mathcal{K}$ and identity morphisms of $\mathcal{L}$ are identity morphisms in $\mathcal{K}$. A category $\mathcal{L}$ is a *full subcategory* of the category $\mathcal{K}$ if, for any $A, B \in Ob\mathcal{L}$, we have $Mor_{\mathcal{L}}(A, B) = Mor_{\mathcal{K}}(A, B)$.

Morphisms. Dual categories

Definition 1.2.8. *A morphism $f \colon A \to B$ in a category $\mathcal{K}$ is said to be a monomorphism in $\mathcal{K}$ if for any two morphisms $\varphi \colon C \to A$*

and $\psi\colon C \to A$, *the equality* $\varphi f = \psi f$ *implies* $\varphi = \psi$.

Epimorphisms are defined dually. A morphism $f\colon A \to B$ *is an epimorphism in* $\mathcal{K}$ *if for any pair* $\varphi\colon B \to C$ *and* $\psi\colon B \to C$, *the equality* $f\varphi = f\psi$ *implies* $\varphi = \psi$.

In any variety of Ω-algebras Θ categorical monomorphisms coincide with injective homomorphism of Ω-algebras. However, surjective homomorphisms of algebras from Θ are always categorical epimorphisms, but not necessarily vice versa.

Definition 1.2.9. *A morphism* $f\colon A \to B$ *is called an isomorphism if there exists a morphism* $f^{-1}\colon B \to A$ *such that* $ff^{-1} = \varepsilon_A$, $f^{-1}f = \varepsilon_B$. *An isomorphism of the form* $A \to A$ *is called an automorphism of the object* A.

Every isomorphism is epimorphic and monomorphic, while a morphism which is both monomorphic and epimorphic may not be an isomorphism.

Definition 1.2.10. *Let* $\mathcal{K}$ *be a category. The dual category (opposite category)* $\mathcal{K}^{op}$ *is defined as follows:*

1. *Objects of* $\mathcal{K}^{op}$ *coincide with objects of* $\mathcal{K}$.
2. *Given objects* A *and* B *in* $\mathcal{K}^{op}$, *we have*

$$Mor_{\mathcal{K}^{op}}(A,B) = Mor_{\mathcal{K}}(B,A).$$

The product of morphisms is taken in a reverse order. Namely, if $f\colon A \to B$ *and* $\varphi\colon B \to C$ *are morphisms in* $\mathcal{K}$, *denote them regarded as morphisms in* $\mathcal{K}^{op}$ *by* $f^{op}\colon B \to A$ *and* $\varphi^{op}\colon C \to B$, *and define their product by*

$$\varphi^{op} f^{op} = (f\varphi)^{op}\colon C \to A.$$

Each notion or construction in the category $\mathcal{K}$ has a mirror in $\mathcal{K}^{op}$ and vice versa. Besides, $\mathcal{K} = (\mathcal{K}^{op})^{op}$.

Functors. Examples

Definition 1.2.11. *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be categories. A covariant functor* $\mathcal{F}\colon \mathcal{K}_1 \to \mathcal{K}_2$ *is a map that takes every object* $A \in Ob\mathcal{K}_1$ *to* $\mathcal{F}(A)$ *in* $Ob\mathcal{K}_2$ *and every morphism* $f \in Mor\mathcal{K}_1$ *to* $\mathcal{F}(f) \in Mor\mathcal{K}_2$. *The functor* $\mathcal{F}$ *satisfies the conditions:*

1. *if* $f \in Mor(A,B)$, *then* $\mathcal{F}(f) \in Mor(\mathcal{F}(A), \mathcal{F}(B))$;
2. $\mathcal{F}(\varepsilon_A) = \varepsilon_{\mathcal{F}(A)}$ *for every* $A \in Ob\mathcal{K}_1$;

3. *$\mathcal{F}(f\varphi) = \mathcal{F}(f)\mathcal{F}(\varphi)$ for any pair of appropriate morphisms in $Mor\mathcal{K}_1$.*

A functor $\mathcal{F}$ is called *contravariant* if the first and the third conditions are replaced by the following two conditions

1'. *if $f \in Mor(A, B)$, then $\mathcal{F}(f) \in Mor(\mathcal{F}(B), \mathcal{F}(A))$;*

3'. *$\mathcal{F}(f\varphi) = \mathcal{F}(\varphi)\mathcal{F}(f)$,*

respectively.

One can view covariant functors as homomorphisms of categories. Correspondingly, contravariant functors are antihomomorphisms of categories.

Example 1.2.12. The identity functor $\mathcal{K}$ to $\mathcal{K}$ is a covariant functor.

Example 1.2.13. For an arbitrary category $\mathcal{K}$, we have a canonical contravariant functor from $\mathcal{K}$ to the dual category $\mathcal{K}^{op}$: $\mathcal{F}(A) = A$, $\mathcal{F}(f) = f^{op}$.

Thus, a contravariant functor from $\mathcal{K}$ to $\mathcal{K}_1$ is a covariant functor from $\mathcal{K}$ to $\mathcal{K}_1^{op}$.

Example 1.2.14. Let $\mathcal{K}_1$ be the category of finite dimensional vector spaces over a field K. Its objects are vector spaces V, morphisms are linear maps $\mu : V_1 \to V_2$.

Given a vector space V denote by V^* the vector space of all linear maps $f : V \to K$, where K is regarded as a one-dimensional vector space. Every linear map $\mu : V_1 \to V_2$ gives rise to a linear map $\mu^*\colon V_2^* \to V_1^*$ defined by the rule $f \to \mu^*(f)$, where $\mu^*(f)(v) = f(\mu(v))$ for every $f \in V_2^*$ and every $v \in V_1$.

Denote by $\mathcal{K}_2$ the category with objects V^* and morphisms μ^*. Since $(\mu\nu)^* = \nu^*\mu^*$ the transition $*$ determines the contravariant functor from $\mathcal{K}_1$ to $\mathcal{K}_2$.

Example 1.2.15. Let $\mathcal{K}$ be a category of Ω-algebras. Given $A \in Ob\mathcal{K}$, denote by $\mathcal{F}(A)$ the underlying set (it can be multi-sorted) of the algebra A. For every homomorphism f of $\mathcal{K}$, we denote by $\mathcal{F}(f)$ the map of sets that acts in the same way as f. The constructed functor $\mathcal{F}$ from $\mathcal{K}$ to the category of sets is called a *forgetful* functor, since it "forgets" the algebraic operations.

Example 1.2.16. Consider an arbitrary category $\mathcal{K}$ and fix an object A of $\mathcal{K}$. We associate with A a covariant functor

$$\mathcal{F}_A\colon \mathcal{K} \to \mathsf{Set}$$

by letting $\mathcal{F}_A(B) = Mor(A, B)$ for any object B of $\mathcal{K}$. For every morphism $\varphi\colon B \to B'$ in $\mathcal{K}$, we define a map of sets

$$\mathcal{F}_A(\varphi)\colon Mor(A, B) \to Mor(A, B')$$

according to the rule: $f \to f\varphi$ for any $f \in Mor(A, B)$.

Similarly, we define a contravariant functor

$$\mathcal{F}^A\colon \mathcal{K} \to \mathsf{Set}.$$

By definition, $\mathcal{F}^A(B) = Mor(B, A)$, and given a morphism $\varphi\colon B' \to B$ in $\mathcal{K}$, the map

$$\mathcal{F}^A(\varphi)\colon Mor(B, A) \to Mor(B', A)$$

is defined according to the rule: $f \to \varphi f$ for any $f \in Mor(B, A)$.

Thus, with every object A of an arbitrary category $\mathcal{K}$, we have associated the functors $\mathcal{F}_A$ and $\mathcal{F}^A$ that act from the category $\mathcal{K}$ to the category of sets. These functors are called *representing functors.*

Natural transformations of functors

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be covariant functors that act from a category $\mathcal{K}$ to another category $\mathcal{K}'$. We introduce the notion of morphism (or *natural transformation*) for such functors. In particular, we shall obtain the notion of isomorphism for functors.

Definition 1.2.17. *A natural transformation f of functors $\mathcal{F}_1$ and $\mathcal{F}_2$ assigns to each object A of $\mathcal{K}$ a morphism $f_A : \mathcal{F}_1(A) \to \mathcal{F}_2(A)$ in the category $\mathcal{K}'$ such that for every morphism $\mu\colon A \to B$ in the category $\mathcal{K}$ we have the commutative diagram*

$$\begin{tikzcd} \mathcal{F}_1(A) \arrow[r, "f_A"] \arrow[d, "\mathcal{F}_1(\mu)"'] & \mathcal{F}_2(A) \arrow[d, "\mathcal{F}_2(\mu)"] \\ \mathcal{F}_1(B) \arrow[r, "f_B"] & \mathcal{F}_2(B). \end{tikzcd}$$

Definition 1.2.18. *If the morphisms f_A are isomorphisms for any object A of $\mathcal{K}$, then $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ is called an isomorphism between the functors $\mathcal{F}_1$ and $\mathcal{F}_2$.*

Two functors $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *isomorphic* if there exists an isomorphism between them. For any functor isomorphism $f\colon \mathcal{F}_1 \to \mathcal{F}_2$, there exists the inverse isomorphism $f^{-1}\colon \mathcal{F}_2 \to \mathcal{F}_1$ given by the class of isomorphisms $f_A^{-1}\colon \mathcal{F}_2(A) \to \mathcal{F}_1(A)$ that are inverse to the isomorphisms $f_A\colon \mathcal{F}_1(A) \to \mathcal{F}_2(A)$ in the category $\mathcal{K}'$.

Given categories $\mathcal{K}_1$ and $\mathcal{K}_2$, one can define the category of functors from $\mathcal{K}_1$ to $\mathcal{K}_2$ with natural transformations of functors as morphisms.

Proceed now to an example of natural transformations that make use of the representing functors. Suppose a category $\mathcal{K}$ is given. We have associated the functor $\mathcal{F}_A$ from $\mathcal{K}$ to Set with any object A of $\mathcal{K}$. Given a morphism $\mu\colon A \to B$ in $\mathcal{K}$, we shall define a natural transformation $f\colon \mathcal{F}_B \to \mathcal{F}_A$. We have to define a map

$$f_C\colon \mathcal{F}_B(C) = Mor(B, C) \to \mathcal{F}_A(C) = Mor(A, C).$$

for any object $C \in Ob\mathcal{K}$. Define it by the rule: $f_C(\varphi) = \mu\varphi$ for any $\varphi\colon B \to C$. It is easy to verify that f is a natural transformation and, therefore, we obtain a contravariant functor from the category $\mathcal{K}$ to the category of functors acting from $\mathcal{K}$ to Set. The same argument shows that the correspondence $A \to \mathcal{F}^{\mathcal{A}}$ gives rise to a covariant functor from $\mathcal{K}$ to the corresponding category of functors.

An isomorphism of objects A and B induces an isomorphism between the functors $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$, and also between $\mathcal{F}^{\mathcal{A}}$ and $\mathcal{F}^{\mathcal{B}}$. The converse statement is also true.

The notion of functor isomorphism enables us to pose the following class of problems related to representing functors: when a given functor to the category of sets could be realized, up to a functor isomorphism, as a representing functor associated with a suitable object of the category. If such a representation exists, then the corresponding object of the category is uniquely determined up to isomorphism.

Definition 1.2.19. *Categories $\mathcal{K}_1$ and $\mathcal{K}_2$ are called equivalent if there exist covariant functors $\mathcal{F}\colon \mathcal{K}_1 \to \mathcal{K}_2$ and $\mathcal{F}'\colon \mathcal{K}_2 \to \mathcal{K}_1$ such that $\mathcal{F}\mathcal{F}'$ is isomorphic to the identity functor of the category $\mathcal{K}_1$, and $\mathcal{F}'\mathcal{F}$ is isomorphic to the identity functor of the category $\mathcal{K}_2$. Such $\mathcal{F}$ and $\mathcal{F}'$ are said to be inverse to each other up to isomorphism.*

Definition 1.2.20. *Categories $\mathcal{K}_1$ and $\mathcal{K}_2$ are called isomorphic if there exist covariant functors $\mathcal{F}\colon \mathcal{K}_1 \to \mathcal{K}_2$ and $\mathcal{F}'\colon \mathcal{K}_2 \to \mathcal{K}_1$ such that $\mathcal{F}\mathcal{F}'$ is the identity functor of the category $\mathcal{K}_1$, and $\mathcal{F}'\mathcal{F}$ is the identity functor of the category $\mathcal{K}_2$. Such $\mathcal{F}$ and $\mathcal{F}'$ are said to be inverse to each other.*

If $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$ then we get the notions of *autoequivalence* and *automorphism* of the category $\mathcal{K}$, respectively.

Definition 1.2.21. *Categories $\mathcal{K}_1$ and $\mathcal{K}_2$ are dually equivalent if $\mathcal{K}_1$ is equivalent to $\mathcal{K}_2^{op}$. Categories $\mathcal{K}_1$ and $\mathcal{K}_2$ are dually isomorphic if $\mathcal{K}_1$ is isomorphic to $\mathcal{K}_2^{op}$.*

Definition 1.2.22. *An automorphism φ of the category $\mathcal{K}$ is called inner if it is isomorphic to the identity automorphism $1_{\mathcal{K}}$.*

According to Definition 1.2.18 this means that if $s : 1_{\mathcal{K}} \to \varphi$ is an isomorphism of functors, then for every object A of the category $\mathcal{K}$ there is an isomorphism $s_A : A \to \varphi(A)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "s_A"] \arrow[d, "\nu"'] & \varphi(A) \arrow[d, "\varphi(\nu)"] \\ B \arrow[r, "s_B"] & \varphi(B) \end{tikzcd}$$

is commutative for any morphism $\nu : A \to B$ in $\mathcal{K}$. So, φ is inner if and only if it can be represented in the form:

$$\varphi(\nu) = s_A^{-1} \nu s_B : \varphi(A) \to \varphi(B).$$

This formula motivates the term "inner automorphism".

1.2.2 Products and coproducts

We start this section with the notion of a universal object of a category.

Definition 1.2.23. *An object A of a category $\mathcal{K}$ is called an initial object of $\mathcal{K}$ if for any object $B \in Ob\mathcal{K}$ there exists a unique morphism $f\colon A \to B$.*

Dually, an object A is a terminal object of a category $\mathcal{K}$ if for every $B \in Ob\mathcal{K}$ there is a unique morphism $f\colon B \to A$.

All initial objects as well as terminal objects are canonically isomorphic. Any object which is isomorphic to an initial object is initial, and the same is true for terminal objects.

Every one-element set is a terminal object in the category of sets. The trivial group is an initial and, at the same time, a terminal object in the category of groups. There exist categories that have neither initial nor terminal objects. Most of the constructions of the universal algebra, like Cartesian, free, amalgamated and other products, like universal cover and universal enveloping algebra, and so on, can be realized in categorical terms as initial (terminal) objects in appropriate categories. This fact guarantees the uniqueness (up to an isomorphism) of these objects.

Let $\mathcal{K}$ be a category and $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ a collection of objects of $\mathcal{K}$. Take a new category $\mathcal{K}_\mathcal{A}$, whose objects are objects B of $\mathcal{K}$ together with morphisms $f_\alpha : B \to A_\alpha$ for all $\alpha \in I$. Morphisms of two objects (B_1, f_α^1), $\alpha \in I$ and (B_2, f_α^2), $\alpha \in I$ are represented by morphisms $\mu : B_1 \to B_2$ which satisfy the commutative diagrams

$$\begin{tikzcd} B_1 \arrow[rr, "\mu"] \arrow[dr, "f_\alpha^1"'] & & B_2 \arrow[dl, "f_\alpha^2"] \\ & A_\alpha & \end{tikzcd}$$

for all $\alpha \in I$.

Definition 1.2.24. *A terminal object of the category $\mathcal{K}_\mathcal{A}$ is called a product of the collection of objects $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ in the category $\mathcal{K}$. In other words, a product of the objects $A_\alpha,\ \alpha \in I$, is an object C supplied with morphisms $\pi_\alpha : C \to A_\alpha$ such that, for any object B and any morphisms $f_\alpha \colon B \to A_\alpha$, there exists a unique morphism $f \colon B \to C$ satisfying the conditions $f_\alpha = f\pi_\alpha$*

The product C of A_α, $\alpha \in I$ is denoted by $\prod_{\alpha \in I} A_\alpha$. Coproducts are defined dually, by reversing direction of arrows.

Let $\mathcal{K}$ be a category and $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ a collection of objects of $\mathcal{K}$. Take a new category $\mathcal{K}^\mathcal{A}$, whose objects are objects B of $\mathcal{K}$ together with morphisms $f_\alpha : A_\alpha \to B$ for all $\alpha \in I$. Morphisms of two objects (B_1, f_α^1), $\alpha \in I$ and (B_2, f_α^2), $\alpha \in I$ are represented by morphisms $\mu : B_1 \to B_2$ which satisfy the commutative diagrams

$$\begin{tikzcd} B_1 \arrow[rr, "\mu"] & & B_2 \\ & A_\alpha \arrow[ul, "f_\alpha^1"] \arrow[ur, "f_\alpha^2"'] & \end{tikzcd}$$

for all $\alpha \in I$.

Definition 1.2.25. *An initial object of the category $\mathcal{K}^\mathcal{A}$ is called a coproduct of the collection of objects $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ in the category $\mathcal{K}$. In other words, a coproduct of the objects $A_\alpha,\ \alpha \in I$, is an object C supplied with morphisms $i_\alpha : A_\alpha \to C$ such that, for any object B and any morphisms $f_\alpha \colon A_\alpha \to B$, there exists a unique morphism $f \colon C \to B$ satisfying the conditions $i_\alpha f = f_\alpha$.*

The coproduct C of A_α, $\alpha \in I$ is denoted by $\coprod_{\alpha \in I} A_\alpha$. Products are often called *direct products*, while coproducts are called *free products*. In these cases morphisms π_α are called projections, while morphisms i_α are called embeddings.

Example 1.2.26. 1. Let $\mathcal{K}$ be the category Set of sets. Products and coproducts exist in Set and coincide with the Cartesian products of sets and the disjoint unions of sets, respectively.

2. Let $\mathcal{K}$ be the category of arbitrary algebras. The direct products coincide with the Cartesian products provided the category $\mathcal{K}$ is a variety. This follows from the fact that a variety is closed under Cartesian products.

3. Let $\mathcal{K}$ be the category (and the variety) of all groups. Then coproducts coincide with free products of groups.

4. Let $\mathcal{K}$ be the category (and the variety) of abelian groups. Then the coproduct of abelian groups A_α, $\alpha \in I$ coincide with their discrete direct product. A discrete direct product of groups A_α, $\alpha \in I$ is a subgroup of $A = \prod A_\alpha$ consisting of elements a such that $a(\alpha)$ is equal to the identity element for all but a finite number of $\alpha \in I$. Using the additive notation for abelian groups, one can write that $\coprod_{\alpha\in I} A_\alpha = \bigoplus \sum_{\alpha\in I} A_\alpha$, the direct sum of abelian groups A_α. Hence, if the set I is finite, then products and coproducts coincide in the category of abelian groups.

5. Let $\mathcal{K}$ be the category of commutative algebras over a commutative ring with unity R. Then the coproduct of a collection of algebras coincides with their tensor product.

Inverse and direct limits

In the constructions of products and coproducts the indexing set I was arbitrary. Assume now that I is a partially ordered set with the order relation $\leq$. Recall that a partially ordered set I is called *directed* if for any $\alpha,\ \beta \in I$ there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$ (see Definition 1.1.3) .

Once again, let $\mathcal{K}$ be a category, I a directed set and $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ a collection of objects of $\mathcal{K}$.

Definition 1.2.27. *A family of morphisms* $\{f^\alpha_\beta\} \in Mor(A_\alpha, A_\beta)$, $\alpha, \beta \in I$, $\alpha \leq \beta$ *is called directed if*

1. f^α_α *is the identity morphism in* A_α *for all* $\alpha \in I$.
2. $f^\alpha_\beta f^\beta_\gamma = f^\alpha_\gamma$ *for all* $\alpha \leq \beta \leq \gamma$. !!! Umnozhenie morfizmov sprava!!!

Take the same category $\mathcal{K}^{\mathcal{A}}$ as is Definition 1.2.25. Distinguish a collection of objects $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ over the directed set I and assume that the family of morphisms $\{f^\alpha_\beta\}$ is directed. An initial object in this category is the *direct limit* of objects $\mathcal{A} = \{A_\alpha\}$. In terms of the category $\mathcal{K}$ a direct limit of $\mathcal{A} = \{A_\alpha\}$ is defined as follows:

Definition 1.2.28. *Let $\mathcal{K}$ be a category, $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ a family of objects, $\{f^\alpha_\beta\}$ a directed family of morphisms between objects of $\mathcal{A}$. An object C of $\mathcal{K}$ supplied with morphisms $i_\alpha : A_\alpha \to C$ is called the direct limit of $\{A_\alpha\}$ if*

1. $i_\alpha = f^\alpha_\beta i_\beta$*, for all* $\alpha \leq \beta$.

2. *Given any object B and morphisms $f_\alpha : A_\alpha \to B$, $\alpha \in I$ such that $f_\alpha = f^\alpha_\beta f_\beta$ for all $\alpha \leq \beta$, there exists a unique morphism $f : C \to B$ such that $i_\alpha f = f_\alpha$.*

The direct limit of $\{A_\alpha,\ \alpha \in I\}$ is denoted by $\varinjlim A_\alpha$. A motivating example for direct limits is an increasing infinite sequence of sets A_α, with the inclusions as morphisms. The direct limit is the union $\bigcup A_\alpha$.

The direct limit of $\{A_\alpha,\ \alpha \in I\}$ of algebras can be represented as

$$\varinjlim A_\alpha = \bigsqcup_\alpha A_\alpha/\rho,$$

where the congruence ρ on $\bigsqcup_\alpha A_\alpha$ is defined by $a_\alpha \rho a_\beta$, $a_\alpha \in A_\alpha$, $a_\beta \in A_\beta$, if there exists $\gamma \in I$ such that $f^\alpha_\gamma(a_\alpha) = f^\alpha_\gamma(a_\beta)$. So, a direct limit of algebras is a quotient algebra of a coproduct of algebras.

Reversing direction of arrows in the previous definitions leads to the notion of an inverse limit.

Definition 1.2.29. *A family of morphisms $\{f^\alpha_\beta\} \in Mor(A_\beta, A_\alpha)$, $\alpha, \beta \in I$, $\alpha \leq \beta$ in $\mathcal{K}_\mathcal{A}$ is called inversely directed or just inverse if*

1. f^α_α *is the identity morphism in* A_α *for all* $\alpha \in I$.

!!! Umnozhenie morfizmov sprava!!!

2. $f^\beta_\gamma f^\alpha_\beta = f^\alpha_\gamma$*, for* $\alpha \leq \beta \leq \gamma$.

Take the same category $\mathcal{K}_\mathcal{A}$ as is Definition 1.2.24. Distinguish a collection of objects $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ over the directed set I and assume that the family of morphisms $\{f^\alpha_\beta\}$ is inversely directed. A terminal object in this category is the *inverse limit* of objects $\mathcal{A} = \{A_\alpha\}$. In terms of the category $\mathcal{K}$ an inverse limit of $\mathcal{A} = \{A_\alpha\}$ is defined as follows:

Definition 1.2.30. *Let $\mathcal{K}$ be a category, $\mathcal{A} = \{A_\alpha,\ \alpha \in I\}$ a family of objects, $\{f^\alpha_\beta\}$ an inversely directed family of morphisms between objects of $\mathcal{A}$. An object C of $\mathcal{K}$ supplied with morphisms $\pi_\alpha : C \to A_\alpha$ is called the inverse limit of $\{A_\alpha\}$ if*

1. $\pi_\alpha = \pi_\beta f^\alpha_\beta$*, for all* $\alpha \leq \beta$.

2. *Given any object B and morphisms $f_\alpha : B \to A_\alpha$, $\alpha \in I$ such that $f_\alpha = f_\beta f^\alpha_\beta$ for all $\alpha \leq \beta$, there exists a unique morphism $f : B \to C$ such that $f_\alpha = f\pi_\alpha$.*

The inverse limit of $\{A_\alpha,\ \alpha \in I\}$ is denoted by $\varprojlim A_\alpha$. Inverse limits exist in many categories, in particular in the categories of sets, groups, rings topological spaces and more general algebras. For example, in the category of groups, the inverse limit of a family of groups G_α, $\alpha \in I$ with homomorphisms $f^\alpha_\beta : G_\beta \to G_\alpha$ is a subgroup of the Cartesian product $\prod_\alpha G_\alpha$ consisting of the sequences (g_α), $\alpha \in I$ such that:

$$\varprojlim G_\alpha = \{(g_\alpha) \mid f^\alpha_\beta(g_\beta) = g_\alpha \text{ for all } \alpha \leq \beta\}.$$

In particular, if we take the inverse limit of finite groups we come up with the notion of a profinite group. The example of groups can be generalized to a very general setting of algebras (see [DMR2] for the exposition useful for the geometric aims).

1.2.3 Constants in algebras

!!! Perepisat'

The next step is devoted to algebras with the distinguished algebra of constants.

Recall that we consider all nullary operations in an algebra G as constants (see Subsection 1.1.2). In this subsection we give a categorical insight on algebras with constants. !!! ???

Let Θ be an arbitrary variety of algebras, G be a fixed algebra in Θ, $|G| > 1$. Consider a new variety, denoted by Θ^G. First we define the category Θ^G. Objects in Θ^G are of the form $h : G \to H$, where $H \in \Theta$, and h is a homomorphism of algebras, not necessarily injective. These objects will be called *G-algebras*, i.e., a G-algebra H is a pair (H, h). Morphisms in Θ^G are presented by commutative diagrams

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "h'"'] & H \arrow[d, "\mu"] \\ & H' \end{tikzcd}$$

where μ, h, h' are homomorphisms in Θ.

An algebra H, treated as a G-algebra, is denoted by (H, h). We view elements $g^h \in G$ as constants, i.e., nullary operations in H. Adding them to the signature Ω, we come up with the extended signature Ω^G.

Identities of a G-algebra are just identities in the signature Ω^G (G-identities). They are presented by identities of Θ and by defining relations of the algebra G. *A variety of G-algebras* is a class of G-algebras determined by a set of G-identities. *A quasivariety of G-algebras* consists of all G-algebras which satisfy a set of G-quasi-identities , i.e., quasi-identities over Ω^G. Other axiomatic classes of

G-algebras and prevarieties of G-algebras are defined in a similar way.

A free algebra $W(X)$ in Θ^G is of the form $G * W_0(X)$, where $W_0(X)$ is the free algebra in Θ over X, $*$ is the free product in Θ and the embedding $i_G : G \to W(X) = G * W_0(X)$ follows from the definition of a free product.

A G-algebra (H, h) is called *faithful* if $h : G \to H$ is an injection. In particular, a free algebra $(W(X), i_G)$ and the G-algebra G with the identical map $G \to G$ are faithful.

Let (H, h) be a G-algebra, and $\mu : H \to H'$ be a homomorphism in Θ. Take $h' = \mu h$, then H' becomes a G-algebra, and μ is a homomorphism of G-algebras. Since one can start from an arbitrary congruence T in H and from the natural congruence H/T, we say that T is *faithful* if the G-algebra H/T is faithful. A congruence T is faithful if and only if $g_1^h = g_2^h$ is equivalent to $g_1 = g_2$.

Let a morphism

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "h'"'] & H \arrow[d, "\mu"] \\ & H' \end{tikzcd}$$

be given, and let (H', h') be a faithful G-algebra. Then (H, h) is a faithful G-algebra. If $T = Ker\mu$, then T is a faithful congruence and H/T is also faithful.

We can assume that homomorphisms of faithful G-algebras leave elements from G unchanged.

Example 1.2.31. A variety $Com - K$ is a variety of the type Θ^G, where Θ is a variety of associative and commutative rings with 1, and G is a field K. In this example, elements of the field K are constants in K-algebras. They are viewed as nullary operations, and, simultaneously, using multiplication, we can look at them as unary operations.

Example 1.2.32. G-group is another example of G-algebras. Here, elements from G also can be viewed as unary operations. Using the analogy with $Com - K$ we can denote the free G-group by $G[X] = G * W_0(X)$ and view its elements as non-commutative polynomials with coefficients from the given group G.

Since Θ^G is a variety of algebras, all constructions like Cartesian and free products, subalgebras and homomorphisms are naturally defined for Θ^G. Note that the free product of two G-algebras H_1 and H_2 is exactly the amalgamated product $H_1 *_G H_2$ in the variety Θ.

Consider special property on Θ^G. Namely, we assume that the G-algebra G generates the whole variety Θ^G, i.e., in G there are no non-trivial identities with coefficients from G. This property is fulfilled in $Com - K$ if the field K is infinite and in $(Grp)^F$, where F is a free group.

Every faithful G-algebra H contains G as a subalgebra. Thus, the property above implies that every faithful G-algebra H generates the whole variety Θ^G, i.e., in Θ^G there are no proper subvarieties containing faithful algebras.

In the category Θ^G along with morphisms, one can consider also semimorphisms. They are of the form

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[d, "\sigma"'] & H \arrow[d, "\mu"] \\ G \arrow[r, "h'"] & H' \end{tikzcd}$$

where $\sigma \in End(G)$. Then, one can consider semi-isomorphic G-algebras.

Another possibility is to vary also the algebra of constants G. This leads to a diagram of the form

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[d, "\sigma"'] & H \arrow[d, "\mu"] \\ G' \arrow[r, "h'"] & H' \end{tikzcd}$$

with componentwise multiplication.

Let us make a remark on equations. Equations of the form $w = w'$ with $w, w' \in W(X) = G * W_0(X)$ are *equations with constants from* G. Consider a system of such equations T. If T is a congruence, then T has a solution in a faithful G-algebra H if and only if T is a faithful congruence on $W(X)$. Thus, a system T has a solution if T is contained in a faithful congruence in $W(X)$. Note that, by definition, all faithful congruences are proper.

In what follows if we speak about the variety of algebras Θ we always keep in mind also the case Θ^G, where $G \in \Theta$ is an algebra of constants. All the constructions above can be transferred to the multi-sorted case (see [Hig]).

Chapter 2

Basics of Universal Algebraic Geometry

!!! V KAKIH TEOREMAH PRIVESTI DOKAZATEL"STVA???RESHIT"

Universal algebraic geometry spreads the ideas of classical algebraic geometry to arbitrary varieties of algebras. So, one of its objectives is to study solutions of equations over a given algebra H from a given variety of algebras Θ. However, in what follows we focus our attention on the other goal of the universal algebraic geometry, namely, on studying geometric invariants of the algebras from Θ.

In this section we sketch the basics of universal geometry mostly avoiding proofs and complicated considerations related to geometry in the particular varieties. In fact, we give a list of general facts and notions which illuminates analogies and distinctions between classical geometry and universal one. Besides, the passage

$$\textit{classical algebraic geometry} \quad \Longrightarrow \textit{universal algebraic geometry}$$

will be extended in Part II to a further one

$$\textit{universal algebraic geometry} \quad \Longrightarrow \textit{logical geometry.}$$

Subsections 2.1.2 – 2.1.4 of Section 2.1 are devoted to basics of classical algebraic geometry. Subsections 2.2.1 – 2.2.5 of the same section are completely parallel to them and treat the same material from the positions of universal algebraic geometry. Subsections 2.2.6 – 2.2.8 deal with the notions which are peculiar for universal algebraic geometry. In the short subsection 2.2.10 there is a bibliography which can help to navigate in the field of universal algebraic geometry.

We start with recalling the elementary (scheme-free) setting of classical algebraic geometry.

2.1 Classical Algebraic Geometry

The material of this section is standard. For the general references the books [Shaf], [Harts], [Haris], [Hulek] could be taken.

2.1.1 Galois correspondence

Let K be a field and $K[X] = K[x_1, \dots, x_n]$ the ring of polynomials in the commuting variables $x_1, \dots, x_n$ with coefficients in K.

Algebraic geometry was grown up from the desire to describe the solutions of the systems of equations of the form

$$f_1(x_1, \dots, x_n) = 0, \dots, f_n(x_1, \dots, x_n) = 0,$$

where all $f_i(x_1, \dots, x_n)$ are the polynomials from $K[x_1, \dots, x_n]$.

Definition 2.1.1. *The set of all n-tuples of the form $(a_1, \dots, a_n)$, $a_i \in K$ is called the affine n-space over the field K.*

The affine n-space is denoted by K^n which means that as a set this is exactly the n-th Cartesian power of the field K. For the elements of K^n the vector notation $\bar{a} = (a_1, \dots, a_n)$ is used.

The space K^n is the place were the solutions of polynomial equations $f(x_1, \dots, x_n) = 0$ live. A point $\bar{a} = (a_1, \dots, a_n) \in K^n$ is *a solution* of the equation $f(x_1, \dots, x_n) = 0$ if $f(a_1, \dots, a_n) = 0$. Given the set of polynomials T in $K[x_1, \dots, x_n]$ denote by $V(T) \subset K^n$ the set of common zeros of the polynomials from T:

$$V(T) = \{\bar{a} = (a_1, \dots, a_n) \in K^n \mid f(\bar{a}) = 0, \text{ for all } f \in T\}.$$

By definition, if $T_1 \subseteq T_2$, then $V(T_1) \supseteq V(T_2)$.

Definition 2.1.2. *A subset A in the affine space K^n is called algebraic set if there exists a set T of polynomials from $K[x_1, \dots, x_n]$ such that A is the set of common zeros of the polynomials from T.*

In other words every element $\bar{a}$ of an algebraic set $A \subset K^n$ is the solution of a system of equations $T \subset K[x_1, \dots, x_n]$, and we have:

$$A = V(T).$$

We view V as the correspondence between subsets in $K[x_1, \dots, x_n]$ and subsets in the affine space K^n which assigns to a set of polynomials T in $K[x_1, \dots, x_n]$ the algebraic set $V(T)$ in K^n. Let $\langle T \rangle$ be the ideal generated by a set T. It is easy to see that $V(T) = V(\langle T \rangle)$. Hence, one can assume that the set T is an ideal and V establishes a correspondence between the ideals in $K[x_1, \dots, x_n]$ and algebraic sets in K^n.

Theorem 2.1.3 (Hilbert basis theorem)**.** *If R is a commutative Noetherian ring (see 2.2.74), then $R[x_1, \dots, x_n]$ is also Noetherian. In particular, $K[x_1, \dots, x_n]$ where K is a field is Noetherian, i.e., every its ideal is finitely generated.* $\square$

This implies that in the correspondence

$$V: \text{ ideals in } K[x_1, \dots, x_n] \to \text{ algebraic sets in } K^n,$$

all ideals are finitely generated. The latter means that there is no need to consider systems with infinite number of equations: every algebraic set can be defined by finitely many polynomials.

Let A be an arbitrary set in the affine space K^n. Define the ideal $I(A)$ in $K[x_1, \dots, x_n]$ as the set of all polynomials vanishing at every point from A:

$$I(A) = \{f \in K[x_1, \dots, x_n] \mid f(\bar{a}) = 0, \text{ for all } \bar{a} \in A\}.$$

By definition, if $A_1 \subseteq A_2$, then $I(A_1) \supseteq I(A_2)$. We view I as the correspondence between subsets in the affine space K^n and ideals in $K[x_1, \dots, x_n]$:

$$I: \text{ sets in } K^n \to \text{ideals in } K[x_1, \dots, x_n].$$

In this correspondence the image $I(\varnothing)$ of the empty set is the whole ring $K[x_1, \dots, x_n]$. Hence, the image of a point is always a maximal ideal in $K[x_1, \dots, x_n]$.

Consider the general notion of a Galois correspondence between partially ordered sets.

Definition 2.1.4. *Let $(P, \leq)$ and $(Q, \leq)$ be two partially ordered sets. A pair of order-reversing functions $\varphi : P \to Q$ and $\psi : Q \to P$ constitutes the Galois correspondence between P and Q if for all $p \in P$ and $q \in Q$ we have:*

$$\psi(\varphi(p)) \geq p, \ \varphi(\psi(q)) \geq q.$$

Elements $\bar{p} = \psi(\varphi(p))$, and $\bar{q} = \varphi(\psi(q))$ are called the Galois closures of p and q, respectively. Elements $p \in P$ and $q \in Q$ are called Galois closed if $\bar{p} = p$ and $\bar{q} = q$.

The maps V and I give rise to the Galois correspondence between the ideals in the polynomial ring and the subsets in the affine space:

$$\begin{cases} V(T) = \{\bar{a} = (a_1, \dots, a_n) \in K^n \mid f(\bar{a}) = 0, \text{ for all} f \in T\}, \\ I(A) = \{f \in K[x_1, \dots, x_n] \mid f(\bar{a}) = 0, \text{ for all } \bar{a} \in A\}. \end{cases}$$

In a Galois correspondence φ and ψ are arbitrary order-reversing maps. So are V and I: they are not necessarily injections or surjections. For example, the ideals $T_1 = \langle x_1, \ldots, x_n\rangle$ and $T_2 = \langle x_1^k, \ldots, x_n^k\rangle$ glue together under the action of V:

$$V(\langle x_1, \ldots, x_n\rangle) = V(\langle x_1^k, \ldots, x_n^k\rangle),$$

and V is not injective. The map V is not surjective since there are lots of infinite non-algebraic sets in K^n. The map I is neither surjective, nor injective as well. For example, if $K = \mathbb{R}$ and $A_1 = 2\mathbb{Z} \subset \mathbb{R}$, and $A_2 = 3\mathbb{Z} \subset \mathbb{R}$ then $I(A_1) = I(A_2) = 0$ in $\mathbb{R}[x]$.

On the other hand, the maps $\varphi\psi$ and $\psi\varphi$ are the closure operators for arbitrary Galois correspondence. Moreover, φ and ψ give rise to a bijection between closed objects. Applying this observation to the maps V and I we come up with the question: what are the closed sets in $K[x_1, \ldots, x_n]$ and K^n, and how to describe the structure of the closures $IV(T)$ and $VI(A)$, where T and A are subsets in $K[x_1, \ldots, x_n]$ and K^n, respectively.

According to Definition 2.1.2 algebraic sets are exactly the Galois closed subsets in K^n. The next aim is to find out what are the Galois closed ideals in $K[x_1, \ldots, x_n]$.

Definition 2.1.5. *The radical $Rad(J)$ of an ideal J in a commutative ring R is defined as*

$$Rad(J) = \{r \in R \mid r^s \in J, \text{ for some } s \in \mathbb{N}\}.$$

It is easy to see that $Rad(J)$ is an ideal, and $J \subseteq Rad(J)$. An ideal J is called *radical* if $J = Rad(J)$. Hence, every maximal ideal is radical.

Theorem 2.1.6 (Hilbert's Nullstellensatz)**.** *Let K be an algebraically closed field. Then:*

1. Every maximal ideal J in $K[x_1, \ldots, x_n]$ is of the form

$$J = \langle x_1 - a_1, \ldots, x_n - a_n\rangle = I(\bar{a}),$$

where $\bar{a} = (a_1, \ldots, a_n)$ is a point in K^n.

2. $V(J) \neq \varnothing$ for every proper ideal J in $K[x_1, \ldots, x_n]$.

3. The Galois closure of an ideal J in $K[x_1, \ldots, x_n]$ is $Rad(J)$:

$$Rad(J) = IV(J). \qquad \square$$

Thus, one can say that if a ground field K is big enough (e.g. algebraically closed), then every (consistent) system of equations has a solution, the Galois closed objects in $K[x_1, \ldots, x_n]$ are exactly the radical ideals, and there is a one-to-one correspondence between maximal ideals in $K[x_1, \ldots, x_n]$ and points in the affine space K^n:

$$\text{points in } K^n \underset{V}{\overset{I}{\rightleftarrows}} \text{maximal ideals in } K[x_1, \ldots, x_n].$$

Since for an arbitrary Galois correspondence there is a bijection between the closed subjects, we have the bijection:

$$\text{algebraic sets in } K^n \underset{V}{\overset{I}{\rightleftarrows}} \text{radical ideals in } K[x_1, \ldots, x_n],$$

over an algebraically closed field K. For the case of an arbitrary field K the one-to-one correspondence is as follows:

$$\text{algebraic sets in } K^n \underset{V}{\overset{I}{\rightleftarrows}} \text{ideals of the form } IV(J) \text{ in } K[x_1, \ldots, x_n],$$

The Galois closure $IV(J)$ for non-algebraically closed fields heavily depends on the ground field K (see [BCR], [Du] for the real Nullstellensatz).

Remark 2.1.7. *The nilradical of a commutative ring is the set of all nilpotent elements of the ring. It can also be characterized as the intersection of all prime ideals of the ring. In these terms Hilbert's Nullstellensatz states that for an algebraically closed field K and arbitrary ideal J, the nilradical of the ring $K[x_1, x_2, \ldots, x_n]/J$ coincides with $IV(J)/J$.*

Remark 2.1.8. *Theorem 2.1.7 contains the so-called weak Nullstellensatz. If K is an algebraically closed field, then the equality $Rad(J) = IV(J)$ implies $V(J) \neq \varnothing$ for every proper ideal J in $K[x_1, \ldots, x_n]$. Hence, if $V(J) = \varnothing$, then $J = K[x_1, x_2, \ldots, x_n]$.*

Now we formulate as a remark another vision of affine spaces. This viewpoint will be a base for many generalizations.

Remark 2.1.9. *Let us identify a point $\bar{a} = (a_1, \ldots, a_n) \in K^n$ with the map $\mu = \mu_a : K[X] = K[x_1, \ldots, x_n] \to K$, defined by $\mu(x_i) = a_i$. Since $K[X] = K[x_1, \ldots, x_n]$ is a free algebra in the variety of associative commutative rings with unity, every map of algebras defined on generators gives rise to the homomorphism of algebras.*

Thus, the affine space K^n can be viewed as the set of all homomorphisms $Hom(K[X], K)$. If $f = f(x_1, \ldots, x_n) \in K[X]$ and $\mu \in Hom(K[X], K)$, then $f^\mu = f(x_1^\mu, \ldots, x_n^\mu) = f(\bar{a})$. A point μ is a root of the polynomial f if $f \in Ker\mu$.

In view of Remark 2.1.9 the Galois correspondence defined above now looks as follows. Let T be a subset in the algebra $K[X]$. Consider T as a system of equations of the form $f_i = 0$, where $f_i \in T$. It corresponds the set of points $A \subset Hom(K[X], K)$, defined by the rule

$$A = V(T) = \{\mu : K[X] \to K \ |T \subset Ker\mu\}.$$

This means that a point μ belongs to $V(T)$ if μ is a root of every polynomial from T.

Let A be an arbitrary set of points $\mu : K[X] \to K$. It corresponds the ideal $I(A)$ in $K[X]$ defined by

$$I(A) = \bigcap_{\mu \in A} Ker(\mu)$$

This is the set of all polynomials f, such that every point $\mu \in A$ is a root of f.

2.1.2 Zariski topology

Recall that a topology $(B, \mathfrak{B})$ on a set B is defined, if there is a distinguished collection $\mathfrak{B}$ of subsets in B subject to conditions:

1. $\varnothing \in \mathfrak{B}$, $B \in \mathfrak{B}$.
2. If $C_1, C_2, C_3, \dots,$ is a family of sets from $\mathfrak{B}$, then $\bigcap\limits_i C_i \in \mathfrak{B}$.
3. If $C_1, \dots, C_n$ is a finite family of sets from $\mathfrak{B}$, then $\bigcup\limits_{i=1}^{n} C_i \in \mathfrak{B}$.

The elements from $\mathfrak{B}$ are called *closed sets*, their complements to B are *open sets*.

Lemma 2.1.10. *The map V satisfies the properties:*

1. $V(0) = K^n$, $V(K[x_1, \dots, x_n]) = \varnothing$.
2. *If $\{T_i\}$ is a family of ideals in $K[x_1, \dots, x_n]$, then* $\bigcap\limits_i V(T_i) = V(\sum\limits_i T_i)$.
3. *If T_1 and T_2 are ideals in $K[x_1, \dots, x_n]$, then* $V(T_1) \cup V(T_2) = V(T_1 \cap T_2) = V(T_1T_2)$. $\square$

In particular, Lemma 2.1.10 states that arbitrary intersections and finite unions of algebraic sets are algebraic sets. Hence, algebraic sets can be considered as closed sets of some topology. This topology is called *the Zariski topology* on the affine space K^n.

Let us collect some facts about the Zariski topology (the field K is assumed to be infinite).

- Every finite set is closed in the Zariski topology over an algebraically closed field, in particular, every point is closed.
- Every two non-empty open subsets of K^n have a non-empty intersection, hence the Zariski topology is not Hausdorff.
- The Zariski topology is weaker than the usual topology where closed sets are the zeros of continuous functions.
- Any open subset of the affine space is the Zariski dense, i.e., its closure in the Zariski topology coincides with the whole affine space K^n.
- Let A be a subset of K^n. The Zariski closure $\bar{A}$ of A coincides with the Galois closure $VI(A)$ of the set A.

Definition 2.1.11. *A topology on K^n is Noetherian, if every descending chain of closed subsets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ stabilizes, that is, there exists s such that $A_k = A_{k+1}$ for all $k \geq s$.*

The Zariski topology on K^n is Noetherian since $K[x_1, \dots, x_n]$ is Noetherian (Theorem 2.1.3) and the map V is order-reversing.

Definition 2.1.12. *An affine set A is called irreducible if it cannot be represented as a union of two proper closed subsets. Otherwise A is reducible.*

Every algebraic set A can be uniquely (up to the order of components) represented as the finite union of irreducible closed subsets. If the affine set A is irreducible, then $I(A)$ is a prime ideal. The algebraic set $V(f)$ corresponding to an irreducible polynomial f in $K[x_1, \dots, x_n]$ is irreducible.

2.1.3 The coordinate ring of an algebraic set

Let A be an algebraic set in K^n and $I(A)$ the corresponding ideal in $K[x_1, \dots, x_n]$.

Definition 2.1.13. *The ring $K[A] = K[x_1, \dots, x_n] \diagup I(A)$ is called coordinate ring of the algebraic set A, or the ring of polynomial functions on A.*

Since $K[x_1, \dots, x_n]$ is Noetherian, the ring $K[A]$ is also Noetherian. Hence, a coordinate ring is finitely generated. If the field K is algebraically closed, then by Hilbert's Nullstelensatz a coordinate ring has no nilpotent elements. Recall that an algebra is *reduced* if it has no nilpotent elements. So, a coordinate ring is a finitely generated reduced algebra over K.

Remark 2.1.14. *This property is characteristic: every finitely generated reduced algebra over K is a coordinate ring of some algebraic set.*

Since the ideal $I(A)$ of an irreducible algebraic set A is prime, the coordinate ring $K[A]$ is an integral domain. This means that $K[A]$ does not contain zero divisors.

Definition 2.1.15. *A map $\varphi : A \to K$ is called a polynomial (regular) function on A if there exists a polynomial $g \in K[x_1, \dots, x_n]$ such that $\varphi(\bar{a}) = g(\bar{a})$ for all $\bar{a} \in A$.*

Two polynomials g_1 and g_2 in $K[x_1, \dots, x_n]$ define the same polynomial function on A if and only if $g_1(\bar{a}) - g_2(\bar{a}) = 0$, for all $\bar{a} \in A$, i.e. $g_1 - g_2 \in I(A)$. Thus, elements of $K[A]$ can be identified with polynomial functions on A.

Definition 2.1.16. *Let A and B be algebraic sets in K^n and K^m respectively. A map $\varphi : A \to B$ is called a polynomial map if there exist polynomials $g_1, \dots, g_m$ in $K[x_1, \dots, x_n]$ such that for all $\bar{a} \in A$*

$$\varphi(\bar{a}) = (g_1(\bar{a}), \dots, g_m(\bar{a})) = \bar{b} \in B.$$

All polynomial maps are continuous in Zariski topology. If A, B and C are algebraic sets and $\varphi : A \to B$, $\psi : B \to C$ are polynomial maps, then the composition $\psi \circ \varphi : A \to C$ is also a polynomial map. A polynomial map $\varphi : A \to B$, is called an *isomorphism of algebraic sets* if there exists a polynomial map $\psi : B \to A$ such that the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity maps on A and B, respectively.

2.1.4 Categories of algebraic sets and coordinate rings

Our next aim is to define the categories of algebraic sets and coordinate algebras.

Denote by $\mathcal{A}(K)$ the *category of algebraic sets* over a field K. The objects of $\mathcal{A}(K)$ are algebraic sets over K. Morphisms of $\mathcal{A}(K)$ are polynomial maps of algebraic sets. According to Remark 2.1.14, the category $\mathcal{C}(K)$ of coordinate rings is the category of finitely generated reduced algebras over K, with homomorphisms as morphisms.

Theorem 2.1.17. *The category of algebraic sets $\mathcal{A}(K)$ is dually equivalent to the category of finitely generated reduced K-algebras $\mathcal{C}(K)$.* $\square$

By Definition 1.2.21 the categories $\mathcal{A}(K)$ and $\mathcal{C}(K)$ are *dually equivalent* if there are contravariant functors $\mathcal{F}_1 : \mathcal{A}(K) \to \mathcal{C}(K)$ and $\mathcal{F}_2 : \mathcal{C}(K) \to \mathcal{A}(K)$ such that $\mathcal{F}_1\mathcal{F}_2$ and $\mathcal{F}_2\mathcal{F}_1$ are isomorphic to identity functors $id_{\mathcal{A}(K)}$ and $id_{\mathcal{C}(K)}$. The functor $\mathcal{F}_1$ on objects is defined by $\mathcal{F}_1(A) = K[A]$, where A is an algebraic set. Let $\varphi : A_1 \to A_2$ be a morphism in $\mathcal{A}(K)$, i.e., a polynomial map of algebraic sets. We shall define $\mathcal{F}_1(\varphi) : K[A_2] \to K[A_1]$. The morphism $\mathcal{F}_1(\varphi) : \mathcal{F}_1(A_2) \to \mathcal{F}_1(A_1)$ is determined by diagram

$$\begin{tikzcd} A_1 \arrow[rr, "\varphi"] \arrow[dr, "\mathcal{F}_1(\varphi)"'] & & A_2 \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

where ψ is a polynomial function on A_2, i.e., an element of $K[A_2]$. Given $\psi \in K[A_2]$, the morphism $\mathcal{F}_1(\varphi)$ acts as $\mathcal{F}_1(\varphi)(\psi) = \psi \circ \varphi \in K[A_1]$. It can be checked that $\mathcal{F}_1(\varphi)$ is indeed a morphism, that is a homomorphism of K-algebras.

Lemma 2.1.18. *Given a morphism $\nu : K[A_2] \to K[A_1]$, there exists a unique $\varphi : A_1 \to A_2$ such that $\nu = \mathcal{F}_1(\varphi)$.* $\square$

This lemma shows that there is a bijection between morphisms in $\mathcal{A}(K)$ and in $\mathcal{C}(K)$.

Let D be a finitely generated reduced K-algebra with the generators $d_1, \dots, d_n$. Since $K[x_1, \dots, x_n]$ is the free K-algebra, the map $\mu(x_i) = d_i$, $i = 1, \dots, n$ gives rise to the homomorphism $\mu : K[x_1, \dots, x_n] \to D$ of K-algebras. Let $J = Ker(\mu)$. Define $\mathcal{F}_2(D) = V(J)$. Using Lemma 2.1.18, it is easy to see that the contravariant functors $\mathcal{F}_1$ and $\mathcal{F}_2$ define an equivalence of the categories.

Remark 2.1.19. *The algebra D is an integral domain if and only if the ideal J is prime. In this case $V(J)$ is an irreducible affine set. So, the category of irreducible affine sets and the category of finitely generated integral domains over the field K are equivalent.*

Theorem 2.1.17 provides back and forth passages from the level of algebraic sets, which is geometrical by its nature, to the level of coordinate rings, which is purely algebraic.

Moreover, since both the categories depend on K, they can be considered as dual algebraic-geometric invariants attached to the field K.

2.2 Universal Algebraic Geometry

From now on let Θ denote an arbitrary variety of universal algebras, i.e., a class of algebras defined by some set of identities. It can be, for example, the variety of groups, semigroups, associative algebras, Lie algebras or any other variety. Our first aim is to set up the system of notions of the universal algebraic geometry and to obtain the setting similar to the classical one, described in Section 2.1.

Each notion of a universal character can be specialized to a particular variety of algebras Θ. Fixing Θ and an algebra H in Θ we come up with the algebraic geometry in the particular Θ over a given H.

If we want to emphasize that this geometry is built with respect to solutions of equations we call it equational geometry. This setting will be later on spread out to logical geometry, where the equations are replaced by arbitrary first-order formulas.

2.2.1 Equations and affine spaces

The classical algebraic geometry starts with consideration of system of polynomial equations. These equations are expressions of the form $f(x_1, \ldots, x_n) = 0$, where f is a polynomial, that is an element of the polynomial algebra $K[X] = K[x_1, \ldots, x_n]$. Looking at the algebra $K[x_1, \ldots, x_n]$ from the positions of universal algebra we note that $K[x_1, \ldots, x_n]$ is the finitely generated free algebra in the variety $Com{-}K$ of commutative associative algebras with unit over the field K. Hence, if we take instead of $Com - K$ an arbitrary variety Θ, we place equations in a finitely generated free in Θ algebra $W(X)$, i.e.,

$$K[X] = K[x_1, \ldots, x_n] \text{ is replaced by } W(X), \quad |X| < \infty.$$

The general form of equation in an arbitrary variety Θ is:

$$w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n), \ w, w' \in W(X), |X| = n.$$

This means that in universal algebraic geometry

$$\text{polynomial equations } f(x_1, \ldots, x_n) = 0$$

are replaced by

$$\text{equations } w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n).$$

Remark 2.2.1. *By abuse of language we will speak about "equations from (or over) $W(X)$", having in mind that they are elements of $W(X) \times W(X)$. Later on we will consider them as formulas of a special kind.*

The next object we shall introduce is the affine space. This is the place were the solutions of equations are situated. In classical algebraic geometry the affine spaces are of the form K^n, where K is a ground field and n is the number of generators in a polynomial algebra. We can also consider affine spaces L^n where L is an extension of the ground field K. In any case both K and L can be viewed as algebras in the variety $Com - K$. Replace $Com - K$ by an arbitrary variety Θ. This leads to the following definition.

Definition 2.2.2. *Let H be an arbitrary algebra in the variety Θ. Affine spaces over H have the form H^n, where n is the number of generators in a free algebra $W(X)$.*

A point $\bar{a}$ of an affine space H^n is an n-tuple $\bar{a} = (a_1, \dots, a_n)$, $a_i \in H$, $i = 1, \dots, n$.

Now we explore the bijection of sets $Hom(W(X), H) \to H^n$.

Let $\bar{a} = (a_1, \dots, a_n)$ be a point in H^n. It corresponds the function $\mu : X \to H$ defined by $\mu(x_i) = a_i$, $i = 1, \dots, n$. Since $W(X)$ is a free algebra, each function $\mu(x_i) = a_i$, $i = 1, \dots, n$, gives rise to the homomorphism μ in $Hom(W(X), H)$. Conversely, let μ be a homomorphism in $Hom(W(X), H)$. It corresponds the point $\bar{a} = (a_1, \dots, a_n)$ in H^n defined by $\mu(x_i) = a_i$.

This correspondence allows us to identify the set of homomorphisms $Hom(W(X), H)$ with H^n and to consider it as an affine space. The next definition is equivalent to Definition 2.2.2.

Definition 2.2.3. *Let H be an arbitrary algebra in the variety Θ. Affine spaces over H have the form $Hom(W(X), H)$, where $W(X)$ is a free algebra in Θ.*

Homomorphisms μ from $Hom(W(X), H)$ are the points of the affine space $Hom(W(X), H)$.

Usually in this book we use Definition 2.2.3 as the definition of an affine space.

So, in universal algebraic geometry in a variety Θ:

affine spaces K^n

are replaced by

affine spaces $H^n \simeq Hom(W(X), H)$,

where $H \in \Theta$, and $W(X)$ is a free finitely generated algebra in Θ.

Let the point $\mu \in Hom(W(X), H)$ be induced by a map $\mu : X \to H$ and corresponds to $\bar{a} = (a_1, \dots, a_n)$ in H^n, where $a_i = \mu(x_i)$. This correspondence gives rise to kernels of points $\mu \in Hom(W(X), H)$.

Definition 2.2.4. *Let $\mu \in Hom(W(X), H)$ be a point in the affine space $Hom(W(X), H)$. The kernel $Ker(\mu)$ of the point μ is the kernel of the homomorphism $\mu : W(X) \to H$.*

Let $w_1 = w_1(x_1, \dots, x_n)$ and $w_2 = w_2(x_1, \dots, x_n)$ be the elements in $W(X)$.

Definition 2.2.5. *A point $\bar{a} = (a_1, \dots, a_n) \in H^n$ is a solution of the equation $w_1 = w_2$ in the algebra H if $w_1(a_1, \dots, a_n) = w_2(a_1, \dots, a_n)$. This is equivalent to: a point $\mu \in Hom(W(X), H)$ is a solution of $w_1 = w_2$ if*

$$w_1^\mu = {w_2}^\mu,$$

where $w_i^\mu = w_i(\mu(x_1), \dots, \mu(x_n))$.

The equality $w_1^\mu = w_2^\mu$ means that the pair (w_1, w_2) belongs to $Ker(\mu)$. In other words, *a point μ is a solution of the equation $w_1 = w_2$ if (w_1, w_2) belongs to the kernel of the point μ.* We will say that $w_1 = w_2$ belongs to the kernel of a point if and only if the pair (w_1, w_2) belongs to this kernel.

The kernel $Ker(\mu)$ is a congruence on the algebra $W(X)$ and, thus, the quotient algebra $W(X)/Ker(\mu)$ is defined. These kernels play an important role in the sequel.

If Θ is the variety Grp of all groups, then $W(X) = F(X)$ is a finitely generated free group. Equations in $F(X)$ have the form $w(x_1, \dots, x_n) = 1$, where $w \in F(X)$, $|X| = n$. For the affine space $H^n = Hom(W(X), H)$ there exists a plenty of choices. If $H = F(X)$, then we come up with equations over the free group, and with the corresponding geometry. If H is a simple group then we are looking for solutions of equations in a specific simple group.

If Θ is the variety $Ass - K$ of associative algebras over a field K, then $W(X)$ is the free algebra of polynomials with non-commuting variables. The affine space is $K^n = Hom(W(X), K)$ or $L^n = Hom(W(X), L)$, where L is an extension of the field K. The obtained geometry is a *non-commutative* algebraic geometry.

If Θ is the variety $Lie - K$ of Lie algebras over a field K, then $W(X)$ is the free algebra of Lie polynomials. The equations have the form $w(x_1, \dots, x_n) = 0$, where $w \in W(X)$ is a Lie polynomial. The choice of H depends on a particular problem and can vary from simple algebras to free algebras.

Example 2.2.6. This case is of special importance. Let Θ be a variety of algebras, G a fixed algebra in Θ. Consider the variety Θ^G of G-algebras (see Subsection 1.2.3). A free in Θ^G algebra $W = W(X)$ is the free product $G * W_0(X)$, where $W_0(X)$ is a free algebra in Θ. The elements from the distinguished algebra G play the role of constants in an equation $w = w'$, $w, w' \in W(X)$. Usual equations in the polynomial algebra $K[x_1, \ldots, x_n]$ are of these type, where the elements of the field K play the role of constants. Another popular example of such kind is the equations from $Grp^{F(X)}$, where $F(X)$ is a free group. Here Θ is the variety of groups Grp and the $G = F(X)$ is the group of constants.

Example 2.2.7. Let $F = F(x, y)$ be the free group with two generators. Take the variety $\Theta = Grp^F$, and consider the equation

$$xybxa = b^{-1}$$

in Θ. It has a solution $x = b^{-1}a$, $y = a^{-3}$. Another example of an equation in $\Theta = Grp^F$ is as follows

$$[x, y] = [a, b],$$

where $[x, y] = xyx^{-1}y^{-1}$. It has two series of solutions $x = ab^n,\ y = b$ and $x = a,\ y = ba^m$, $n, m = 0, 1, 2, ...$

In fact, with each variety of algebras Θ the following algebraic geometries are associated:

- *Algebraic geometry in Θ, that is coefficient-free algebraic geometry.* In this geometry equations have the form:

  $$w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n).$$

  Solutions of equations lie in the affine space $Hom(W(X), H)$, where H is an algebra in Θ.

- *Algebraic geometry in Θ^G with coefficients in the algebra $G \in \Theta$.* The solutions lie in the affine space $Hom(W(X), H)$, where $H \in \Theta^G$ is a G-algebra. The elements of $W(X) = G * W_0(X)$, where $W_0(X)$ is the free algebra in Θ can be viewed as words in variables $x_1, \ldots, x_n$ with coefficients in G. Equations in $W(X)$ have the form:

  $$w(x_1, \ldots, x_n; g_1, \ldots, g_k) = w'(x_1, \ldots, x_n; g_1, \ldots, g_k),$$

  which means that every word w in $W(X)$ involves the variables $x_1, \ldots, x_n$ and the constants $g_1, \ldots, g_k$ from G.

- *Diophantine algebraic geometry.* This is a particular case of the previous item, i.e., the geometry in Θ^G with the solutions of equations in the affine space $Hom(W(X), G)$.

!!! See Sela for details, v chem raznitsa ?

2.2.2 Galois correspondence in the universal case

Let T be a system of equations of the form $w(x_1, \dots, x_n) = w'(x_1, \dots, x_n)$, where $w, w' \in W(X)$. With the vocabulary of Section 2.2.1 one can define the operators:

$$V: \text{ systems of equations } \to \text{ subsets of the affine space,}$$

and

$$I: \text{ subsets of the affine space} \to \text{ systems of equations,}$$

exactly in the same way as it is done for the case of classical algebraic geometry:

$$\begin{cases} V(T) = \{\bar{a} = (a_1, \dots, a_n) \in H^n \mid w(\bar{a}) = w'(\bar{a}), \text{ for all } w = w' \in T\}, \\ I(A) = \{w = w', w, w' \in W(X) \mid w(\bar{a}) = w'(\bar{a}), \text{ for all } \bar{a} \in A \subset H^n\}. \end{cases}$$

As we know , the set of polynomials T and the ideal $\langle T \rangle$ have the same set of common zeros in the affine space. Analogously, the set of equations T and the congruence $\langle T \rangle$ generated by T have the same set of solutions. Thus, the correspondence above can be viewed as a correspondence between congruences on $W(X)$ and subsets in the affine space.

Warning 2.2.8. *There is no reason to think that some analogue of the Hilbert's basis theorem holds for arbitrary Θ. Hence, the set T of equations in $W(X)$ can be infinite. We will discuss the Noetherian properties in Subsection 2.2.8.*

Remark 2.2.9. *For the sake of convenience we will use the stroke notation instead of V and I operators. The direction the stroke acts is clear from the context. For example, $A = T'_H$ is the set of common solutions of the equations from T. The index specifies the affine space we are dealing with.*

In the new notation the Galois correspondence can be rewritten as

$$\begin{cases} T'_H = A = \{\bar{a} = (a_1, \dots, a_n) \in H^n \mid w(\bar{a}) = w'(\bar{a}), \text{ for all } w = w' \in T, \} \\ A'_H = T = \{w = w', w, w' \in W(X) \mid w(\bar{a}) = w'(\bar{a}), \text{ for all } \bar{a} \in A \subset H^n\}. \end{cases}$$

Rewrite this Galois correspondence once again using the bijection between H^n and $Hom(W(X), H)$ and considering points of the affine space as homomorphisms $\mu : W(X) \to H$ (cf. Remark 2.1.9 in the classical case). Recall that a point μ is a solution of the equation $w = w'$ if (w, w') belongs to the kernel of the point μ. Then

$$T'_H = A = \{\mu : W(X) \to H \mid T \subset Ker(\mu)\},$$

$$A'_H = T = \{(w = w') \mid (w, w') \in \bigcap_{\mu \in A} Ker(\mu)\}.$$

The set T'_H consists of the points μ satisfying each equation from T. The set A'_H consists of all equations $w = w'$ in $W(X)$ which satisfy every point from A. This is always a congruence being the intersection of kernels of homomorphisms. The congruence A'_H is an ideal for the varieties $Com - K$, $Ass - K$, $Lie - K$, and a normal subgroup for the variety Grp.

Definition 2.2.10. *A set A in the affine space $Hom(W(X), H)$ is called an algebraic set if there exists a system of equations T in $W(X)$ such that each point μ of A satisfies all equations from T, i.e., $A = T'_H$.*

A congruence T in $W(X)$ is called H-closed if there exists A such that $T = A'_H$.

Example 2.2.11. The affine space $Hom(W(X), H)$ is an algebraic set being the solution of the system T of equations $w = w$, $w \in W(X)$.

A point $\mu : W(X) \to H$, $\mu(x_i) = h_i$, $h_i \in H$ is not necessarily an algebraic set for arbitrary Θ. However this is the event for classical algebraic geometry and for Diophantine geometry. In the latter case the point μ is defined by the system T of equations $x_i = h_i$, $i = 1, \dots, n$.

Let $\Theta = Grp^G$. Consider an equation $[x, a] = 1$, where $x \in W(X)$, $a \in G$. Let H be a G-group. Then the centralizer $C_H(a)$ is an algebraic set in H.

The Galois closures of arbitrary sets A and T are defined as $A''_H = ((A)'_H)'_H$ and $T''_H = ((T)'_H)'_H$, respectively. Then A''_H is always an algebraic set, while T''_H is an H-closed congruence.

The one-to-one correspondence:

$$\text{algebraic sets} \quad \rightleftarrows \quad H\text{-closed congruences}$$

takes place for every Θ.

Remark 2.2.12. *Hilbert's Nullstellensatz exactly tells us what is the description of the H-closed congruences if $\Theta = Com - K$. It also hints the following definition:*

Definition 2.2.13. *Let T be a system of equations. The H-closed congruence T''_H is called the radical of T and denoted by $Rad_H(T)$.*

Given a set T, the congruence $Rad_H(T)$ is the minimal H-closed congruence containing T. A congruence T is called *radical congruence* if $T = Rad_H(T)$. Thus, the term radical congruence is just another name for H-closed congruences and referring to the classical case, we can rewrite the one-to-one correspondence as:

$$\text{algebraic sets} \quad \rightleftarrows \quad \text{radical congruences.}$$

2.2.3 Zarisky topology in arbitrary variety Θ

The Galois correspondence between congruences in $W(X)$ and algebraic sets in the affine space $Hom(W(X), H)$ possesses the following properties:

Lemma 2.2.14. *Given a set I, the maps $'$ satisfy*

1. $\bigcap\limits_{i\in I}(T_i)'_H = (\bigcup\limits_{i\in I} T_i)'_H$.
2. $\bigcup\limits_{i\in I}(T_i)'_H \subset (\bigcap\limits_{i\in I} T_i)'_H$.
3. $\bigcap\limits_{i\in I}(A_i)'_H = (\bigcup\limits_{i\in I} A_i)'_H$.
4. $\bigcup\limits_{i\in I}(A_i)'_H \subset (\bigcap\limits_{i\in I} A_i)'_H$.

   If all A_α and T_α are H-closed sets, then

5. $(\bigcup T'_\alpha)''_H = (\bigcap T_\alpha)'_H$.
6. $(\bigcup A'_\alpha)''_H = (\bigcap A_\alpha)'_H$. $\square$

Thus, the intersection of algebraic sets is an algebraic set, and the intersection of closed congruences is again a closed congruence.

Warning 2.2.15. *To the contrary with the classical case the finite union of algebraic sets is not necessarily an algebraic set.*

Example 2.2.16. Let $\Theta = Grp$ and H be a cyclic group. Take equations $x^2 = 1$ as T_1 and $x^3 = 1$ as T_2. The corresponding algebraic sets $(T_1)'_H$, $(T_2)'_H$ in H consist of elements of exponent 2 and 3, respectively. Then $((T_1)'_H \cup (T_2)'_H) \neq (T_1 \cap T_2)'_H$, because the latter algebraic set consists of elements of exponent 6. Moreover, it is easy to see that if H is an abelian group, then any algebraic set $A \subset H^n$ is always a subgroup of H^n. Hence, $((T_1)'_H \cup (T_2)'_H) \neq (T_1 \cap T_2)'_H$ for arbitrary T_1 and T_2, since the union of subgroups is not a subgroup (assuming $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$).

Definition 2.2.17. *The algebra H is said to be geometrically stable if for every $W(X) \in \Theta$ and every two algebraic sets A and B in the space $Hom(W(X), H)$ the union $A \cup B$ is also an algebraic set.*

Remark 2.2.18. *Geometrically stable algebras are also called equational domains ([DMR4]).*

In particular,

$$((T_1)'_H \cup (T_2)'_H) = (T_1 \cap T_2)'_H,$$

for geometrically stable algebras. A general criteria for an algebra H from a variety Θ or Θ^G to be geometrically stable is given in terms of Ω-groups (i.e., multioperator groups, see [Hig], [Ku1]). It turns out, that most of the interesting cases: non-abelian free groups, free commutative and associative algebras, free associative algebras and free Lie algebras, simple groups, fields and skew fields are geometrically stable (see [BPP]).

If an algebra H in the variety Θ is geometrically stable, then one can introduce a Zariski topology in the affine space $Hom(W(X), H)$ in a usual way, declaring affine sets to be closed sets in this topology. Indeed, arbitrary intersections and finite unions of algebraic sets are algebraic sets in geometrically stable algebras.

We shall make some remarks about trivial cases. A congruence T corresponding to equalities $w = w$ is called *zero congruence*. T is called a *unity or a non-proper congruence*, if $(w, w') \in T$ is fulfilled for every $w, w' \in W(X)$. These congruences are denoted as $T = \mathbf{0}$, $T = \mathbf{1}$, respectively.

Define a zero subalgebra of H as a subalgebra consisting of one element which is distinguished by a unique nullary operation. For example, zero element in an associative algebra is considered as a zero subalgebra. Analogously, unit element is a zero subalgebra in any group.

We have $\mathbf{0}'_H = Hom(W(X), H)$ and, therefore, $Hom(W(X), H)$ is an algebraic set. As for $\mathbf{1}'_H$, this is either an empty set, or a zero point in $Hom(W(X), H)$, sending $W(X)$ to the zero subalgebra in H, if the latter exists.

Remark 2.2.19. *If we consider equations in Θ^G (see Example 2.2.6), i.e., equations with constants from an algebra G, then always $\mathbf{1}'_H = \varnothing$. If we consider equations without constants, then $\mathbf{1}'_H$ can be a zero subalgebra. For example, if $\Theta = Grp$, then any set T of equations of the form $w(x_1, \ldots, x_n) = 1$ has a common solution $\mu \in Hom(W(X), H)$ such that $x_i^\mu = 1$, $i = 1, \ldots, n$.*

Zero congruence is not necessarily closed in $W(X)$ because $(\mathbf{0})''_H = \big(Hom(W(X), H)\big)'_H = T$, where T is the congruence of all identities of H in the free algebra $W(X)$. This is the minimal closed congruence in $W(X)$. If an algebra H generates the whole variety Θ, then $\mathbf{0}''_H = \mathbf{0}$, and $\mathbf{0}$ is a closed congruence. We use the following agreement regarding empty sets: if $T = \varnothing$, then $T'_H = Hom(W(X), H)$; if $A = \varnothing$ then $A'_H = \mathbf{1}$.

The constructed Zariski topology on the affine space $Hom(W(X), H)$, where $H \in \Theta$ is a geometrically stable algebra, maintains many properties of the Zariski topology for the classical geometry. In particular, it introduces the Zariski topology on algebraic subsets in the affine space, the Zariski closure $\bar{A}$ of an arbitrary set $A \subset Hom(W(X), H)$ coincides with its Galois closure A''_H, etc. However,

Warning 2.2.20. *To the contrary with the classical case the affine space $Hom(W(X), H)$ is not necessarily Noetherian for arbitrary $H \in \Theta$ (cf. Warning 2.2.8).*

The problem whether the Zariski topology over $H \in \Theta$ is Noetherian, is one of the key problems of the whole theory. As soon as geometry for the particular $H \in \Theta$ turns to be Noetherian, there is a reasonable basis for the use of geometric methods (see Subsection 2.2.8 for details).

Now we define the Zariski topology for not necessarily stable algebras. Given A and B the algebraic sets in $Hom(W(X), H)$, define

$$A \overline{\cup} B = (A \cup B)''_H.$$

If T_1 and T_2 are closed congruences in $W(X)$, then we set

$$T_1 \overline{\cup} T_2 = (T_1 \cup T_2)''_H.$$

Definition 2.2.21. *Let Θ be a variety of algebras and H an algebra in Θ. Closed sets in the Zariski topology on the affine space $Hom(W(X), H)$ are represented by finite unions of algebraic sets and their arbitrary intersections.*

In topological terms this definition means that algebraic sets form a pre-base of closed sets for Zariski topology. In other words, the Zariski topology on $Hom(W(X), H)$ is a topology with respect to the generalized union operation $\overline{\cup}$. If the algebra H is geometrically stable, then

$$A \overline{\cup} B = A \cup B,$$

and we obtain the usual Zariski topology of algebraic sets.

For a given $H \in \Theta$ and $W = W(X)$, denote by $Alv_H(W)$ the set of all algebraic sets in $Hom(W(X), H)$, and by $Cl_H(W)$ the set of all H-closed congruences in $W(X)$. The sets $Alv_H(W)$ and $Cl_H(W)$ constitute lattices with respect to operations $\overline{\bigcup}$ defined above.

Proposition 2.2.22. *Lattices $Alv_H(W)$ and $Cl_H(W)$ are dual.* □

The duality is determined by the transition $A \to A'$ and the properties 5, 6 from Lemma 2.2.14. If H is a stable algebra, then every lattice $Alv_H(W)$ is distributive. In this case it is a sublattice in the lattice $Bool_\Theta(W, H)$ of all subsets in $Hom(W(X), H)$. A dual lattice $Cl_H(W)$ is also distributive. In general, the lattice $Alv_H(W)$ is not necessarily distributive. Hence, $Alv_H(W)$ is not a sublattice in $Bool_\Theta(W, H)$.

There is another approach to the Zariski topology on $Hom(W(X), H)$, which hints further generalizations. Let us look at the formulas

$$w_1 = v_1 \vee \cdots \vee w_n \equiv v_n, \quad w_i, v_i \in W(X),$$

called *pseudo-equalities*. Considering them as pseudo-equations, we say that a point $\mu \in Hom(W(X), H)$ is a solution of a pseudo-equation if there exists $1 \leq i \leq n$ such that $w_i^\mu = v_i^\mu$. One can build the Galois correspondence with respect to pseudo-equations, and to define *pseudo-algebraic sets* as sets of common solutions of systems of pseudo-equations.

Proposition 2.2.23. *Closed sets in the Zariski topology on the space $Hom(W(X), H)$ coincide with pseudo-algebraic sets.*

According to definitions, every algebraic set is a pseudo-algebraic. Conversely, if an algebra H is stable, then every pseudo-algebraic set is algebraic.

2.2.4 Coordinate algebras

Our next aim is to imitate Definition 2.1.13 of the coordinate ring for the case of arbitrary variety Θ.

Definition 2.2.24. *Let A be an algebraic set, $T = A'_H$ the corresponding H-closed congruence. Algebra W/T is called the coordinate algebra of the algebraic set A.*

Coordinate algebras have many algebraic faces. First of all, similarly to the classical case they can be treated as algebras of regular functions on A.

Definition 2.2.25. *Let A be an algebraic set, $H \in \Theta$. A function $\alpha : A \to H$ is said to be regular if there is $w \in W(X)$ satisfying $\alpha(\mu) = w^\mu$, for every point $\mu \in A$.*

The function α can also be defined via another element $w_0 \in W(X)$. Then $w^\mu = w_0^\mu$ for every $\mu \in A$. This means that $(w, w_0) \in A'_H$. Hence, the coordinate algebra W/A'_H is isomorphic to the algebra of regular functions on A.

The problem of characterization of H-closed congruences for a specific variety Θ is, in fact, a Nullstellensatz-type problem for this Θ and $H \in \Theta$. It is a challenge, since each particular variety Θ and $H \in \Theta$ has its own Nullstellensatz. However, let us point out some algebraic properties of the closed congruences and coordinate algebras which can be viewed through the prism of the general Nullstellensatz.

First of all, let us make a comment on the structure of the "general solution" of equations from T. We look for solutions in the algebra $H \in \Theta$. Consider the natural homomorphism $\mu_0 : W \to W/T$. This homomorphism gives rise to the commutative diagram

$$\begin{tikzcd} W \arrow[r, "\mu_0"] \arrow[d, "'"'] & W/T \arrow[d, "'"] \\ Hom(W,H) & Hom(W/T,H) \arrow[l, "\widetilde{\mu}_0"'] \end{tikzcd}$$

where $\widetilde{\mu}_0$ is defined by $\widetilde{\mu}_0(\nu) = \nu\mu_0$, for $\nu \in Hom(W/T,H)$. Denote

$$Hom(W/T,H)\mu_0 = \{\nu\mu_0 : W \to H \ \mid \ \nu \in Hom(W/T,H)\}.$$

Commutativity of the diagram implies

Proposition 2.2.26.

$$T'_H = Hom(W/T,H)\mu_0,$$

for any $T \in W$. Moreover, $\tilde{\mu}_0 : Hom(W/T,H) \to T'_H$ is a bijection.
$\square$

Thus, the set of solutions A of the system of equations T, where T is a congruence, can be presented as:

$$T'_H = A = Hom(W/T,H)\mu_0.$$

For arbitrary algebras H and G, denote

$$(H - Ker)(G) = \bigcap_{\nu : G \to H} Ker\nu.$$

Let T be a congruence in W and take $\mu_0 : W \to W/T$. Using Proposition 2.2.26 we have

Proposition 2.2.27.

$$T''_H = \mu_0^{-1}\big((H - Ker)(W/T)\big).$$

Proof. Let $\tau = (H - Ker)(G)$. Consider $W \overset{\mu_0}{\to} G \overset{\mu_1}{\to} G/\tau$, where μ_1 is the natural homomorphism, and set $\widetilde{T} = Ker(\mu_0\mu_1)$. Then $w\widetilde{T}w'$ means that $w^{\mu_0} Ker(\mu_1) w'^{\mu_0}$, i.e. $w^{\mu_0}\tau w'^{\mu_0}$. So, $\widetilde{T} = \mu_0^{-1}(\tau)$. We shall verify that $\widetilde{T} = T''_H$.

Assume that $w\widetilde{T}w'$. By definition of τ, we have $(w^{\mu_0}, w'^{\mu_0}) \in Ker(\nu)$ and $w^{\mu_0\nu} = w'^{\mu_0\nu}$ for every $\nu \colon G \to H$. By Proposition 2.2.26, $\mu_0\nu$ is an element of $T'_H = A$, and $(w, w') \in Ker(\mu_0\nu)$. Therefore, $(w, w') \in T''_H$, and thus $\widetilde{T} \subset T''_H$.

Now assume that $wT''_H w'$. Then $w^{\mu_0\nu} = w'^{\mu_0\nu}$ for every $\nu \colon G \to H$, and $(w^{\mu_0}, w'^{\mu_0}) \in \bigcap_\nu Ker(\nu) = \tau$. This implies that $w^{\mu_0\mu_1} = w'^{\mu_0\mu_1}$ and $w\widetilde{T}w'$. So, $T''_H \subset \widetilde{T}$. Thus, $\widetilde{T} = T''_H$ for every G. Take $G = W/T$.

□

Proposition 2.2.27 can be viewed as one of the forms of Hilbert's Nullstellensatz.

Example 2.2.28. Let us derive the classical Hilbert's Nullstellensatz (Theorem 2.1.6) from Proposition 2.2.27. We use two general facts (see [AM], [E]).

The first one says that if H is a finitely generated associative and commutative algebra, then its Jacobson radical $\mathfrak{J}(H)$ is, at the same time, the nill-radical of H, i.e., it coincides with the set of all nilpotent elements of H.

The other fact is as follows: if T is a proper ideal of the ring $R = K[x_1, \ldots, x_n]$, and L is the algebraically closed extension of the field K, then there is a homomorphism $\mu : R \to L$, such that $T \subset Ker\mu$. A property like this could serve as a general definition of the algebraic closeness of arbitrary universal algebras.

We shall check that:

$$((L - Ker)(W/T)) = \mathfrak{J}(R/T).$$

The radical $\mathfrak{J}(R/T)$ is the intersection of all maximal ideals. Suppose that T_0/T is a maximal ideal of R/T. Then T_0 is a maximal ideal of R, and there is a homomorphism $\mu : R \to L$ with $T_0 \subset Ker\mu$. It follows from the maximality condition that $T_0 = Ker\mu$. Since $T \subset Ker\mu$, the homomorphism μ induces another homomorphism $\nu : R/T \to L$ and here $T_0/T = Ker\nu$. Therefore, every maximal ideal of R/T is realized as the kernel of some ν. This means that the inclusion

$$((L - Ker)(W/T)) \subset \mathfrak{J}(R/T)$$

holds. Every element of $\mathfrak{J}(R/T)$ is nilpotent, and every nilpotent element of R/T belongs to the kernel of any $\nu : R/T \to L$. Hence the converse inclusion is true. The Hilbert's theorem (see Theorem 2.1.3) now follows from Proposition 2.2.27.

Since T''_H is the minimal H-closed congruence containing T, it can be represented as the intersection of all H-closed congruence containing T, i.e., $T = \bigcap T_\alpha$, where all T_α are H-closed. Every W/T_α lies in H. Then, by Remak's theorem

Proposition 2.2.29. *A congruence T in W is H-closed if and only if for some set I there is an injection*

$$W/T \to H^I. \qquad \Box$$

From Proposition 2.2.29 follows that an algebra $G \in \Theta$ can be presented as a coordinate algebra of an algebraic set A over given algebra $H \in \Theta$ if and only if G is finitely generated algebra and there is an injection $G \to H^I$ for some set I.

Recall, that the class of algebras $\mathfrak{X}$ is called a *prevariety* if $\mathfrak{X}$ is closed under Cartesian products and subalgebras (see Subsection 1.1.6).

For an arbitrary class $\mathfrak{X}$ the corresponding closure up to prevariety is $SC(\mathfrak{X})$ (Proposition 2.2.30). Here S and C are closure operators on classes of algebras: C under Cartesian products and S under subalgebras. Proposition 2.2.29 implies

Proposition 2.2.30. *A congruence T in W is H-closed if and only if $W/T \in SC(H)$.* $\Box$

Besides, if T is an arbitrary binary relation in W, then T''_H is an intersection of all congruences T_α with $T \subset T_\alpha$ and $W/T_\alpha \in SC(H)$.

Definition 2.2.31. *An algebra G is called residually H (or H separates G) if for every pair of elements g_1 and g_2 in G, $g_1 \neq g_2$, there exists a homomorphism $\varphi : G \to H$ such that $\varphi(g_1) \neq \varphi(g_2)$.*

An algebra is residually H if and only if there exists a set of congruences ρ_α, such that $\bigcap \rho_\alpha = 1$ and for every α there is a monomorphism $G/\rho_\alpha \to H$.

We collect the properties of coordinate algebras in

Proposition 2.2.32. *Let T be a congruence in $W(X)$. Algebra $W/T \in \Theta$ is a coordinate algebra of an algebraic set A in $Hom(W(X), H)$ if and only if one of the following conditions hold:*

- *W/T is embedded in H^I, for some set I,*
- *W/T belongs to the prevariety generated by the algebra H,*
- *W/T is a residually H algebra.* $\square$

2.2.5 Categories of coordinate algebras and algebraic sets

Let Θ^0 be the category of all free algebras $W = W(X)$ in Θ, where X is finite. Homomorphisms of algebras are morphisms in Θ^0.

Let us introduce the *category of affine spaces* $K^0_\Theta(H)$. Objects of this category are affine spaces

$$Hom(W(X), H).$$

Morphisms

$$\widetilde{s} : Hom(W(X), H) \to Hom(W(Y), H)$$

of $K^0_\Theta(H)$ are induced by homomorphisms $s : W(Y) \to W(X)$ according to the rule $\widetilde{s}(\nu) = \nu s$ for every $\nu : W(X) \to H$.

The correspondence $W(X) \to Hom(W(X), H)$ and $s \to \widetilde{s}$ gives rise to a contravariant functor

$$\mathcal{F} : \Theta^0 \to K^0_\Theta(H).$$

Proposition 2.2.33. *The functor $\mathcal{F} : \Theta^0 \to K^0_\Theta(H)$ determines the duality of categories if and only if $Var(H) = \Theta$.*

Proof. The condition of duality means that if $s_1 \neq s_2$ for the given morphisms $s_1, s_2 : W(Y) \to W(X)$, then $\widetilde{s_1} \neq \widetilde{s_2}$.

Let assume that $Var(H) = \Theta$ and the categories are not dual, so there are morphisms s_1 and s_2 such that $s_1 \neq s_2$ and $\widetilde{s_1} = \widetilde{s_2}$. Take some $y \in Y$ such that $s_1(y) = w_1$, $s_2(y) = w_2$ and $w_1 \neq w_2$. We will show that in the algebra H there is the non-trivial identity $w_1 \equiv w_2$. Take an arbitrary homomorphism $\nu : W(X) \to H$. The equality $\widetilde{s_1} = \widetilde{s_2}$ implies $\widetilde{s_1}(\nu) = \widetilde{s_2}(\nu)$ or $\nu s_1 = \nu s_2$. We apply this morphism to the variable y:

$$\nu s_1(y) = \nu s_2(y) \text{ or } \nu(w_1) = \nu(w_2).$$

Since $\nu : W(X) \to H$ is an arbitrary homomorphism, then $w_1 \equiv w_2$ is an identity of the algebra H. But $Var(H) = \Theta$, which means that there are no non-trivial identities in $Var(H)$. So, we have a contradiction and the condition $Var(H) = \Theta$ implies duality of the given categories.

Now we show that if $Var(H) \subset \Theta$ then there is no duality. Let $w_1 \equiv w_2$ be some non-trivial identity of the algebra H. Take $Y = \{y_0\}$ and let $s_1(y_0) = w_1$, $s_2(y_0) = w_2$. For any $\nu : W(X) \to H$ we have

$$\nu(w_1) = \nu(w_2),\ \nu s_1(y_0) = \nu s_2(y_0),\ \widetilde{s}_1(\nu)(y_0) = \widetilde{s}_2(\nu)(y_0).$$

Since the set Y contains only one element y_0, then $\widetilde{s}_1(\nu) = \widetilde{s}_2(\nu)$. Since ν is arbitrary homomorphism, then $\widetilde{s}_1 = \widetilde{s}_2$ and there is no duality of the categories. $\square$

Definition 2.2.34. *A map of affine spaces $\alpha : Hom(W(X), H) \to Hom(W(Y), H)$ is called regular (polynomial) if it coincides with some $\widetilde{s} : Hom(W(X), H) \to Hom(W(Y), H)$.*

Proceed now to the *category $K_\Theta(H)$ of algebraic sets.* Its objects have the form (X, A), where A is an algebraic set in the space $Hom(W(X), H)$.

Let us call $s : W(Y) \to W(X)$ admissible with respect to algebraic sets $A \subset Hom(W(X), H)$ and $B \subset Hom(W(Y), H)$ if $\nu \in A$ implies $\widetilde{s}(\nu) = \nu s \in B$. Given admissible $s : W(Y) \to W(X)$, a morphism $[s] : (X, A) \to (Y, B)$ is defined by

$$[s](A) = \{\mu \mid \mu = \widetilde{s}(\nu) = \nu s,\ \nu \in A\}.$$

Definition 2.2.35. *A map $\alpha : A \to B$ of algebraic sets is called regular if there exists $[s] : A \to B$, such that $\alpha(\nu) = \widetilde{s}(\nu)$, for all $\nu \in A$.*

Morphisms of $K_\Theta(H)$ are regular maps of algebraic sets. So, morphisms of the category $K_\Theta(H)$ of algebraic sets are defined via regular maps of affine spaces.

The category $K_\Theta(H)$ is the full subcategory of the category Set_H, whose objects have the form (X, A), where A is an arbitrary subset in the space $Hom(W(X), H)$ while morphisms coincide with morphisms of $K_\Theta(H)$.

Let us define the category $C_\Theta(H)$. Its objects are coordinate algebras and have the form W/T, where W is an abject of the category Θ^0 and T is an H-closed congruence in W. Morphisms of $C_\Theta(H)$ are the homomorphisms of algebras in the variety Θ.

Our next aim is to relate the categories of algebraic sets $K_\Theta(H)$ and of coordinate algebras $C_\Theta(H)$.

Let T_2 and T_1 be congruences in $W(Y)$ and $W(X)$, respectively. A homomorphism $s : W(Y) \to W(X)$ is admissible with respect to congruences T_2 and T_1 if for $w \equiv w' \in T_2$ we have $s(w) \equiv s(w') \in T_1$.

Suppose that $A = (T_1)'_H$, $B = (T_2)'_H$ and consider $s : W(Y) \to W(X)$.

Lemma 2.2.36. *A homomorphism $s : W(Y) \to W(X)$ is admissible with respect to congruences if and only if s is admissible with respect to algebraic sets.*

Proof. Assume that $\nu s \in B$ for every $\nu \in A$ and that $w\, T''_2\, w'$. We need to check that $s(w) T''_1 s(w')$. We have $T''_1 = A'$ and $T''_2 = B'$. Moreover, $A' = \bigcap_{\nu \in A} Ker(\nu)$. Check that, for all $\nu \in A$, $(s(w), s(w')) \in Ker(\nu)$ or, in other words, $\nu s(w) = \nu s(w')$. By definition, $T''_2 = B' = \bigcap_{\mu \in B} Ker(\mu)$. Hence, $w T''_2 w'$ means that $\mu(w) = \mu(w')$. In particular, this is true for $\mu = \nu s$, and then $\nu s(w) = \nu s(w')$.

Conversely, assume that $w\, T''_2\, w'$ implies $s(w)\, T''_1\, s(w')$. Given $\nu \in A$, check that $\nu s \in B$ i.e., $\nu s(w) = \nu s(w')$ whenever $w\, T''_2\, w'$. The latter condition implies $s(w)\, T''_1\, s(w')$. So, if $\nu \in A$, then $\nu s(w) = \nu s(w')$, that is $\nu s \in B$. □

Lemma 2.2.37. *Every morphism $[s]\colon A \to B$ induces a homomorphism $\overline{s}\colon W(Y)/B'_H \to W(X)/A'_H$. Conversely, every homomorphism $\sigma\colon W(Y)/B'_H \to W(X)/A'_H$ gives rise to a morphism $[s]\colon A \to B$.*

Proof. Suppose we have a morphism $[s]\colon A \to B$, where $A = T'_1$ and $B = T'_2$. The homomorphisms $s\colon W(Y) \to W(X)$ and $\sigma_0\colon W(X) \to W(X)/T''_1$ define $s\sigma_0\colon W(Y) \to W(X)/T''_1$. By Lemma 2.2.36, the congruence T''_2 lies in $Ker(s\sigma_0)$. Hence, the homomorphism $\sigma = \overline{s}\colon W(Y)/T''_2 \to W(X)/T''_1$ is defined. It remains to observe that $T''_2 = B'$ and $T''_1 = A'$.

Conversely, suppose a homomorphism $\sigma\colon W(Y)/T''_2 \to W(X)/T''_1$ is given. We have a commutative diagram

$$\begin{tikzcd}
W(Y) \arrow[r, "s"] \arrow[d, "\sigma_1"'] & W(X) \arrow[d, "\sigma_0"] \\
W(Y)/T''_2 \arrow[r, "\sigma"] & W(X)/T''_1.
\end{tikzcd}$$

Assume that wT''_2w'. This means that $\sigma_1(w) = \sigma_1(w')$. Then $\sigma\sigma_1(w) = \sigma\sigma_1(w')$ and $\sigma_0 s(w) = \sigma_0 s(w')$, whence $s(w)T''_1s(w')$. By Lemma 2.2.36 we have a morphism $s\colon A \to B$. □

Corollary 2.2.38. *If s_1 and s_2 are admissible with respect to A and B and $[s_1] = [s_2] : A \to B$, then $\overline{s}_1 = \overline{s}_2$.*

Proof. Follows from the construction of $\overline{s}$ in Lemma 2.2.37. See also **??** (Part II). $\square$

The correspondence $[s] \to \overline{s}$ and $(X, A) \to W(X)/A'_H$ determines the contravariant functor $\mathcal{F}$ from $K_\Theta(H)$ to $C_\Theta(H)$. The correspondence $\overline{s} \to [s]$ and $W/T \to (X, T'_H)$ gives rise to the contravariant functor $\mathcal{F}' : C_\Theta(H) \to K_\Theta(H)$. Lemma 2.2.37 and Corollary 2.2.38 yield that any regular map $\alpha = [s] : A \to B$ induces the homomorphism $\overline{s} : W(Y)/B'_H \to W(X)/A'_H$ in a unique way. Hence, the pair of functors $\mathcal{F}$ and $\mathcal{F}'$ determines the duality of the categories $K_\Theta(H)$ and $C_\Theta(H)$. We shall state this important fact as a theorem.

Theorem 2.2.39 ([Pl-7L],[DMR2])**.** *The category of algebraic sets $K_\Theta(H)$ is dually isomorphic to the category of coordinate algebras $C_\Theta(H)$.*

Proof. Let $[s_1], [s_2] : A \to B$ be given. Suppose that $[s_1] = [s_2]$. We have to check that $\mathcal{F}([s_1]) = \mathcal{F}([s_2])$. The latter means that $\overline{s_1} = \overline{s_2}$, which follows from Corollary 2.2.38. $\square$

Corollary 2.2.40. *The category of algebraic sets $K_\Theta(H)$ is dually equivalent to the category of residually H algebras.*

Proof. By Proposition 2.2.32 every residually H-algebra is isomorphic to a coordinate algebra.

$\square$

Remark 2.2.41. *Theorem 2.2.39 is completely parallel to Theorem 2.1.17 from the classical geometry. The general regular maps are converted for the variety $Com - K$ to usual polynomial maps between algebraic sets. Theorem 2.1.17 is a particular case of Theorem 2.2.39.*

Recall that a subcategory $\mathcal{L}$ of a category $\mathcal{K}$ is a skeleton of $\mathcal{K}$ if the inclusion functor is an equivalence, and no two objects of $\mathcal{L}$ are isomorphic. Two categories are equivalent if and only if their skeletons are isomorphic.

The skeleton of the category $K_\Theta(H)$ is denoted by $\widetilde{K}_\Theta(H)$. The objects of $\widetilde{K}_\Theta(H)$ are called *algebraic varieties* over H.

2.2.6 Geometrically equivalent algebras

Suppose we have algebras H_1 and H_2 from the same variety Θ. We want to compare their abilities with respect to solving systems of equations. This point of view hints the following definition.

Definition 2.2.42. *Algebras H_1 and H_2 in Θ are called geometrically equivalent if for every finite X and every system of equations T in the free algebra $W(X)$, the equality*

$$T''_{H_1} = T''_{H_2}$$

takes place.

This condition is equivalent to the following one: $T''_{H_1} = T$ if and only if $T''_{H_2} = T$, i.e., any congruence T in $W(X)$ is H_1-closed if and only if it is H_2-closed. Hence, the corresponding coordinate algebras coincide, and

Proposition 2.2.43. *If the algebras H_1 and H_2 in Θ are geometrically equivalent, then the categories of algebraic sets $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic.* $\square$

So, the geometrical equivalence of algebras H_1 and H_2 is a sufficient condition which provides isomorphism of the categories of algebraic sets $K_\Theta(H_1)$ and $K_\Theta(H_2)$. Necessary and sufficient conditions for isomorphism of $K_\Theta(H_1)$ and $K_\Theta(H_2)$ will be considered in Section 2.2.7.

For the classical case $\Theta = Com - K$ geometrical equivalence of algebras looks as follows. Two extensions L_1 and L_2 of the field K are geometrically equivalent if for every finite X and every ideal T in the polynomial algebra $K[X]$ the equality

$$T''_{L_1} = T''_{L_2}$$

takes place.

If L_1 and L_2 are arbitrary algebraically closed extensions of K, then they are geometrically equivalent. Note that finite extensions L_1 and L_2 of K are geometrically equivalent if and only if they are isomorphic.

Theorem 2.2.44. *If the field K is algebraically closed, then all its extensions are geometrically equivalent. If every two extensions of K are geometrically equivalent, then K is algebraically closed.*

Proof. If K is an algebraically closed field, then Hilbert's Nullstellensatz implies

$$T''_{L_1} = Rad(T) = \{t \in K[x_1, \ldots, x_n] \mid t^s \in T, s \in \mathbb{N}\} = T''_{L_2},$$

for every ideal T in $K[X]$. Hence, every two extensions of the algebraically closed field are geometrically equivalent.

Conversely, suppose that every two extensions of K are geometrically equivalent. Let L be the algebraic closure of K. Then K

and L are geometrically equivalent. Hence $T''_K = T\text{''}''_L = Rad(T)$. In view of Remark 2.1.8, $T''_K = Rad(T)$ implies that if $V(J) = \varnothing$ for an ideal $J \in K[x_1, \dots, x_n]$, then $J = K[x_1, x_2, \dots, x_n]$. Hence, the field K is algebraically closed since over a non-algebraically closed field K there exists a non-constant polynomial without roots in K. $\square$

In the general case geometrical equivalence of algebras heavily depends on the ground field K.

A field K is called *formally real* if -1 is not a sum of squares in K. The class of such fields coincides with the class of fields that admit an ordering. A formally real field which has no formally real algebraic extensions is called a *real closed field.* From Nullstellensatz for real closed fields follows that two real extensions of a real closed field are geometrically equivalent. However,

Theorem 2.2.45 ([Berzins-GeomEquiv])**.** *Two real closed extensions L_1 and L_2 of a field K are geometrically equivalent if and only if they induce equal orders on K.* $\square$

Example 2.2.46. Let $K = \mathbb{Q}(\alpha)$, $\alpha^2 = 2$. There exist two different embeddings of K into the field of real numbers $\mathbb{R}$:

$$\mathbb{Q}(\alpha) \to \mathbb{R}, \quad \alpha = \sqrt{2},$$

$$\mathbb{Q}(\alpha) \to \mathbb{R}, \quad \alpha = -\sqrt{2},$$

Let us check that these extensions L_1 and L_2 are not geometrically equivalent. Indeed, take the ideal $T = \langle x^2 - \alpha \rangle$ in $K[x]$. In the first extension the polynomial $x^2 - \alpha$ has two roots, and so $T''_{L_1} = I$. In the second extension it has no roots, and so $T''_{L_2} = \langle 1 \rangle$. This yields that these extensions are not geometrically equivalent and hence, the condition from Theorem 2.2.45 is necessary.

The problem of geometric equivalence for arbitrary fields is difficult (see [Berzins-GeomEquiv] for discussions).

Geometric equivalence relation behaves well with respect to Cartesian products of algebras. Assume that there is a nullary operation 0 among the ground operations of the variety Θ which singles out of every $G \in \Theta$ a one-element subalgebra.

Proposition 2.2.47. *Suppose that algebras H_α and H'_α, $\alpha \in I$ are geometrically equivalent. Then the Cartesian products $H_1 = \prod_\alpha H_\alpha$ and $H_2 = \prod_\alpha H'_\alpha$ are also geometrically equivalent. Conversely, if $\prod_\alpha H_\alpha$ and $\prod_\alpha H'_\alpha$ are geometrically equivalent, then H_α and H'_α, $\alpha \in I$ are geometrically equivalent too.*

Proof. We shall prove that $T''_{H_1} = T''_{H_2}$. Check, first, that

$$(\prod_\alpha H_\alpha - Ker)(G) = \bigcap_\alpha (H_\alpha - Ker)(G),$$

where $(H - Ker)(G) = \bigcap_{\nu: G\to H} Ker(\nu)$. Let τ_1, τ_2 stand for the left and right hand congruences, respectively. Assume that $g_1\tau_2 g_2$ for $g_1, g_2 \in G$. Thus, $g_1^\nu = g_2^\nu$ for all $\alpha \in I$ and ν: $G \to H_\alpha$. Let us take $\mu: G \to \prod_\alpha H_\alpha$ and verify that $g_1^\mu = g_2^\mu$. This equality means that $g_1^\mu(\alpha) = g_2^\mu(\alpha)$ for every $\alpha \in I$. We use the projections $\pi_\alpha: \prod_\alpha H_\alpha \to H_\alpha$ and denote $\mu\pi_\alpha = \nu_\alpha$. Then $g_1^\mu(\alpha) = g_1^{\nu_\alpha} = g_2^{\nu_\alpha} = g_2^\mu(\alpha)$, i.e. $g_1^\mu = g_2^\mu$ and, further, $g_1\tau_1 g_2$.

Conversely, let $g_1\tau_1 g_2$. Given $\alpha \in I$ and $\nu: G \to H_\alpha$, define μ by the rule: $g^\mu(\alpha) = g^\nu$, and $g^\mu(\beta)$ is the zero if $\beta \neq \alpha$. Then $\mu: G \to \prod_\alpha H_\alpha$ and $g_1^\mu = g_2^\mu$. But then $g_1^\mu(\alpha) = g_1^\nu = g_2^\mu(\alpha) = g_2^\nu$. Therefore, $g_1\tau_2 g_2$.

Hence,

$$(\prod_\alpha H_\alpha - Ker)(G) = \bigcap_\alpha (H_\alpha - Ker)(G) =$$

$$= \bigcap_\alpha (H'_\alpha - Ker(G)) = (\prod_\alpha H'_\alpha - Ker)(G).$$

If $G = W/T$, then $T''_{H_1} = T''_{H_2}$ by Proposition 2.2.27. □

Corollary 2.2.48. *For every algebra $H \in \Theta$ and every set I, the algebras H and H^I are geometrically equivalent.* □

Our next goal is to introduce a logical criterion for geometrically equivalent algebras. Recall, (see Subsection 1.1.6), that a quasi-identity in Θ has the form

$$w_1 \equiv w'_1 \wedge \cdots \wedge w_n \equiv w'_n \to w_0 \equiv w'_0,$$

where w_i, w'_i, w, w' belong to $W(X)$ with finite X.

We consider also more general quasi-identities of the form

$$(\bigwedge_{(w,w')\in T} w \equiv w') \to w_0 \equiv w'_0,$$

or, for short,

$$T \to w_0 \equiv w'_0,$$

where the set T is not necessarily finite.

Definition 2.2.49. *A quasi-identity is called infinitary if the set T is infinite, and finitary in the opposite case.*

Recall that, a *quasivariety* (see Subsection 1.1.6), is a class of algebras defined by finitary quasi-identities. A class $qVar(H)$ is the minimal quasivariety containing the algebra H. All algebras from $qVar(H)$ have the same quasi-identities as H.

Observe that

Proposition 2.2.50. *$w_0 \equiv w_0' \in T_H''$ if and only if the quasi-identity $T \to w_0 \equiv w_0'$ holds in the algebra H.*

Proof. By definition, $w_0 \equiv w_0' \in T_H''$ if and only if $(\neg T \vee (w_0 \equiv w_0'))$ holds true for every $\mu : W(X) \to H$.

□

This remark provides one more point of view on H-closed congruences and, thus, on general Hilbert Nullstellensatz. It implies

Proposition 2.2.51. *Algebras H_1 and H_2 in Θ are geometrically equivalent, if and only if each quasi-identity $T \to w_0 \equiv w_0'$ (finitary or infinitary), which holds in H_1 is a quasi-identity of the algebra H_2, and vice versa.* □

Hence,

Proposition 2.2.52. *If the algebras H_1 and H_2 are geometrically equivalent, then they generate the same quasivariety*

$$qVar(H_1) = qVar(H_2).$$

In particular, varieties generated by geometrically equivalent algebras H_1 and H_2 coincide

$$Var(H_1) = Var(H_2). \qquad \square$$

Corollary 2.2.53. *If two groups H_1 and H_2 are geometrically equivalent and one of them is torsion-free, then the second one is also torsion-free.* □

Warning 2.2.54. *For an arbitrary variety Θ the set of equations can be not reduced to a finite set, and the converse statement to Proposition 2.2.51 is expected to be false. Indeed,*

Theorem 2.2.55 ([MR])**.** *There exists the variety of algebras Θ, algebra $H_1 \in \Theta$ and an ultrapower H_2 of H_1, such that the algebras H_1 and H_2 are not geometrically equivalent.* □

Algebras H_1 and H_2 are called *elemetarily equivalent* if they satisfy the same first-order sentences (see Section 3.1.1). By theorem of Łoś's ([Marker], [Lo]), an algebra and its ultrapower have the same elementary theory and, hence, the same quasi-identities. Thus, Theorem 2.2.55 yields an example of non-geometrically equivalent algebras which generate the same quasivariety (cf., Theorem 2.2.108). Moreover, counter-examples to converse of Proposition 2.2.51 exist among finitely generated groups:

Theorem 2.2.56 ([MR]). *There exists a group H_1 and a finitely generated group H_2, such that $qVarH_1 = qVarH_2$, but H_1 and H_2 are not geometrically equivalent.* □

Compare now the notions of elementary equivalence and geometric equivalence of algebras. Elementarily equivalent algebras H_1 and H_2 satisfy the same sentences. On the other hand, if H_1 and H_2 are geometrically equivalent, then they have the same (infinitary in general) quasi-identities. Since infinitary quasi-identities are not a part of first-order formulas, these two notions should be distinct. This is the case, indeed.

In one direction, Theorem 2.2.55 provides an example of elementary equivalent algebras which are not geometrically equivalent.

In the other one, an example of non-elementary equivalent geometrically equivalent algebras can be found in the classical variety $\Theta = Com - K$. Let K be an algebraically closed field and L be its non-algebraically closed extension. Then L and K are geometrically equivalent (see Theorem 2.2.44), while they are not elementary equivalent.

Let $f(x) = \alpha_0 + \alpha_1 x \cdots + \alpha_n x^n$ be a polynomial with the coefficients in L and without roots in L. Take a polynomial over K

$$\varphi(x, y_0, \ldots, y_n) = y_0 + y_1 x + \cdots + y_n x^n$$

and consider a formula

$$\forall y_0 \ldots y_n \exists x (\varphi(x, y_0, \ldots, y_n) = 0).$$

This formula holds in K and does not hold in L.

Warning 2.2.54 is not relevant if the algebras H_1 and H_2 are geometrically Noetherian (see Definition 2.2.78). This is the case, when every infinite set of equations T can be replaced by a finite set T_0. Proposition 2.2.51 immediately implies the following.

Proposition 2.2.57. *Geometrically Noetherian algebras H_1 and H_2 are geometrically equivalent if and only if*

$$qVar(H_1) = qVar(H_2). \qquad \square$$

Since coincidence of elementary theories of algebras implies coincidence of their quasi-identities, we have

Corollary 2.2.58. *If two geometrically Noetherian algebras H_1 and H_2 are elementary equivalent, then they are geometrically equivalent.* $\square$

In particular

Corollary 2.2.59. *Two extensions L_1 and L_2 of a field K are geometrically equivalent if and only if they satisfy the same quasi-identities. If extensions L_1 and L_2 of a field K are elementary equivalent, then they are geometrically equivalent.* $\square$

Geometric equivalence of algebras is tightly connected with prevarities of algebras.

For every class of algebras $\mathfrak{X}$ define a local operator L as follows: $H \in L\mathfrak{X}$ if every finitely generated subalgebra H_0 of H belongs to $\mathfrak{X}$.

Definition 2.2.60. *A prevariety of algebras $\mathfrak{X}$ is called locally closed if it is closed under the local operator L.*

If H is an algebra then the class $LSC(H)$ is the locally closed prevariety generated by H. Here, S and C are the standard closure operators on classes of algebras, used in the characterization of prevarieties (see Section 1.1.6).

Proposition 2.2.61 ([PPT])**.** *Two algebras H_1 and H_2 are geometrically equivalent if and only if the corresponding locally closed prevarities coincide:*

$$LSC(H_1) = LSC(H_2). \qquad \square$$

We shall add that according to Proposition 2.2.30 finitely generated algebras in the prevariety $SC(H)$ are coordinate algebras of algebraic sets over H.

For any class $\mathfrak{X}$ the locally closed prevariety $LSC(\mathfrak{X})$ is contained in the quasivariety, generated by $\mathfrak{X}$ [PPT],[MR].

The class $LSC(\mathfrak{X})$ is not a quasivariety and, moreover, not an axiomatized class (see [MR],[Mal1]). In this sense, the relation of geometric equivalence of algebras is not an axiomatizable relation. This relation is axiomatizable in terms of infinitary quasi-identities.

Classification of algebras with respect to geometric equivalence is a difficult but challenging problem. For example, geometric equivalence of the abelian groups looks as follows:

Theorem 2.2.62. *Two abelian groups H_1 and H_2 are geometrically equivalent if and only if the following two conditions hold*

1. $VarH_1 = VarH_2$.
2. *For every prime p and the corresponding Sylow subgroups $(H_1)_p$ and $(H_2)_p$,*
$$Var(H_1)_p = Var(H_2)_p.$$

Since for abelian groups H_1 and H_2 the coincidence of exponents is equivalent to $VarH_1 = VarH_2$, this theorem can be reformulated in terms of exponents.

Proof. Let, first, H_1 and H_2 be finitely generated abelian groups. Any finitely generated abelian group A is isomorphic to $\mathbb{Z}^n \oplus A_0$, where $A_0 = A_{p_1^{k_1}} \oplus \cdots \oplus A_{p_n^{k_n}}$, and $A_{p_i^{k_i}}$ is a primary Sylow subgroup.

Necessity of the first condition in Theorem 2.2.62 follows from Proposition 2.2.51. Let us check that the exponents of the Sylow subgroups $(H_1)_p$ and $(H_2)_p$ coincide. Since $(H_1)_p$ and $(H_2)_p$ are geometrically equivalent, they have the same quasi-identities. Denote by n_1, n_2,$\cdots$, n_k the exponents of the Sylow subgroups. Then the quasi-identity
$$x_1^{n_1} = 1 \wedge \cdots \wedge x_k^{n_k} = 1 \to y^{n_1 n_2 \cdots n_k} = 1,$$
holds in H_1. Since H_1 and H_2 are geometrically equivalent, this quasi-identity holds also in H_2, which implies necessity of the second condition.

Conversely, one has to prove that under conditions 1-2 the groups H_1 and H_2 are geometrically equivalent. Use again the presentation of a finitely generated abelian group A as $\mathbb{Z}^n \oplus A_{p_1^{k_1}} \oplus \cdots \oplus A_{p_n^{k_n}}$, where $A_{p_i^{k_i}}$ is a primary Sylow subgroup. The torsion-free parts of H_1 and H_2 have the form $\mathbb{Z}^m$ and $\mathbb{Z}^s$. Thus, they are geometrically equivalent by Proposition 2.2.48.

Let us check that condition 2 implies geometric equivalence of torsion parts. Suppose that the exponent of a finite abelian group A_1 equals to the exponent of a finite abelian group A_2 and equals p^n. Both groups are directs sums of cyclic ones. Embed cyclic groups of A_1 into a group A_1' of order p^n, $n > 0$. In the same manner take a group A_2' isomorphic to A_1', containing all cyclic subgroups of A_2. Then, since p^n is the exponent of both A_1 and A_2, we have
$$(A_1 - Ker)(G) = (A_1' - Ker)(G);$$
and
$$(A_2 - Ker)(G) = (A_2' - Ker)(G);$$
for the arbitrary G. Hence,
$$(A_1 - Ker)(G) = (A_2 - Ker)(G),$$

and A_1 and A_2 are geometrically equivalent.

Applying this observation to Sylow subgroups of the same exponent $(A_1)_p$ and $(A_2)_p$ we get their geometrical equivalence. Hence, if the exponents of the Sylow subgroups coincide, then by Proposition 2.2.47 the torsion parts of two finitely generated abelian groups are geometrically equivalent. Then H_1 and H_2 are geometrically equivalent as a direct sums of the geometrically equivalent groups. $\square$

The proof for arbitrary abelian groups is similar. In order to avoid technical details we refer to the paper [Vino], where the following general result is proved.

Theorem 2.2.63. *Let $p_1, p_2, p_3, \ldots$ be an enumeration of the set of all primes. With a quasivariety of abelian groups $\mathfrak{X}$ associate an infinite sequence $(\alpha_0, \alpha_1, \alpha_2, \ldots)$, defined as follows: $\alpha_0 = 1$ if $\mathfrak{X}$ contains the infinite cyclic group, and $\alpha_0 = 0$ otherwise; for $n > 0$, $\alpha_n = m$ if $\mathfrak{X}$ contains the cyclic group of order p_n^m, but not the cyclic group of order $p_n^{(m+1)}$, and $\alpha_n = 1$ if $\mathfrak{X}$ contains the cyclic group of order p_n^m for all m. Two quasivarieties are equal if and only if their associated sequences are identical.* $\square$

Let us compare conditions providing geometric equivalence of abelian groups with the ones making abelian groups elementary equivalent. The classical theorem of W.Szmielew [Sz], see also [Ek], [EkF], [Hod], classifies abelian groups up to elementary equivalence. Namely, distinguish the following kinds of abelian groups:

- $Z(p^k)$ denotes the cyclic group of order p^k,
- $Z(p^\infty)$ denotes the locally cyclic (i.e., every finite set of elements generates a cyclic group) p-group. This group can be represented as the direct limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$.
- $Z_{(p)}$ denotes the localisation of the integers $\mathbb{Z}$ at the prime p, i.e., the additive group of all rational numbers with denominator not divisible by p.
- $\mathbb{Q}$ denotes the additive group of all rational numbers.

Then,

Theorem 2.2.64 ([Sz]). *Any abelian group A is elementarily equivalent to a group of the form*

$$\oplus_p[\oplus_n Z(p^n)^{(\alpha_{p,n})} \oplus Z_{(p)}^{(\beta_p)} \oplus Z(p^\infty)^{(\gamma_p)}] \oplus \mathbb{Q}^{(\delta)},$$

where $\alpha_{p,n}, \beta_p, \gamma_p, \delta$ are finite or countable. $\square$

Groups defined in Theorem 2.2.64 are called the Szmielew groups. So, the elementary equivalence of abelian groups can be recognized by comparison with an appropriate Szmielew group. This observation gives rise to the Szmielew invariants: $Exp(p,n,A)$, $U(p,n,A)$, $D(p,n,A)$, $Tf(p,n,A)$, where Exp originates from "exponent", U from "Ulm", D from "divisible", and Tf from "torsion free" (see [Ek], [Hod] for the precise definitions). Each of them is responsible for the elementary equivalence of the abelian group in question with the appropriate Szmielew's group.

Geometric equivalence of abelian groups is recognized using a part of Szmielew invariants and is a much weaker condition than elementary equivalence. This distinction between geometric and elementary equivalence modeled on abelian groups allows us to visualize what is the part of quasi-identities inside the whole elementary theory of an abelian group.

For the varieties of nilpotent and solvable groups the situation with elementary equivalence and geometric equivalence is much more complicated. The elementary classification of nilpotent groups is surveyed in [MS1][MS2]. In full generality this problem for finitely generated nilpotent groups is open. Despite that, there exists an advanced theory on this subject, which is especially developed for free finitely generated groups (see [MS2], [Be], [MR1]-[MR3], [Og], [Mys1], [Mys2], etc.). Geometric equivalence of nilpotent groups is studied in [Ts2], [Ts2], [BG]. Elementary equivalence of free solvable groups is considered in [RSS], see also [Ch1],[Ch].

2.2.7 Geometric equivalence and correct isomorphism

Definition 2.2.65. *Let $K_\Theta(H_1)$ and $K_\Theta(H_2)$ be categories of algebraic sets. An isomorphism φ of these categories is called correct if it induces an isomorphism of the lattices of algebraic sets $Alv_{H_1}(W)$ and $Alv_{H_2}(W)$.*

We view the existence of this strong version of isomorphism of categories of algebraic sets as the sameness of equational geometries over the algebras H_1 and H_2.

For every algebra $H \in \Theta$, consider a (contravariant) functor $Cl_H : \Theta^0 \to Set$. If $W = W(X)$ is an object of Θ^0, then $Cl_H(W)$ is the set of all H-closed congruences T in W. If, further, $s : W(Y) \to W(X)$ is a morphism of Θ^0, then the mapping of sets $Cl_H(s) : Cl_H(W(X)) \to Cl_H(W(Y))$ is defined by the rule: if T is an H-closed congruence in $W(X)$, then $Cl_H(s)(T) = s^{-1}T$. It is always an H-closed congruence in $W(Y)$. Here, $w(s^{-1}T)w'$ if $w^sTw'^s$.

Let φ be an automorphism of Θ^0. Consider the commutative diagram

$$\begin{tikzcd} \Theta^0 \arrow[rr, "\varphi"] \arrow[dr, "Cl_{H_1}"'] & & \Theta^0 \arrow[dl, "Cl_{H_2}"] \\ & PoSet & \end{tikzcd}$$

Commutativity of this diagram means that there exists an isomorphism of functors Cl_{H_1} and $Cl_{H_2} \cdot \varphi$. Denote this isomorphism by $\alpha(\varphi)$.

Definition 2.2.66. *Let H_1 and H_2 be algebras in Θ. Algebras H_1 and H_2 are called geometrically similar if*

1. *There exists an automorphism $\varphi : \Theta^0 \to \Theta^0$, such that:*
2. *The functors Cl_{H_1} and $Cl_{H_2}\varphi$ are isomorphic through the isomorphism $\alpha(\varphi)$ depending on φ.*

Given variety Θ and the category Θ^0, consider a function β which assigns to every congruence T in W_2 a binary relation $\beta = \beta_{W_1,W_2}(T)$ in $Hom(W_1, W_2)$ defined as follows: $s_1 \beta s_2$ holds for $s_1, s_2 : W_1 \to W_2$ if and only if $w^{s_1} T w^{s_2}$ for every $w \in W_1$.

The isomorphism condition yields that given an automorphism φ there exists a function

$$\alpha(\varphi) : Cl_{H_1} \to Cl_{H_2} \cdot \varphi,$$

with the following properties:

1. To every $W = W(X) \in Ob\, \Theta^0$ it corresponds the bijection
$$\alpha(\varphi)_W : Cl_{H_1}(W) \to Cl_{H_2}(\varphi(W))$$
2. The function $\alpha(\varphi)$ is compatible (in the sense of natural transformation of functors) with the automorphism φ.

The last condition means that

$$\varphi(\beta_{W_1,W_2}(T)) = \beta_{\varphi(W_1),\varphi(W_2)}(\alpha(\varphi)_{W_2}(T)).$$

Here W_1, W_2 are objects in Θ_1^0, T is an H_1-closed congruence in W_2, and for every relation ρ in $Hom(W_1, W_2)$ the relation $\varphi(\rho)$ is defined by the rule: $s_1' \varphi(\rho) s_2'$ holds for $s_1', s_2' : \varphi(W_1) \to \varphi(W_2)$ if there are $s_1, s_2 : W_1 \to W_2$ such that $\varphi(s_1) = s_1'$, $\varphi(s_2) = s_2'$ and $s_1 \rho s_2$.

We say that the automorphism φ determines similarity of algebras. Properties of this φ determine properties of similarity.

For the identical φ geometrical equivalence and geometrical similarity coincides, since in this case $\alpha(\varphi)$ yields the equality $Cl_{H_1} = Cl_{H_2}$.

Theorem 2.2.67 ([Pl-VarAlg-AlgVar-Categ]). *Categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if the algebras H_1 and H_2 are geometrically similar.*

Proof. Part II, ??? □

Thus, geometric similarity of algebras provides coincidence of the geometries over these algebras.

However, we know that there exists an example of isomorphic categories of algebraic sets $K_\Theta(H_1)$ and $K_\Theta(H_2)$ with non-geometrically equivalent algebras H_1 and H_2 (see Example 2.2.46).

The following theorem reveals the role of an inner automorphism with respect to geometrical equivalence and geometrical similarity of algebras:

Theorem 2.2.68 ([Pl-St],[Pl-VarAlg-AlgVar-Categ]). *Algebras H_1 and H_2 are geometrically equivalent if and only if:*

1. *They are geometrically similar.*
2. *The automorphism φ of the category Θ^0 is inner.*

Proof. Part II, ??? □

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Corollary 2.2.69. *If every automorphism of the category Θ^0 is inner, then geometrically similar algebras are geometrically equivalent and vice versa.*

Our next aim is to find out how Theorems 2.2.67 and 2.2.68 look for specific varieties.

1. Variety $\Theta = Grp$. Every automorphism φ of Θ^0 is inner (see [Pl-VarAlg]). Let $Var(H_1) = Var(H_2) = Grp$. The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if the algebras H_1 and H_2 are geometrically equivalent .

Let algebra H belong to $\Theta = Com-P$, or $Ass-P$ or $Lie-P$ and $\sigma \in Aut(P)$. Define a new algebra H^σ. In H^σ the multiplication on a scalar $\circ$ is defined through the multiplication in H by the rule:

$$\lambda \circ a = \lambda^\sigma \cdot a, \quad \lambda \in P, \quad a \in H.$$

2. Variety $\Theta = Com - P$. Let $\Theta = Com - P$ with P infinite.

Theorem 2.2.70 ([BPP],[Pl-St],[Pl-IJAC]). *Let H_1 and H_2 be algebras from $\Theta = Com-P$. The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if for some $\sigma \in Aut(P)$ the algebras H_1^σ and H_2 are geometrically equivalent.*

3. Variety $\Theta = Ass-P$**.** Let $\Theta = Ass-P$, and $H \in \Theta$. Denote by H^* the algebra with the multiplication $*$ defined as follows: $a*b = b \cdot a$. The algebra H^* is called opposite to H.

Theorem 2.2.71 ([Pl-St],[KBL], [Pl-IJAC],[Ber2])**.** *Let H_1 and H_2 be algebras from $\Theta = Ass - P$ such that $Var(H_1) = Var(H_2) = \Theta$. The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if for some $\sigma \in Aut(P)$ the algebras H_1^* and H_2 are geometrically equivalent, where $(H_1^\sigma)^*$ is opposite to either H_1 or to H_1^*.*

4. Variety $\Theta = Lee - P$**.** Let $\Theta = Lee - P$ with P infinite.

Theorem 2.2.72 ([MPP],[Pl1], [Pl-St])**.** *Let H_1 and H_2 be algebras from $\Theta = Lee - P$ such that $Var(H_1) = Var(H_2) = \Theta$. The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if for some $\sigma \in Aut(P)$ the algebras (H_1^σ) and H_2 are geometrically equivalent.*

5. Variety $\Theta = Mod-K$**.** Let $\Theta = Mod-K$, where K is a ring, not necessarily commutative, but with IBN property. This means that if KX and KY are free K-modules with the finite X and Y, then they are isomorphic if and only their cardinalities coincide, i.e., $|X| = |Y|$. In particular, K can be a group algebra PG of the group G or the universal enveloping algebra $U(L)$ of the Lie algebra L over the field P.

For a given K-module H take its annihilator U in K. Consider an ideal V such that there is an isomorphism $\tau : K/U \to K/V$. If V coincides with U, then τ is an automorphism of K/U. The K-module H we can consider as a K/U-module and, using τ, as a K/V-module. This K/V-module can be lifted to a K-module. Denote it by H^τ. The ideal V is the annihilator of H^τ.

Theorem 2.2.73 ([Pl-AG-Mod], [Pl-IJAC])**.** *The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if for some τ the modules H_1^τ and H_2 are geometrically equivalent.*

The similar results are valid for the varieties of semigroups, inverse semigroups, for Grp^F variety, where F is a finitely generated free group, playing the role of constants, and for some other varieties.

2.2.8 Noetherian properties

Notherianity plays a crucial role in many problems related to universal algebraic geometry. To the contrary with the case of classical algebraic geometry, where Hillbert's basis theorem provides the

Noetherianity "for free", there is no reason to think that in arbitrary variety Θ and H in Θ the topological space $Hom(W(X), H)$ is Noetherian. So, as soon as $Hom(W(X), H)$ possesses a kind of the Noetherianity conditions, there are solid grounds to look for a rich geometric theory.

Definition 2.2.74. *A commutative ring R is Noetherian if it satisfies the ascending chain condition on ideals, i.e., given a chain of ideals:*

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq \ J_k \subseteq J_{k+1} \subseteq \cdots$$

there exists a positive integer n such that: $J_n = J_{n+1} = \ldots$.

Equivalently, a ring R is Noetherian if all its ideals admit a finite basis.

If a ring R is not commutative, then replacing ideals in Definition 2.2.74 by left- (right-) side ideals we come up with the notions of left (right) Noetherian rings. Usually, a non-commutative ring is called Noetherian, if it satisfies ascending chain condition with respect to left- and right-side ideals.

Definition 2.2.75. *An algebra $H \in \Theta$ is Noetherian if it satisfies the ascending chain condition on congruences.*

Hilbert basis theorem (see Theorem 2.1.3) states that if R is a Noetherian ring then the polynomial ring $R[x_1, \ldots, x_n]$ is also Noetherian. In particular, $K[x_1, \ldots, x_n]$, where K is a field, is Noetherian. Hence, every its ideal is finitely generated.

Definition 2.1.11 of Noetherian topological spaces admits a useful reformulation.

Definition 2.2.76. *A topological space X is Noetherian if every ascending chain of open subsets of X has a maximal element. Equivalently, the space X is Noetherian if every descending chain of closed subsets have a minimal element.*

It is well-known that,

Proposition 2.2.77. *Let X be a Noetherian topological space. Every non-empty closed subset Y of X can be represented as a finite union $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_s$, where each Y_i is irreducible. If $Y_i \not\subset Y_j$ for $i \neq j$ then this decomposition is unique up to a permutation of components. Each Y_i is called an irreducible component of Y. A Noetherian topological space X has only a finite number of distinct irreducible components $X_1, X_2, \ldots, X_n$.* $\square$

If a finitely generated free algebra $W(X) \in \Theta$ is Noetherian with respect to H-closed congruences, then the affine space $Hom(W(X), H)$ equipped with Zariski topology is Noetherian.

Definition 2.2.78. *An algebra $H \in \Theta$ is called geometrically Noetherian if for every finite set X and every system of equations T in $W(X)$, there exists a finite subsystem $T_0 \subset T$, such that*

$$T_H'' = (T_0)_H''.$$

In this case we say that T and T_0 are equivalent systems of equations. So, if the algebra is geometrically Noetherian, then any system of equations is equivalent to a finite subsystem.

Remark 2.2.79. *In many papers devoted to universal algebraic geometry geometrically Noetherian algebras are called equationally Noetherian algebras. These terms are synonyms.*

Proposition 2.2.80. *An algebra $H \in \Theta$ is geometrically Noetherian if and only if for every free algebra $W(X) \in \Theta$ the ascending chain condition for H-closed congruences holds.* $\square$

Proof. Let $H \in \Theta$ be geometrically Noetherian. Suppose that U_α, $\alpha \in I$ is an ascending chain of H-closed congruences. Take $U = \bigcup_{\alpha \in I} U_\alpha$. $\square$

Proposition 2.2.81. *An algebra $H \in \Theta$ is geometrically Noetherian if and only if for every free algebra $W(X) \in \Theta$ the lattices $Cl_H(W)$ of H-closed congruences and $Alv_H(W)$ of algebraic sets satisfy ascending and descending chain conditions, respectively.*

Proof. Because of duality, it is enough to prove the fact for the lattice of algebraic sets $Alv_H(W)$. Let $H \in \Theta$ be geometrically Noetherian. Suppose that

$$\ldots \subset A_i \subset \ldots \subset A_2 \subset A_1$$

is a descending chain of algebraic sets $A_i = T_i'$ in $Hom(W(X), H)$. It corresponds the ascending chain of H-closed congruences

$$T_1'' \subset T_2'' \subset \ldots \subset T_i'' \subset \ldots,$$

where $T_i'' = A_i'$. Take $T = \bigcup_i T_i''$. There exists a finite $T_0 \subset T$ such that $T'' = T_0''$. Since $T_0 \subset T$, then $T_0' \supseteq A_k = T_k'$, for some k. We have

$$A_k \supseteq A = \bigcap_i A_i = T' = T_0' \supseteq A_k.$$

Thus, $A = A_k$ and an ascending chain of algebraic sets satisfies the ascending chain condition.

Conversely, let algebraic sets in $Hom(W(X,)H)$ satisfy descending chain condition. Then H-closed congruences satisfy ascending chain conditions. Denote the set of all H-closed congruences by M. Then any subset of M has a maximal element.

Denote by M_0 the subset of all H-closed congruences T such that $T \neq T_0''$ for every finite T_0. Suppose M_0 is not empty. Let $\overline{T}$ be a maximal element in M_0. Take any finite S outside M_0. Denote $T_1 = \overline{T} \bigcup S$. Then $T_1'' = \overline{T}'' \bigcup S''$. Since $\overline{T} = \overline{T}'' \neq T_0''$ for any finite T_0 and S is finite, then $T_1'' = \overline{T} \bigcup S''$ belongs to M_0. Contradiction with maximality of $\overline{T}$. Hence M_0 is empty.

□

Proposition 2.2.81 implies,

Proposition 2.2.82. *The Zariski topology in $Hom(W(X), H)$ is Noetherian if and only if H is geometrically Noetherian.* □

In view of Proposition 2.2.77, the latter means that the following theorem holds.

Theorem 2.2.83. *Let H be a geometrically Noetherian algebra. Then any algebraic set A in $Hom(W(X), H)$ is a finite union of irreducible algebraic sets $A_1, \ldots, A_n$, i.e., $A = A_1 \cup \ldots \cup A_n$. If $A_i \not\subset A_j$ for $i \neq j$, then this decomposition is unique up to a permutation of components.* □

Thus, for geometrically Noetherian algebras most of the problems can be reduced to the case of coordinate algebras corresponding to irreducible components.

Definition 2.2.84. *An algebra G is called fully residually H (or H discriminates G) if for every finite set of elements G_0 in G, there exists a homomorphism $\varphi : G \to H$ such that the restriction φ to G_0 is injective.*

See [DMR1]-[DMR4], for fully residual properties of algebras.

Proposition 2.2.85 ([DMR1], [DMR2])**.** *Let T be a congruence in $W(X)$. Algebra $W(X)/T \in \Theta$ is a coordinate algebra of an irreducible algebraic set A in $Hom(W(X), H)$ if and only if $W(X)/T$ is a fully residually H algebra.* □

The class of geometrically Noetherian algebras is rather wide.

Definition 2.2.86. *We call a variety Θ geometrically Noetherian if every finitely generated free algebra $W(X)$ in Θ is geometrically Noetherian.*

If Θ is a Noetherian variety then every algebra $H \in \Theta$ is geometrically Noetherian.

Example 2.2.87.

1) A classical variety $Com - K$ is geometrically Noetherian (Theorem 2.1.6).

2) The variety $\mathfrak{N}_c$ of all nilpotent groups of the nilpotency class c is geometrically Noetherian (Example 2.2.89).

3) Every variety consisting of locally finite groups is geometrically Noetherian.

4) A variety of the form $\mathfrak{N}_c\Theta$, where Θ is a locally finite variety, is geometrically Noetherian.

5) Finitely-dimensional associative and Lie algebras are geometrically Noetherian.

!!!
Perepisat'???Ostavit'???

Warning 2.2.88. *One should be careful with the choice of the variety Θ in the definition of geometrically Noetherian algebras. If we want to know wether a system of equations T with the coefficients from the given algebra $G \in \Theta$ is equivalent to a finite subsystem $T_0 \subset T$, then we should consider the variety Θ^G of G-algebras instead of the variety Θ. If we consider coefficient-free equations, then we work inside the variety Θ. For example:*

Example 2.2.89. Let $\Theta = \mathfrak{N}_c$ be the variety all nilpotent groups of the nilpotency class c. Then every group G in Θ is geometrically Noetherian (cf. Example 2.2.87). Indeed, the free finitely generated nilpotent group $W(X)$ satisfies the ascending chain condition for subgroups (see [Ku2]). In particular, it satisfies this condition for G-closed normal subgroups. Then, by Proposition 2.2.80 the group G is geometrically Noetherian in $\mathfrak{N}_c$, regardless G is finitely or infinitely generated.

Now choose an infinitely generated nilpotent group $G \in \Theta$, and consider the variety Θ^G of G-groups. Any free G-group in Θ^G has the form of free product $W(X) = G * W_0(X)$, where $W_0(X)$ is the free finitely generated group from Θ. Then there is no reason for $W(X)$ to be Noetherian with respect to G-closed normal subgroups. Correspondingly, there is no reason for G to be geometrically Noetherian in the variety Θ^G. This observation is confirmed by the example in [GuR], where an infinitely generated nilpotent group G of class 2 which is not geometrically Noetherian in the variety Θ^G, is constructed.

Note that if G-algebra H is geometrically Noetherian in the variety Θ^G, i.e., in the variety of G-algebras, then H is geometrically Noetherian in Θ. For $\Theta = Grp$ the following converse statement holds:

Proposition 2.2.90 ([MR])**.** *Let H be a G-group from Θ^G. If G is finitely generated and H is geometrically Noetherian in Θ then H is geometrically Noetherian in Θ^G.* $\square$

Another main source for obtaining geometrically Noetherian algebras is linearity. The following theorem plays an exceptional role. Let $\Theta = Grp$ be the variety of groups, G an arbitrary group.

Theorem 2.2.91 ([Gu],[Br])**.** *Let G be a linear group over a commutative Noetherian ring with unity. Then G is geometrically Noetherian (in the variety Grp^G, and, thus, in Grp).* $\square$

Corollary 2.2.92. *Free groups, polycyclic groups [Au], finitely generated metabelian groups [Re1], finitely generated nilpotent groups, free nilpotent or free metabelian groups [W], are geometrically Noetherian.* $\square$

However, there are lots of non-linear geometrically Noetherian groups.

Example 2.2.93. The following groups G are geometrically Noetherian in Grp^G:

- abelian groups [BMR],
- free solvable groups [GuR],
- rigid groups [Ro2],
- torsion free hyperbolic groups [Se7].

The class of geometrically Noetherian G-algebras is closed under taking subalgebras, finite direct products, direct powers, utrapowers (see [BMR],[DMR2], [Pl-7L] for details).

Moreover, the class of geometrically Noetherian G-groups is closed under free products:

Theorem 2.2.94 ([Se9])**.** *Let A, B be geometrically Noetherian groups. Then the free product $G = A*B$ is geometrically Noetherian in Grp^G.* $\square$

This theorem provides additional possibilities for constructing geometrically Noetherian groups.

There are several ways to construct non-geometrically Noetherian algebras. In particular:

Example 2.2.95 ([BMRo])**.** The wreath product G of any non-abelian group and any infinite group is not a geometrically Noetherian group in Grp^G.

Example 2.2.96 ([BMR])**.** Let $BS_{m,n}$ be a Baumslag-Solitar group:

$$BS_{m,n} =< a, t \mid t^{-1}a^m t = a^n >, \ (m, n > 0).$$

The group $BS_{m,n}$ is geometrically Noetherian if and only if $m = 1$, or $n = 1$, or $m = n$.

For the details of these and other examples see [BMR], [BMRo], [MR], [GS], [LP], etc.

The general account of properties of geometrically Noetherian algebras is formulated in [DMR2] in terms of the so-called Unification Theorems.

It turns out that the notion of geometrical Noetherianity is redundant for many purposes. Our next aim is to weaken it preserving most of geometrical applications.

First of all reformulate Definition 2.2.78. It is equivalent to the following one.

Definition 2.2.97. *An algebra $H \in \Theta$ is geometrically Noetherian if for every free algebra $W(X)$ and every set of equations T in $W(X)$ there exists a finite subset T_0 in T, such that every $(w_0, w_0') \in T''_H$ belongs to $(T_0)''_H$.*

In terms of quasi-identities this means that an algebra $H \in \Theta$ is geometrically Noetherian if and only if for every $W(X)$ and T in $W(X)$ there exists a finite subset $T_0 \subset T$ such that the quasi-identity

$$\Big(\bigwedge_{(w,w')\in T} (w \equiv w')\Big) \to w_0 \equiv w_0'$$

holds in H if and only if the quasi-identity

$$\Big(\bigwedge_{(w,w')\in T_0} (w \equiv w')\Big) \to w_0 \equiv w_0'$$

holds in H. Here T_0 is *independent* from (w_0, w_0').

In case when T_0 depends on (w_0, w_0') we call H *weakly geometrically Noetherian.* Thus,

Definition 2.2.98. *An algebra $H \in \Theta$ is called weakly geometrically Noetherian if for every free algebra $W(X)$, every set of equations T in $W(X)$ and for every pair $(w_0, w_0') \in T''_H$ there exists a finite subset T_0 in T, depending, generally, on (w_0, w_0'), such that $(w_0, w_0') \in (T_0)''_H$.*

Remark 2.2.99. *In many papers devoted to universal algebraic geometry (see [MR], [DMR3], and others) weakly geometrically Noetherian algebras are called $(q)_w$-compact algebras. These terms are synonyms.*

The class of weakly geometrically Noetherian algebras possesses many important properties. The next proposition follows from Definition 2.2.98:

Proposition 2.2.100. *The algebra $H \in \Theta$ is weakly geometrically Noetherian if every infinitary quasi-identity in H is reduced in H to a finite quasi-identity.* $\square$

Moreover,

Proposition 2.2.101. *The algebra H is weakly geometrically Noetherian if and only if the union of any directed system of H-closed congruences is also an H-closed congruence for every $W(X)$.* $\square$

Proof. Let the algebra H be logically Noetherian and T a union of some directed system of H-closed congruences T_α, $\alpha \in I$. T is a congruence. We need to check that it is H-closed.

Take T''_H and let it contain the pair (w, w'). Find a finite subset T_0 in T with $(w, w') \in T''_{0H}$. We have T_α with $T_0 \subset T_\alpha$. Then $(w, w') \in T''_{0H} \subset T''_{\alpha H} = T_\alpha \subset T$. Thus, $(w, w') \in T$, $T = T''_H$.

To prove the opposite, assume the condition of directed systems of H-closed congruences.

Take an infinite set T in W. Consider in T all possible finite subsets T_α. All $T''_{\alpha H}$ constitute a directed system of H-closed congruences. Let T_1 be the union of all congruences of this system. $T \subset T_1 \subset T''_H$. Since T_1 is H-closed, then $T_1 = T''_H$. If $(w, w') \in T''_H = T_1$, then $(w, w') \in T''_{\alpha H}$ for some α. This means that the algebra H is logically Noetherian. $\square$

Here the set of congruences is directed with respect to the embedding relation. It is clear that geometrical Noetherianity of algebras implies their logical Noetherianity. Show that the opposite is not true for the case of groups. Consider a free group $F = F(X)$, where X is finite and consider all invariant subgroups U in F. Denote by H the discrete direct product (Example 1.2.26) of all $F(X)/U$. We have injections $F(X)/U \to H$. Therefore, all invariant subgroups in $F(X)$ are H-closed. From this it follows that the group H is not geometrically Noetherian. However, it is logically Noetherian by Proposition 2.2.101.

Theorem 2.2.102 ([MR])**.** *The equality $LSC(H) = qVar(H)$ holds if and only if the algebra H is weakly geometrically Noetherian.* □

Remark 2.2.103. *In fact, Theorem 2.2.102 solves the following Malcev-type problem: for which groups H the class of finitely generated groups from the prevariety $pVar(H)$ coincides with the class of finitely generated groups from the quasivariety $qVar(H)$. Malcev showed [Mal1] that for a given class of groups $\mathfrak{X}$ the prevariety $pVar(\mathfrak{X})$ is an axiomatizable class if and only if $pVar(\mathfrak{X}) = qVar(\mathfrak{X})$. So, he asked what are the classes $\mathfrak{X}$ such that $pVar(\mathfrak{X}) = qVar(\mathfrak{X})$ (see [Gor], [MR] for the solution and details). It remains to note Theorem B1 of [MR] which states that the class of finitely generated groups from the prevariety $pVar(H)$ coincides with the class of finitely generated groups from the quasivariety $qVar(H)$ if and only if H is weakly geometrically Noetherian, that is if $LSC(H) = qVar(H)$.*

According to definitions, inside the class of weakly geometrically Noetherian algebras the geometric equivalence of algebras means the coincidence of their quasi-identities:

Theorem 2.2.104. *Weakly geometrically Noetherian algebras H_1 and H_2 are geometrically equivalent, if and only if they have the same quasi-identities, that is*

$$qVar(H_1) = qVar(H_2). \qquad \square$$

Moreover,

Theorem 2.2.105 ([MR])**.** *Let H be a weakly geometrically Noetherian algebra. Then any two algebras from $qVar(H)$ are geometrically equivalent. If any two algebras from $qVar(H)$ are geometrically equivalent then H is weakly geometrically Noetherian.* □

The class of weakly geometrically Noetherian algebras is rather wide and includes, in particular, all geometrically Noetherian algebras. Hence, linear groups and all G-groups from the example 2.2.93 are weakly geometrically Noetherian. In fact,

G-equationally Noetherian ???

Theorem 2.2.106 ([MR],[DMR3])**.** *If H is a weakly geometrically Noetherian algebra then every algebra in $qVar(H)$ is weakly geometrically Noetherian.* □

At the same time, there are groups and algebras which are not weakly geometrically Noetherian [MR], [GS], [LP], [BG], [BMRo]. For instance:

Example 2.2.107 ([MR]). A nilpotent group G of class 2 given by the presentation (in the variety of class ≤ 2 nilpotent groups)

$$G = \langle a_i, b_i,\ i \in \mathbb{N} \ \mid \ [a_i, a_j] = 1,\ [b_i, b_j] = 1,\ [a_i, b_j] = 1, i \neq j \rangle,$$

is not weakly geometrically Noetherian. Indeed, the infinitary quasi-identity

$$\forall x \forall y (\bigwedge_{i \in \mathbb{N}} ([x, a_i] = 1 \bigwedge_{j \in \mathbb{N}} [x, b_j] = 1) \to [x, y] = 1)$$

holds in G, but for any finite subsets I and J of $\mathbb{N}$ the following quasi-identity

$$\forall x \forall y (\bigwedge_{i \in I} ([x, a_i] = 1 \bigwedge_{j \in J} [x, b_j] = 1) \to [x, y] = 1)$$

does not hold in G. To see this take an element $x = a_m$, such that $m \notin I \cup J$. All x of such kind commute with a_i, $i \in I$ and b_j, $j \in J$, but not central in G. The constructed group G is infinitely generated.

There are also examples of finitely generated not weakly geometrically Noetherian groups ([MR],[BMRo]). Moreover,

Theorem 2.2.108. *Let the finitely generated group $H_1 \in \Theta$ be not weakly geometrically Noetherian. Then there exists a group H_2 such that*

$$qVar(H_1) = qVar(H_2),$$

but H_1 and H_2 are not geometrically equivalent. $\square$

Let the algebra $H_1 \in \Theta$ be not weakly geometrically Noetherian.

At this point we stop a sketchy exposition of the basics of universal algebraic geometry. Our aim is to provide the reader with the facts, which reveal the passages from classical algebraic geometry to the universal one and hint the ways to extending universal algebraic geometry to logical geometry.

2.2.9 A table: classical and universal geometry

This self-explaining table visualizes relations between the parallel notions in classical and universal geometry.

Classical AG	**Universal AG**
Variety	
$Com - K$	Θ
Free algebra	
$K[X]$, $\|X\| = n$	$W(X)$, $\|X\| = n$
Elements of the free algebra	
$f(x_1, \dots, x_n) \in K[X]$	$w(x_1, \dots, x_n) \in W(X)$
Equations	
$f(x_1, \dots, x_n) \equiv 0$	$w \equiv w'$
Ground field	*Algebra in* Θ
K	H
Affine space	
$K^n \cong Hom(K[X], K)$	$H^n \cong Hom(W(X), H)$
Points	
$\mu = (a_1, \dots, a_n)$	$\mu = (a_1, \dots, a_n)$
$\mu \in Hom(K[X], K)$	$\mu \in Hom(W(X), H)$
Solutions	
$f(a_1, \dots, a_n) = 0$	$w(a_1, \dots, a_n) = w'(a_1, \dots, a_n)$
or	
$\mu(f) = 0$	$\mu(w) = \mu(w')$
μ is a solution of f	μ is a solution of $w_i \equiv w_j$
$\Leftrightarrow f \in Ker(\mu)$	$\Leftrightarrow (w_i, w_j) \in Ker(\mu)$
Galois correspondence	
ideal T	congruence T
$\Updownarrow$	$\Updownarrow$
algebraic set A	algebraic set A
Galois closed objects	
radical ideal $I(A)$	closed congruence A'_H
algebraic set $V(A)$	algebraic set T'_H
Topology	
Zariski topology	Zariski topology for geometrically stable algebras
Coordinate algebras	
coordinate ring	coordinate algebra
$K[X]/I(A)$	$W(X)/A'_H$
Category of algebraic sets	
$\mathcal{A}(K)$	$K_\Theta(H)$
Morphisms	
polynomial (regular) maps	

One can extend this table substantially, considering geometrically noetherian algebras and noetherian affine spaces.

2.2.10 Short bibliographical guide

The material included in Section 2 is a short introduction to universal algebraic geometry. Its choice is stipulated by the forthcoming needs of logical geometry. So, a lot of important staff remains untouched. In particular, we did not consider the notions of Krull dimension, domains, limit groups, algebraically closed algebras and many others. The themes dealing with irreducible components, separation and discrimination are only outlined. The deep questions related to geometries over specific algebras require a special attention and left also beyond the scopes of this section.

In order to make life of the interested reader easier we conclude with citations, which can help to navigate in the area of universal algebraic geometry until a special book on this subject will be published.

General principles and problems of universal algebraic geometry are illuminated and surveyed in [BMR], [BPP], [Dan1], [DMR1], [DMR2], [DMR3], [DMR4],[DMR5], [KMR], [Ko], [MPP1], [MR], [NP], [Pi1], [Pi2], [Pl-AG], [Pl-St], [Pl-7L], [Pl-IJAC], [Pl-VarAlg], [Pl-VarAlg-AlgVar-Categ], [Pl1], [Pl2], [Pl5], [Pl6], [Pl7], [PZ1]–[PZ3], [Sc]. All main notions of Section 2.2 can be found in one of these papers.

There is a huge list of works devoted to solving equations over free groups and, therefore, to algebraic geometry over a free group. This area was pioneered by the works [Ap], [Ly],[CE], [Stol],[Br],[Gu] followed by the seminal papers [Ma], [Razb1], [Razb2]. The modern geometry of free and hyperbolic groups grounds on algebraic-geometric-logic ideas proposed by V.Remeslennikov and E.Rips. The achieved results are exposed in [BGM], [CK3], [CR], [CG], [GriKu], [Gro1], [Gro2], [Gui], [KhM-1]- [KhM-6], [Pa], [PS], [RS], [Se1]–[Se6],[Se7], [Se8], [Se9], etc. This list is fairly incomplete.

Algebraic geometry over arbitrary G-groups is thoroughly depicted in the series of papers [BMR0] [BMR], [MR], [KMR]. Various concrete results are contained also in [BMRo], [Berzins-GeomEquiv], [BG], [GS], [MResS], [MS], [Pl3], [Pl-7L], [PPT], [Tsl],[Ts2], etc.

The algebraic geometry over free metabelian groups gives rise to a very consistent theory [Ch], [Re2], [Re3], [RemRo1], [RemRo2], [RemS1], [RemS2], [RemTi], [Ro1]. Geometry over solvable groups is treated in [GuR], [MR], [Ro1].

Universal algebraic geometry specified to varieties of non-commutative and non-associative algebras yield non-commutative and non-associative geometries which can be of utmost importance from the viewpoint of applications. To the contrary with the case of free groups, solutions of equations over free non-commutative algebras are much less understood. For the results see [CS], [Dan1], [Dan2], [DKR1]– [DKR3], [DR], [KLP],[LP], [MPP], [RemS3], [RoSh].

The algebraic geometry over partially commutative groups is studied in [CK1] – [CK3], [GuT], see also [MS], [Sh1]-[Sh3] for other algebraic structures.

We should emphasize once again that this bibliography does not pretend to be complete.

Chapter 3

Basics of Algebraic Logic and Model Theory

Algebraic logic goes throughout the book as a basic tool which makes all necessary considerations with logic and model theory as algebraic as possible. In this chapter we focus our attention on two logically-algebraic structures: Boolean algebras and polyadic algebras. Boolean algebras were introduced already in Section 1.1.3. Now we view Boolean algebras as an algebraic counter-part of the propositional calculus. In more appropriate terms Boolean algebras serve as an algebraization of the propositional calculus. Polyadic algebras are less known objects than Boolean algebras. These algebras naturally arise under the process of algebraization of first-order calculus.

For the aims of logical geometry we will need a multi-sorted variant of polyadic algebras specialized in a given variety of algebras. We call these algebras Halmos algebras. They will be defined in Chapter **??**. !!! Chapter 4

There are many detailed sources related to different parts of algebraic logic and model theory. We refer to the books [BarnesMack], [CKeis], [Halm], [HalGiv], [Hamilton], [HilbAcker], [Hod], [Pl-Datab], [Marker], [Mendelson], [Vereshchagin] for the proofs and complementary information.

3.1 Logical calculus. Syntax and semantics

3.1.1 Syntax of a logical calculus

We shall start with a formal syntactic description of a logical calculus.

Definition 3.1.1. *A logical calculus is a tuple* $\mathfrak{C} = (\mathbb{L}, \mathbb{F}, \mathbb{A}, \mathbb{D})$ *consisting of*

- *a language* $\mathbb{L}$*;*
- *a set* $\mathbb{F}$ *of finite words, constructed using the language* $\mathbb{L}$*, called formulas;*
- *a set* $\mathbb{A}$ *of particular formulas, called axioms;*
- *a finite set* $\mathbb{D}$ *of "derivation rules" which enable us to derive new formulas from axioms and other formulas.*

Each language assumes some stock of variables, which serve as an alphabet, and a number of rules which allow one to construct words from a given alphabet. Formalizing all this, under a language $\mathbb{L}$ we mean the following.

Definition 3.1.2. *A language* $\mathbb{L}$ *is given by specifying the following data.*

1. *A set of variables* $X = \{x_1, x_2, \dots, x_n, \dots\}$*. This set can be finite or infinite. The generic situation is an infinite* X*.*
2. *A set* $\mathcal{F}$ *of function symbols* f *given together with their arities* $n_f \geq 0$*.*
3. *A set* $\mathcal{R}$ *of relation symbols* r *given together with their arities* $n_r \geq 1$*. Relation symbols* $r \in \mathcal{R}$ *are also called predicate symbols.*
4. *A set* $\mathcal{C}$ *of constant symbols. These symbols are treated as function symbols of zero arity.*
5. *The symbols of logical connectives* $\neg$, $\vee$*.*
6. *The symbol of existential quantifier* $\exists$*.*
7. *The punctuation symbols "(", ")", ",".*

The sets $\mathcal{F}$, $\mathcal{R}$, $\mathcal{C}$ together with purely logical symbols $\neg$, $\vee$ and $\exists$ produce the *signature* of a logical calculus. The logical part of a signature is often suppressed in notation.

For some languages the sets $\mathcal{F}$, $\mathcal{R}$, $\mathcal{C}$ may be empty. One can consider a language without quantifiers. This is the case for the propositional calculus, where the language consists of only variables $x_1, x_2, \dots$, the connectives $\neg$, $\vee$ and the punctuation symbols.

Remark 3.1.3. *There is a lot of flexibility hidden in Definition 3.1.2. For example:*

1. *We include in the language the connectives* $\neg$ *and* $\vee$. *The connectives* $\wedge$, $\to$, $\leftrightarrow$ *can be produced using* $\neg$, $\vee$. *Namely,*

$$\begin{array}{rcl} (u_1 \wedge u_2) & \text{is an abbreviation for} & (\neg(\neg u_1 \vee \neg u_2)), \\ (u_1 \to u_2) & \text{is an abbreviation for} & (\neg u_1 \vee u_2), \\ (u_1 \leftrightarrow u_2) & \text{is an abbreviation for} & ((u_1 \to u_2) \wedge (u_2 \to u_1)). \end{array}$$

2. *One can consider a language with the other connectives, for example, with* $\neg$ *and* $\to$. *Then* $\wedge$, $\vee$, $\leftrightarrow$ *can be defined using* $\neg$ *and* $\to$.

3. *There is only the existential quantifier in the language. The universal quantifier* $\forall$ *can be defined in terms of the existential quantifier* $\exists$ *and the connective* $\neg$. *Namely,*

$$(\forall x_i u) \quad \text{is an abbreviation for} \quad (\neg(\exists x_i(\neg u))).$$

Now we need to define the set of formulas $\mathbb{F}$ of a logical calculus $\mathfrak{C}$.

Definition 3.1.4. *Terms in a language* $\mathbb{L}$ *are defined inductively:*

1. *variables are terms;*
2. *constant symbols of* $\mathbb{L}$ *are terms;*
3. *if* $t_1, \ldots, t_{n_f}$ *are terms and* f *is a function symbol of arity* n_f, *then* $f(t_1, \ldots, t_{n_f})$ *is a term;*
4. *there are no other terms.*

Definition 3.1.5. *An atomic formula is a formula of the form* $r(t_1, \ldots, t_{n_r})$, *where* r *is a relation symbol of arity* n_r *and* $t_1, \ldots, t_{n_r}$ *are terms.*

Definition 3.1.6. *Formulas in a language* $\mathbb{L}$ *are defined inductively:*

1. *atomic formulas are formulas;*
2. *if* u_1 *and* u_2 *are formulas, then* $\neg u_1$, $(u_1 \vee u_2)$ *are formulas;*
3. *if* u *is a formula, then* $\exists x_i u$ *is a formula, where* x_i *is a variable;*
4. *there are no other formulas.*

We assume unique readability of formulas using/not using punctuation symbols. We also assume familiarity with the concept of a free variable, that is one not bound by quantifiers. Bound variables are exactly variables which are not free in a formula. A *sentence* is a formula without free variables, that is each occurrence of a variable lies in the scope of some quantifier.

Our goal is only first-order logic which means that no predicate or function symbols can serve as variables, i.e., quantifiers over predicates are not permitted. Besides, all formulas are finite and only finitely many quantifiers appear in each formula.

We do not pretend to consider the whole world of the first-order logical calculi. Our aim is to have a clear picture of algebraization of a propositional calculus and predicate calculus. We confine ourselves with the Hilbert-style deductive systems which means that there is a bunch of schemes of axioms and few deductive laws. Alltogether they allow one to build syntactical consequences from some premises.

Definition 3.1.7. *A formula u is derivable from a set of formulas T if and only if there exists a finite sequence of formulas*

$$u_0, u_1, \dots, u_n = u,$$

whose last term u_n is u, such that u_0 either belongs to T or is an axiom, and every formula u_i, $1 \leq i \leq n$, is either an axiom, or an element of T, or the result of applying a derivation rule to some of preceding formulas in the sequence.

If u is derivable from T, we will write $T \vdash u$. If u is derivable from axioms we will write $\vdash u$. In the latter case we say that u is a *theorem* of the logical calculus.

Definition 3.1.8. *A set of formulas T is syntactically consistent if for any formula u, if $T \vdash u$, then $\neg u$ is not derivable from T.*

A set of sentences T is called *a theory*. The sentences of T play a role of axioms of the theory. Thus, $T \vdash u$ means that u is a theorem of the theory T. Theories are often assumed to be closed under consequences from axioms. A logical calculus is syntactically consistent if there is no formula u such that both u and $\neg u$ are theorems.

Let us mention some well-known theories.

Example 3.1.9.

1. **The theory of semigroups.** Let $\mathbb{L} = \{X, \cdot, =, \neg, \wedge, \rightarrow, \forall\}$ be a language, where X is a set of variables, "$\cdot$" is a binary function symbol and "$=$" is a binary relation symbol of equality. Axioms of the theory of semigroups look as follows:

   1.1 $\forall x_1 \; x_1 = x_1$, (reflexivity of "$=$");

   1.2 $\forall x_1 \forall x_2 \; (x_1 = x_2 \rightarrow x_2 = x_1)$, (symmetry of "$=$");

1.3 $\forall x_1 \forall x_2 \forall x_3 \ \big((x_1 = x_2 \wedge x_2 = x_3) \to x_1 = x_3\big)$, (transitivity of "="");

1.4 $\forall x_1 \forall x_2 \forall x_3 \ x_2 = x_3 \to (x_1 \cdot x_2 = x_1 \cdot x_3 \wedge x_2 \cdot x_1 = x_3 \cdot x_1)$, (substitutivity of "="");

1.5 $\forall x_1 \forall x_2 \forall x_3 \ x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$.

2. **The theory of inverse semigroups.** Let now $\mathbb{L} = \{X, \cdot, ^{-1}, =, \neg, \wedge, \to, \forall\}$ be the same language as before, where "$^{-1}$" is an additional unary function symbol. The theory of inverse semigroups consists of axioms 1.1–1.5 above and the following sentences (see, for example, [Klun], [PZ3]):

   2.1 $\forall x_1 \forall x_2 \ (x_1 \cdot x_2)^{-1} = x_2^{-1} \cdot x_1^{-1}$,

   2.2 $\forall x_1 \ (x_1^{-1})^{-1} = x_1$,

   2.3 $\forall x_1 \ x_1 \cdot x_1^{-1} \cdot x_1 = x_1$,

   2.4 $\forall x_1 \forall x_2 \ x_1^{-1} \cdot x_1 \cdot x_2^{-1} \cdot x_2 = x_2^{-1} \cdot x_2 \cdot x_1^{-1} \cdot x_1$.

3. **The theory of groups**. Let $\mathbb{L} = \{X, \cdot, ^{-1}, 1, =, \neg, \wedge, \to, \forall\}$. The theory of groups consists of sentences 1.1–1.5 above and the sentences:

   3.1 $\forall x_1 \ (x_1 \cdot 1 = x_1 \wedge 1 \cdot x_1 = x_1)$,

   3.2 $\forall x_1 \ (x_1 \cdot x_1^{-1} = 1 \wedge x_1^{-1} \cdot x_1 = 1)$.

3.1.2 Semantics of a logical calculus

Suppose we are given with some logical calculus. This means that we have some infinite set of finite words consisting of symbols of different kind. These are purely formal expressions which we intend to endow with some meaning. With this end one has to choose a set where all these words will be interpreted as expressions constructed on the base of elements of this set with the help of rules written in our logical calculus. So, we need to formalize a notion of interpretation of a logical calculus.

Definition 3.1.10. *An interpretation $\mathbb{M} = (A, \varphi)$ is a pair consisting of a non-empty set A, called the domain of the interpretation (variables $x_i \in X$ are thought of as ranging over A), and a realization φ which assigns:*

- *to each function symbol f (of arity n_f) an n_f-ary operation $f^{\mathbb{M}}$ on A (i.e., a function $f^{\mathbb{M}} : A^{n_f} \to A$);*
- *to each relation symbol r (of arity n_r) an n_r-ary relation $r^{\mathbb{M}}$ on A (i.e., a subset of A^{n_r});*

- *to each constant symbol c some fixed element $c^{\mathbb{M}}$ of A.*

As a rule we omit superscript $\mathbb{M}$ and write simply f instead of $f^{\mathbb{M}}$, etc.

An interpretation $\mathbb{M} = (A, \varphi)$ gives rise to the notion of *a model* of the logical calculus. Grounding more on algebra than on model theory we consider a model as a triple $\mathbb{M} = (A, \mathcal{R}, \varphi)$ where A is an algebra in some variety Θ, $\mathcal{R}$ is a set of symbols of relations and φ is a realization which makes symbols $r \in \mathcal{R}$ into relations in A^{n_r}. In this triple realization function symbols from $\mathcal{F}$ are hidden in the signature of operations related to the variety Θ. We use the same notation for an interpretation and for the associated model.

Now we shall define what means for a formula $u(x_1, \dots, x_n)$ to be valid on a tuple $\overline{a} = (a_1, \dots, a_n) \in A^n$ under the interpretation $\mathbb{M}$. In other words one has to define the value of the formula $u(x_1, \dots, x_n)$ at the point $\overline{a} = (a_1, \dots, a_n)$.

First, we interpret a term t built using variables from $x_1, \dots, x_n$ as a function $t : A^n \to A$ such that:

1. if t is a variable x_i, then $t(\overline{a}) = a_i$;

2. if t is a constant symbol c, then $t(\overline{a}) = c$;

3. if t is the term $f(t_1, \dots, t_{n_f})$, where f is a function symbol and $t_1, \dots, t_{n_f}$ are terms, then $t(\overline{a}) = f(t_1(\overline{a}), \dots, t_{n_f}(\overline{a}))$.

To define the value of a formula $u(x_1, \dots, x_n)$ on a tuple $(a_1, \dots, a_n)$ we should specify a function

$$u : A^n \to \mathbf{2}, \tag{3.1}$$

where $\mathbf{2}$ is a two-element set $\{0, 1\}$. The element 1 is treated as "truth", while 0 means "false". Define this function inductively and start from atomic formulas.

Let $r(t_1, \dots, t_{n_r})$ be an atomic formula. We say that $r(\overline{a}) = 1$ if and only if the tuple $(t_1(\overline{a}), \dots, t_{n_r}(\overline{a}))$ belongs to the relation $r^{\mathbb{M}} \subset A$, and say that $r(\overline{a}) = 0$ otherwise.

Let now $u(x_1, \dots, x_n)$ and $v(x_1, \dots, x_n)$ be arbitrary formulas, then

1. $\neg u(a_1, \dots, a_n) = 1$ if $u(a_1, \dots, a_n) = 0$.

2. $(u \vee v)(a_1, \dots, a_n) = 1$ if $u(a_1, \dots, a_n) = 1$ or $v(a_1, \dots, a_n) = 1$.

3. $\exists x_i u(a_1, \dots, a_n) = 1$ if there is a tuple $(b_1, \dots, b_n) \in A^n$ such that $a_j = b_j$ for all $j \neq i$, $1 \leq j \leq n$, and $u(b_1, \dots, b_n) = 1$.

If $u(a_1, \ldots, a_n) = 1$ we say that the formula $u(x_1, \ldots, x_n)$ is *satisfied* on the tuple $(a_1, \ldots, a_n)$. If for every tuple $(a_1, \ldots, a_n) \in A^n$ we have $u(a_1, \ldots, a_n) = 1$, then the formula $u(x_1, \ldots, x_n)$ is *true* under the interpretation $\mathbb{M}$, or, what is the same, the formula $u(x_1, \ldots, x_n)$ is satisfied on the model corresponding to $\mathbb{M}$. The corresponding notation is $\mathbb{M} \models u$.

Pick up now a set of formulas T. An interpretation $\mathbb{M} = (A, \varphi)$ gives rise to a model for T if every formula $u \in T$ is true under $\mathbb{M}$. Suppose we are given with a language $\mathbb{L}$ and a set of sentences T in $\mathbb{L}$, i.e., a theory T is given.

Definition 3.1.11. *A formula u is a semantical consequence of T, that is $T \models u$, if $\mathbb{M} \models u$ for every model $\mathbb{M}$ of T.*

One would like to have a coincidence of syntactical and semantical derivability for the class of first-order calculi. The following (Gödel's) theorems state:

Theorem 3.1.12. *Let T be a theory, u be a formula in a language. Then*

$$T \vdash u \text{ if and only if } T \models u. \qquad \square$$

Theorem 3.1.13. *A theory T has a model if and only if T is consistent.* $\square$

The next theorem is well-known as *the compactness theorem.*

Theorem 3.1.14. *If every finite subset of T is satisfiable then the set T is also satisfiable.* $\square$

This means that if every finite subset of T has a model, then T has a model.

Suppose now we have a model $\mathbb{M} = (A, \mathcal{R}, \varphi)$. The set of all sentences T valid on $\mathbb{M}$ is called *the theory (the elementary theory)* of $\mathbb{M}$. Usually one speaks just on the elementary theory of an algebra A.

3.1.3 Algebraization of a logical calculus

There are several ways to define an algebraization of a logical calculus, see, for example, discussions in [BP], [ANS], [FJP]. One of the key points of these studies is to find out which logical calculi can be algebraizable and in which sense.

Our goal is more utilitarian with the main destination to construct algebraizations of first-order calculi which will be most appropriate for the aims of logical geometry.

Intuitively, a process of algebraization of logical calculus means that we want to replace the study of a logical calculus by the study of a special algebra associated with this calculus. Moreover, syntactical and semantical properties of the calculus can be reformulated in terms of purely algebraic properties of the corresponding algebra. This idea was formalized by Tarski, who proved that Boolean algebras serve as the algebraization of propositional calculus [Ta], [Ta1]. Tarski's method works also for first-order calculi (see [Halm], [HMT], [Pl-Datab], etc.) and we will use it in what follows.

Let a logical calculus $\mathfrak{C} = (\mathbb{L}, \mathbb{F}, \mathbb{A}, \mathbb{D})$ be given. Define a relation τ on the set of formulas $\mathbb{F}$ by $u\tau v$ if and only if

$$\vdash (u \to v) \wedge (v \to u),$$

where $u, v \in \mathbb{F}$. In other words, two formulas u and v are claimed equivalent if each of them is derivable from the other. It is easy to see that τ is an equivalence relation on $\mathbb{F}$.

Analogously, if T is a theory then define a relation τ_T by $u\tau_T v$ if and only if

$$T \vdash (u \to v) \wedge (v \to u).$$

Denote by $\mathfrak{L}$ the absolutely free algebra constructed over atomic formulas of $\mathbb{L}$ in the signature of operations $\neg$, $\vee$, $\exists x$, where x is a variable. Then,

Proposition 3.1.15. *τ is a congruence of the algebra $\mathfrak{L}$.* $\square$

For the proof see Subsection 3.3.8.

Definition 3.1.16. *The quotient algebra $\mathfrak{L}/\tau$ is called the Lindenbaum-Tarski algebra of the logical calculus $\mathfrak{C}$.*

The described procedure is called *the Lindebaum-Tarski algebraization process*, the congruence τ is the Lindenbaum-Tarski congruence. Identities and structure of the Lindenbaum-Tarski algebra depend heavily on axioms and derivation rules of the logical calculus and can be quite complicated. However, in any case the Lindenbaum-Tarski algebra $\mathfrak{L}/\tau$ is a model of the initial logical calculus.

Example 3.1.17. The Lindenbaum-Tarski algebra of a propositional calculus is a free Boolean algebra (see Subsection 3.2.3).

Note that the Lindebaum-Tarski algebraization process in the form of Definition 3.1.16 has some disadvantages since the resulting algebra can have an unclear structure and identities. One of the ways to bypass this difficulty is to extend the signature $\neg$, $\vee$, $\exists x$

by additional operations. This is the case of algebraizations of first-order calculi and the reason for appearing in parallel polyadic, cylindric and other algebras that provide different algebraizations of a first-order calculus (see Subsection 3.3 for details).

From the geometric point of view the most important case of the algebraization process is Lindenbaum-Tarski algebras specialized in some variety of algebras Θ. Suppose that the language of a theory T contains the equality predicate $\equiv$ and, for simplicity, this is the only relational symbol. Fix a set of variables $X = \{x_1, \dots, x_n\}$. Formulas of the form $w \equiv w'$, where w and w' are terms, define a variety of algebras Θ. Denote by $W(X)$ the free algebra in the variety Θ. Then w and w' belong to $W(X)$. Let M_X be the set of all formulas $w \equiv w'$, $w, w' \in W(X)$, and $\mathfrak{L}_X$ the absolutely free algebra over the generators from M_X in the signature $\neg$, $\vee$, $\exists x$, $x \in X$. The Lindenbaum-Tarski algebra specialized in Θ is the algebra

$$\mathfrak{L}_X/\tau_T.$$

3.2 Propositional calculus and Boolean algebras

Propositional calculus plays an exceptional role among logical calculi. We will show that Boolean algebras are exactly the algebraic structures associated with the propositional calculus.

3.2.1 Propositional calculus. Syntax and semantics

A propositional calculus is a tuple $\mathfrak{C} = (\mathbb{L}, \mathbb{F}, \mathbb{A}, \mathbb{D})$, where

$\mathbb{L}$: The language $\mathbb{L}$ consists of an infinite set of variables $X = \{x_1, x_2, \dots\}$, the logical connectives $\neg$ and $\vee$, and the punctuation symbols.

$\mathbb{F}$: The set of formulas is constructed according to Definition 3.1.6 except for the item 3.

Since the language of a propositional calculus does not include quantifiers, all its formulas are sentences called *propositions*.

$\mathbb{A}$: The set of axioms consists of the following ones:

1. $x_1 \vee x_1 \to x_1$;
2. $x_1 \to x_1 \vee x_2$;
3. $x_1 \vee x_2 \to x_2 \vee x_1$;
4. $(x_1 \to x_2) \to ((x_3 \vee x_1) \to (x_3 \to x_2))$,

where x_1, x_2, x_3 are propositional variables.

There are many ways to choose the sets of axioms and derivation rules for a propositional calculus. The obtained propositional calculus should satisfy the conditions of consistency and completeness. A theory T is *complete* if for every sentence u either u or its negation $\neg u$ belongs to T. The chosen set of axioms $\mathbb{A}$ goes back to [HilbAcker] (see also [HalGiv]). One can describe the propositional calculus using other logical connectives, axioms and derivation rules (for details and examples see [BarnesMack], [HilbAcker], [Hamilton], [Mendelson], etc.)

$\mathbb{D}$: The set of derivation rules for this choice of axioms consists of two rules: substitution rule and modus ponens.

- *The substitution rule* allows to replace all occurrences of a given variable x in a sentence u by an arbitrary sentence v. We use the following notation: if u_1 and u_2 are sentences and x is a variable, then the sentence $u_1[x/u_2]$ is the sentence obtained from u_1 by replacing each occurrence of x in u_1 by u_2. For example,

$$(x_1 \to x_1 \vee x_2)[x_1/u] = u \to u \vee x_2.$$

- *The modus ponens rule* states that the sentences u_1 and $(u_1 \to u_2)$ imply u_2. In this rule u_1 plays the role of a premise and u_2 is a conclusion.

The substitution is not necessary if we consider *axiom schemata* instead of axioms, which means that axioms may be built on the base of arbitrary formulas. For example, if we replace in the axioms above propositional variables x_1, x_2, x_3 by arbitrary sentences u_1,u_2, and u_3, then we obtain a schemata of axioms for the propositional calculus.

It is also useful to view a substitution as a function $s : X \to \mathbb{F}$, which can be extended in a unique way up to $s : \mathbb{F} \to \mathbb{F}$. This s preserves logical connectives and parentheses. The substitution rule says that the set of sentences $\mathbb{F}$ is closed under substitutions.

The syntax of the propositional calculus possesses the necessary property.

Theorem 3.2.1. *The propositional calculus is syntactically consistent.* $\square$

According to the definition of the function 3.1, each sentence of the propositional calculus has the value 0 or 1. If the truth value of a formula u is true, then the truth value of its negation is false and vice versa. The truth value of a disjunction $u \vee v$ is true if and only if the truth value of at least one of u and v is true; otherwise, the truth value of $u \vee v$ is false. These rules determine the truth value of any sentence.

The procedure described above, yielding an interpretation of a propositional calculus in $\mathbf{2} = \{0, 1\}$, can be defined more algebraically through the evaluation map. Denote by $\mathfrak{L}_p(X)$ the set of all sentences of a propositional calculus.

Definition 3.2.2. *A value map*

$$f_{\mathfrak{L}_p(X)} : \mathfrak{L}_p(X) \to \{0, 1\}$$

associates with each sentence one of the two numbers 0 (false) and 1 (true) in such a way that

$$f_{\mathfrak{L}_p(X)}(u_1 \vee u_2) = max(f_{\mathfrak{L}_p(X)}(u_1), f_{\mathfrak{L}_p(X)}(u_2)),$$

and

$$f_{\mathfrak{L}_p}(\neg u_1) = 1 - f_{\mathfrak{L}_p(X)}(u_1),$$

where u_1, u_2 are arbitrary elements of $\mathfrak{L}_p(X)$.

Each value map can be defined using the map from the set of propositional variables X to the set $\{0, 1\}$. We will see that every map of such kind can be extended in a unique way up to the value homomorphism of Boolean algebras.

A sentence is called *a tautology* if its truth value is identically 1 regardless of an interpretation of variables from X. All axioms of a propositional calculus are tautologies. The modus ponens rule can be written as a tautology

$$(u_1 \wedge (u_1 \to u_2)) \to u_2,$$

where u_1, u_2 are sentences.

Theorems 3.1.12–3.1.14 have their counter-parts for the particular case of a propositional calculus. For instance,

Theorem 3.2.3. *A formula u of a propositional calculus is a theorem if and only if u is a tautology.* □

Theorem 3.2.4. *The set of sentences T is consistent if and only if T is satisfiable.* □

Theorem 3.2.5. *If every finite subset of T is satisfiable then the set T is also satisfiable.* □

3.2.2 Boolean algebras

Boolean algebras were already introduced in Subsection 1.1.3. In order to make exposition self-contained we repeat here some material from this section. Let us start with yet another system of axioms for a Boolean algebra. Here the notation $\vee$, $\wedge$ and $\neg$ is used instead of $+, \cdot, ^-$, respectively. Once again recall that we freely switch between the notation $\vee$, $\wedge$, $\neg$ and the compact notation $+$, $\cdot$, $^-$.

The following definition is equivalent to Definition 1.1.8.

Definition 3.2.6. *A Boolean algebra is a set B considered together with two binary operations $\vee$, $\wedge$ and one unary operation $\neg$. These operations satisfy the following rules:*

1. $b_1 \vee b_2 = b_2 \vee b_1, \quad b_1 \wedge b_2 = b_2 \wedge b_1$ *(commutative laws);*
2. $(b_1 \vee b_2) \vee b_3 = b_1 \vee (b_2 \vee b_3), \quad (b_1 \wedge b_2) \wedge b_3 = b_1 \wedge (b_2 \wedge b_3)$ *(associative laws);*
3. $b_1 \vee (b_1 \wedge b_2) = b_1, \quad b_1 \wedge (b_1 \vee b_2) = b_1$ *(absorption laws);*
4. $b_1 \wedge (b_2 \vee b_3) = (b_1 \wedge b_2) \vee (b_1 \wedge b_3), \quad b_1 \vee (b_2 \wedge b_3) = (b_1 \vee b_2) \wedge (b_1 \vee b_3)$ *(distributive laws);*
5. $(b_1 \vee (\neg b_1)) \wedge b_2 = b_2, \quad (b_1 \wedge (\neg b_1)) \vee b_2 = b_2,$

where b_1, b_2, b_3 are elements of B.

We can single out elements $0 = b_1 \wedge (\neg b_1)$ and $1 = b_1 \vee (\neg b_1)$. So,

$$b_1 \vee 0 = b_1, \quad b_1 \wedge 1 = b_1,$$

$$b_1 \vee 1 = 1, \quad b_1 \wedge 0 = 0.$$

The operation $\vee$ can be expressed in terms of the operations $\wedge$ and $\neg$, and the operation $\wedge$ can be expressed in terms of $\vee$ and $\neg$. Thus, the system of axioms above can be written in the signature of just two operations: $\vee$ and $\neg$ or $\wedge$ and $\neg$.

Define $a \to b$ to be the formula $\neg a \vee b$, $a, b \in B$. Then " $\to$ " can be viewed as a new operation on the Boolean algebra derived from the old ones. Since $a \vee b = \neg a \to b$ and $a \wedge b = \neg(a \to \neg b)$, any Boolean algebra can be considered also in the signature $(\neg, \to)$ with the corresponding system of axioms.

Recall that every Boolean algebra is a lattice. The order relation $\leq$ is defined on a Boolean algebra as follows: $b_1 \leq b_2$ if $b_1 \wedge b_2 = b_1$ (or, equivalently, if $b_1 \vee b_2 = b_2$).

A subset I of a Boolean algebra B is *an ideal* if for every $a_1, a_2 \in I$ and $b \in B$ we have $a_1 \vee a_2 \in I$ and $a_1 \wedge b \in I$. One can check that I is an ideal of B if and only if I is closed with respect to the operation $\vee$ and $a_1 \in I$ implies that $a_2 \in I$ for any $a_2 \leq a_1$.

The concept of a filter of a Boolean algebra is dual to the concept of an ideal. Namely, a subset F of a Boolean algebra B is *a filter* of B if it is closed with respect to the operation $\wedge$ and $a \vee b \in F$ whenever $a \in F$ and $b \in B$. Note that F is a filter of B if and only if F is closed with respect to the operation $\wedge$ and $a_1 \in F$ implies that $a_2 \in F$ for any $a_2 > a_1$.

As a rule, speaking about ideals and filters we mean proper ones. A proper ideal does not contain the unit, and a proper filter does not contain the zero. If I is an ideal, then the set $F = \neg I$ consisting of all $\neg a, a \in I$, is a filter. Similarly, a filter F corresponds to the ideal $I = \neg F$. An ideal of a Boolean algebra is called *maximal* if it is not included in a greater ideal distinct from B. Along with maximal ideals, we consider *maximal filters*, these are known as *ultrafilters*. We have the following

Proposition 3.2.7. *A proper Boolean ideal U of a Boolean algebra H is maximal if and only if either $a \in U$ or $\bar{a} \in U$ for every $a \in H$. A proper Boolean filter F of a Boolean algebra H is maximal if and only if either $a \in F$ or $\bar{a} \in F$ for every $a \in H$.*

Proof. Let U be a maximal ideal and suppose that both a and $\bar{a}$ are outside U. Then take U_1 to be the ideal generated by U and a. Since U is maximal, $U_1 = H$. Then $\bar{a} \in U_1$. Moreover, since $U_1 = H$ there exist $u \in U$ and $h \in H$ such that $u + ah = 1$. Hence $\bar{a} = \bar{a}u + \bar{a}ah = \bar{a}u$. Thus, $\bar{a} \leq u$, and $\bar{a} \in U$. Contradiction.

Suppose that for an ideal U either a or $\bar{a}$ lies in U for any $a \in H$. Suppose U lies in a bigger proper ideal U_1. Take $a \in U_1 \setminus U$. Then $\bar{a} \in U$. Thus, $\bar{a} \in U_1$. Hence, both a and $\bar{a}$ lie in U_1, which means that $U_1 = H$. Contradiction.

The proof for filters follows by duality. □

Recall that an algebra is *simple* if it has no proper non-zero ideals, and *semisimple* if the intersection of all its maximal ideals is zero-element set. An ideal I of a Boolean algebra B is maximal if and only if B/I is a simple algebra.

Structure of Boolean algebras is described by the following theorems.

Theorem 3.2.8. *Every simple Boolean algebra is isomorphic to the two-element algebra* **2**. □

Theorem 3.2.9. *Every Boolean algebra is semisimple.* □

The latter theorem is, in fact, equivalent to Theorem 1.1.12 stating that every Boolean algebra is isomorphic to a subalgebra of a power Boolean algebra.

Now we repeat with the proof the important Proposition 1.1.27.

Proposition 3.2.10. *A subset F of a Boolean algebra B is a filter if and only if the following conditions hold:*

1. *if $a_1 \in F$ and $a_1 \to a_2 \in F$, then $a_2 \in F$,*
2. *$1 \in F$.*

Proof. Let F be a filter. Then, obviously, $1 \in F$. Now let $a \in F$ and $a \to b \in F$. We have $ab = a(\bar{a} + b) \in F$, and then $b = ab + b \in F$.

Conversely, assume that F fulfills the two conditions. To check that F is a filter, we first show that $a + b \in F$ if $a \in F$ and $b \in A$. Since

$$a \to (a + b) = \bar{a} + a + b = 1 + b = 1 \in F,$$

it follows that $a + b \in F$ whenever $a \in F$.

Furthermore, assume that $a_1, a_2 \in F$. Then

$$a_2 = (a_1 + \bar{a}_1)a_2 = a_1a_2 + \bar{a}_1a_2 =$$

$$a_1a_2 + \overline{a_1 + \bar{a}_2} = a_1 + \bar{a}_2 \to a_1a_2 \in F.$$

Since $a_1 \in F$, also $a_1 + \bar{a}_2 \in F$. Hence, $a_1a_2 \in F$. □

The next proposition describes the ideal and the filter generated by a subset T of a Boolean algebra B.

Proposition 3.2.11. *The ideal generated by T consists of the elements of the form*

$$(a_1 + \cdots + a_n)b, \quad a_i \in T,\ b \in B.$$

The filter generated by T consists of the elements of the form

$$(a_1 \cdots a_n) + b, \quad a_i \in T,\ b \in B.$$

Proof. It is sufficient to note that the indicated collections of elements make up an ideal and a filter, respectively. □

Now we are able to relate filters with the concept of derivability in Boolean algebras.

Definition 3.2.12. *An element $a \in B$ is derivable from T if there is a sequence of elements (named a derivation),*

$$a_0, a_1, \ldots, a_n = a,$$

where $a_0 \in T$ *or* $a_0 = 1$ *and, for any* $a_i, 0 < i \leq n$*, either* $a_i \in T$ *or* $a_i = 1$*, or there are elements* a_k *and* a_s *with* $k, s < i$ *such that* $a_s = a_k \to a_i$*.*

Proposition 3.2.13. *The filter generated by* T *coincides with the set of all elements derivable from* T*.*

Proof. The proposition follows immediately from Proposition 3.2.11. $\square$

In particular, according to Proposition 3.2.13 an element b belongs to the filter generated by the elements a and $(a \to b)$. In this sense, the element b is a consequence of a and $(a \to b)$.

Since Boolean algebras are defined by identities, one can speak about the variety of all Boolean algebras.

Proposition 3.2.14. *The variety of Boolean algebras is generated by the algebra* $\mathbf{2}$ *and it does not contain proper non-zero subvarieties.*

Proof. Let $\mathfrak{X}$ be the variety of all Boolean algebras, then $Var(\mathbf{2}) \subset \mathfrak{X}$. Now, let B be an algebra from $\mathfrak{X}$, then $B \cong \mathbf{2}^S$ for some set S (see Theorem 1.1.12). So, by construction, the algebra B satisfies all identities of the algebra $\mathbf{2}$. Hence, $\mathfrak{X} \subset Var(\mathbf{2})$. Thus, $\mathfrak{X} = Var(\mathbf{2})$. Assume now that $\mathfrak{X}$ contains a proper subvariety $\mathfrak{X}_1$ and let $B \cong \mathbf{2}^S$ be an algebra from $\mathfrak{X}_1$. The algebra B contains the two-element subalgebra isomorphic to $\mathbf{2}$. So, $\mathbf{2}$ is in $\mathfrak{X}_1$. But the algebra $\mathbf{2}$ generates the whole variety $\mathfrak{X}$. Hence, $\mathfrak{X} = \mathfrak{X}_1$ and $\mathfrak{X}$ does not contain proper non-zero subvarieties. $\square$

Every variety possesses free algebras. Construct now a free algebra in the variety of Boolean algebras. Let J be a set and $\mathcal{P}(J)$ be the power set of J. Since $\mathcal{P}(J) \cong Fun(J, \mathbf{2}) = \mathbf{2}^J$, we will consider elements of $\mathcal{P}(J)$ as functions–strings $g : J \to \mathbf{2}$. Let $Fun(\mathbf{2}^J, \mathbf{2}) = \mathbf{2}^{\mathbf{2}^J}$ be the Boolean algebra of all functions from $\mathbf{2}^J$ to $\mathbf{2}$.

Let X be the set of all functions x_α from $Fun(\mathbf{2}^J, \mathbf{2})$ such that $x_\alpha(g) = g(\alpha)$, where $\alpha \in J$, $g \in Fun(J, \mathbf{2})$. Each x_α can be viewed as a variable, accepting the value $g(\alpha)$, for every $g \in Fun(J, \mathbf{2})$.

Theorem 3.2.15. *The subalgebra* $\mathcal{F}(X)$ *of* $Fun(\mathbf{2}^J, \mathbf{2})$ *generated by* X *is the free Boolean algebra over* X*.* $\square$

The proof is based on a characterization of elements from $\mathcal{F}(X)$ as finite support functions on $Fun(\mathbf{2}^J, \mathbf{2})$ and on Theorem 1.1.12 (for more details see, for example, [Si], [Pl-Datab]). In particular, $\mathcal{F}(X) = \mathbf{2}^{\mathbf{2}^J}$ if and only if J is finite. It consists of 2^{2^n} Boolean functions, where $n = |J|$.

In the propositional calculus the concept of a value map $f_{\mathfrak{L}_p}$ plays an important role. Now we define similar notion for Boolean algebras.

Definition 3.2.16. *A value homomorphism of a Boolean algebra B is a homomorphism from B to the Boolean algebra* $\mathbf{2}$.

Definition 3.2.17. *An element of a Boolean algebra B is called a tautology if it is true (mapped to 1) under every value homomorphism. An element is called a contradiction if it is false (mapped to 0) under every value homomorphism.*

Take the free Boolean algebra $\mathcal{F}(X)$ with the free generating set X. The set X plays the role of propositional variables. Every map $X \to \mathbf{2}$ can be extended in a unique way up to the value homomorphism of the Boolean algebras

$$Val_{prop} : \mathcal{F}(X) \to \mathbf{2}.$$

Later on we will specify the homomorphism of such kind for the logic of predicates and for the multi-sorted logic which is one of the main objects of this book.

3.2.3 Algebraization of the propositional calculus

Now we will show that Boolean algebras appear as a result of algebraization of the propositional calculus. With this end, we apply the Lindenbaum-Tarski algebraization process to the case of a propositional calculus.

For the sake of convenience consider the propositional calculus with respect to the signature $(\neg, \vee)$. Choose the scheme of axioms from Section 3.2, and plug in $\neg x_1 \vee x_2$ for $x_1 \to x_2$ in it.

Let $\mathfrak{L}_p(X)$ be the set of all formulas of the propositional calculus viewed as the absolutely free algebra over X with respect to the operations $(\neg, \vee)$.

Relying on $\mathfrak{L}_p(X)$, one can obtain the free Boolean algebra $\mathcal{F}(X)$. To do that, rewrite a system of identities defining a Boolean algebra in the signature $(\neg, \vee)$. Let ρ be the verbal congruence in $\mathfrak{L}_p(X)$ corresponding to the chosen set of identities. Then the algebra $\mathfrak{L}_p(X)/\rho$ is the free algebra $\mathcal{F}(X)$ in the variety of Boolean algebras (see Proposition 1.1.42).

Denote by τ the Lindenbaum-Tarski equivalence on the set of formulas $\mathfrak{L}_p(X)$. Given $u, v \in \mathfrak{L}_p(X)$, define $u\tau v$ if and only if

$$\vdash (u \to v) \wedge (v \to u).$$

Proposition 3.2.18. *τ is a congruence on $\mathfrak{L}_p(X)$.*

Proof. Follows from the axioms of a propositional calculus (Subsection 3.2.1). Indeed, one has to check that

1. $(u_1\tau v_1) \wedge (u_2\tau v_2) \vdash (u_1 \vee u_2)\tau(v_1 \vee v_2)$,
2. $u\tau v \vdash (\neg u)\tau(\neg v)$.

1. $(u_1 \to v_1) \wedge (v_2 \to u_2) \vdash (u_1 \to v_1)$. Hence $\vdash u_1 \to (v_1 \vee v_2)$. Symmetrically, $\vdash u_2 \to (v_2 \vee v_1)$. Thus, $\vdash (u_1 \vee u_2) \to (v_1 \vee v_2)$. Starting from $\vdash (v_1 \to u_1)$ we obtain in a similar way $\vdash (v_1 \vee v_2) \to (u_1 \vee u_2)$, that is $(u_1\tau v_1) \wedge (u_2\tau v_2) \vdash (u_1 \vee u_2)\tau(v_1 \vee v_2)$,.

2. $u\tau v$ means that $\vdash (\neg u \vee v) \wedge (\neg v \vee u)$, while $(\neg u)\tau(\neg v)$ is $\vdash (\neg v \vee u) \wedge (\neg u \vee v)$, which is the same according to commutative law. □

Sentences u and v are called *tautologically equivalent* if and only if they have the same truth values under any interpretation. Straightforward check yields that $u\tau v$ if and only if u and v are tautologically equivalent.

Proposition 3.2.19. *The Lindenbaum-Tarski algebra $\mathfrak{L}_p(X)/\tau$ is a free Boolean algebra.*

Proof. Check, first, that $\rho \subseteq \tau$. It is enough to show that $\mathfrak{L}_p(X)/\tau$ is a Boolean algebra. Thus, we have to verify that the identities of a Boolean algebra are fulfilled in $\mathfrak{L}_p(X)/\tau$. Denote the class of a formula u by $[u]_\tau$. The elements of $\mathfrak{L}_p(X)/\tau$ are equivalence classes with the operations

$$[u]_\tau \vee [v]_\tau = [u \vee v]_\tau, \qquad \neg[u]_\tau = [\neg u]\tau.$$

These operations satisfy Boolean identities $(1-5)$ from Definition 3.2.6. For example, we have

$$[u]_\tau \vee [v]_\tau = [u \vee v]_\tau, \quad [v]_\tau \vee [u]_\tau = [v \vee u]_\tau.$$

Since $u \vee v$ and $v \vee u$ have the same truth values, $u \vee v$ and $v \vee u$ are tautologically equivalent. This means that $(u \vee v)\tau(v \vee u)$ and

$$[u]_\tau \vee [v]_\tau = [v]_\tau \vee [u]_\tau.$$

The identity (1) is fulfilled in $\mathfrak{L}_p(X)/\tau$.

The same routine check can be done for other identities. One can use the list of tautological equivalences in [Hi].

Since ρ is the smallest congruence such that the corresponding quotient algebra is Boolean, we have $\rho \subseteq \tau$.

Check the opposite, $\tau \subseteq \rho$. Let u and v be elements in $\mathfrak{L}_p(X)$ and $u\tau v$. Take the natural homomorphism $\mu : \mathfrak{L}_p(X) \to \mathfrak{L}_p(X)/\rho$. Since $u\tau v$, $(u \to v) \vee (v \to u)$ is a tautology and

$$((u \to v) \wedge (v \to u))^\mu = 1,$$

in $\mathfrak{L}_p(X)/\rho$. Then

$$(u^\mu \to v^\mu) \wedge (v^\mu \to u^\mu) = 1.$$

However, this equality is possible in a Boolean algebra if and only if □

!!! Chto-to propushcheno!!!

Recall that if T is a set of formulas in $\mathfrak{L}_p(X)$ then the Lindenbaum-Tarski equivalence τ_T is defined as: $u\tau_T v$ if and only if

$$T \vdash (u \to v) \wedge (v \to u).$$

The next theorem shows that every theory T can be modeled as a Boolean algebra defined by generators and defining relations.

Theorem 3.2.20. *Every Boolean algebra is the Lindenbaum–Tarski algebra of a collection of formulas T.*

Proof. Suppose B is a Boolean algebra. Then it is isomorphic to $\mathcal{F}/I$ where $\mathcal{F} = \mathcal{F}(X)$ is the free Boolean algebra and I is an ideal in $\mathcal{F}$. We pass to the filter $F = \neg I$ and denote by T the full inverse image of F in $\mathfrak{L}_p(X)$ with respect to the epimorphism $\mu\colon \mathfrak{L}_p(X) \to \mathcal{F}$. It is easy to see that the algebra $\mathfrak{L}_p(X)/\tau_T$ is isomorphic to $\mathcal{F}/I$. □

3.3 Predicate calculus and polyadic algebras

In this section we recall what the predicate calculus is and define polyadic algebras which serve as an algebraization of the predicate calculus.

3.3.1 Predicate calculus. Syntax and semantics

The predicate calculus can be viewed as a purely logical first-order theory, which is not associated with a specific algebraic system. In particular, this means that the set of function symbols is empty. On the other hand, one can say that the predicate calculus constitutes the logical core of any first-order theory.

Before going over the definition of the predicate calculus we shall make some preparations and recall some well-known definitions, which are related to any first-order theory.

A consecutive part of a formula in a first-order language which is itself a formula is called *a subformula. The scope* of quantifiers $\forall$ and $\exists$ is defined as a subformula which starts from these quantifiers. An occurrence of the variable x in the formula u is *free*, if x does not belong to the scope of the quantifiers $\forall x$ and $\exists x$. All other occurrences x in u are called bound.

A variable x is called *a parameter* of a formula u if there exists a free occurrence of x in u. As for propositional calculus, denote by $u[x/t]$ the result of replacing all occurrences of x in u by t. Let two variables x and y be given. A substitution of y instead of x in a formula u is called *proper* if after replacing all free occurrences of x in u by occurrences of y, the variable y will not become bound for u. If t is a term, then the substitution of t instead of x is called *proper* if it is proper for every variable occurring in t. More precisely, if $u = u(x_1, \ldots, x_i, \ldots, x_n)$ is an atomic formula, then a substitution of t for x_i results in $u[x_i/t)] = u(x_1, \ldots, t, \ldots, x_n)$.

The predicate calculus is a tuple $\mathfrak{C} = (\mathbb{L}, \mathbb{F}, \mathbb{A}, \mathbb{D})$, where

$\mathbb{L}$: The language $\mathbb{L}$ consists of an infinite set of variables $X = \{x_1, x_2, x_3, \ldots\}$, the set of relation (predicate) symbols $\mathcal{R} = \{r_1^{n_1}, r_2^{n_2}, r_3^{n_3} \ldots\}$, the logical connectives $\neg$ and $\vee$, the symbol of existential quantifier $\exists$ and punctuation symbols.

$\mathbb{F}$: The set of formulas is constructed according to Definition 3.1.6.

$\mathbb{A}$: The set of axioms consists of the following ones:

1. $x_1 \vee x_1 \to x_1$,
2. $x_1 \to x_1 \vee x_2$,
3. $x_1 \vee x_2 \to x_2 \vee x_1$,
4. $(x_1 \to x_2) \to ((x_3 \vee x_1) \to (x_3 \to x_2))$,
5. $\forall x u(x) \to u[x/t]$, where $u[x/t]$ is proper,
6. $u[x/t] \to \exists x u(x)$, where $u[x/t]$ is proper,
7. $\forall x(u \to v) \to (u \to \forall x v)$, where x is not a parameter of u,
8. $\forall x(v \to u) \to (\exists v \to u)$, where x is not a parameter of u.

Here x_1, x_2, x_3 are variables, t is a term, u and v are formulas, $\forall x u$ is an abbreviation for $\neg(\exists x(\neg u))$.

Axioms (1)-(4) are related to a propositional part of the predicate calculus. For the sake of uniformity they are the same as in Section 3.2.1, and chosen according to [HilbAcker]. They can be replaced by any other set of axioms for the propositional calculus.

Axioms (5) – (8) provide, in fact, a scheme of axioms. As for the propositional calculus, there are many ways to choose a scheme of axioms and a set of derivation rules for the predicate calculus.

$\mathbb{D}$: The set of derivation rules for this set of axioms consists of:

- *The substitution rule* allows one to replace all occurrences of a given variable x in a formula u by a term t in case the substitution $u[x/t]$ is proper (see [Vereshchagin] for the detailed and clear exposition; below we follow this source).

  For logical connectives a proper substitution works as follows.

  1. A substitution of t for x in $\neg u$ is proper if it is proper for the formula u itself:
  $$(\neg u)[x/t] = \neg(u[x/t]).$$
  2. A substitution of t for x in $u \wedge v$ is proper if it is proper for each of u and v:
  $$(u \wedge v)[x/t] = u[x/t] \wedge v[x/t].$$
  3. The similar rule for $u \vee v$:
  $$(u \wedge v)[x/t] = u[x/t] \vee v[x/t].$$

  For quantified formulas the rules for proper substitutions are more complicated and use some conditions.

  4. Let $u = \forall xv$ (the case $u = \exists xv$ is treated in a similar way). A substitution of t for y in u is proper if there are no free occurrences of y in u (that is either y is a parameter for v or $y = x$). Thus,
  $$u[y/t] = u.$$
  5. Let u be as above. A substitution of t for y in u is proper if y is bound for u but x does not occur in t and the substitution of t for y in v is proper. Thus,
  $$u[y/t] = (\forall xv)[y/t] = \forall x(v[y/t]).$$

- *The modus ponens rule*, as before, states that the derivable formulas u_1 and $(u_1 \to u_2)$ imply u_2.

- *The generalization rule* says that u implies $\forall xu$.

Remark 3.3.1. *If we treat a substitution as a function $s : \mathbb{F} \to \mathbb{F}$, then items (1)-(3) say that s induces an endomorphism of the corresponding Boolean algebras. The relations between s and quantifiers are ruled by (4)-(5), which will result in axioms of the Halmos algebras (see Definition 3.3.17).*

The following logical equivalences and implications of first-order formulas are well known (see, for example, [Hi]) and can be deduced from the axioms and derivation rules.

Proposition 3.3.2.

1. $\neg(\exists x u) \leftrightarrow (\forall x \neg u)$,
2. $\neg(\forall x u) \leftrightarrow (\exists x \neg u)$,
3. $\forall x(u \wedge v) \leftrightarrow \forall x u \wedge \forall x v$,
4. $\exists x(u \vee v) \leftrightarrow \exists x u \vee \exists x v$,
5. $\exists x \exists y u \leftrightarrow \exists y \exists x u$,
6. $\forall x \forall y u \leftrightarrow \forall y \forall x u$,
7. $\exists y \forall x u \to \forall x \exists y u$,

If the variable x is not free for the formula u, then

8. $u \leftrightarrow \exists x u$,
9. $u \leftrightarrow \forall x u$,
10. $\forall x(u \vee v) \leftrightarrow u \vee \forall x v$,
11. $\exists x(u \wedge v) \leftrightarrow u \wedge \exists x v$. $\square$

We finish an excerpt from the syntax of the predicate calculus to the following well-known observation (see, [Vereshchagin]): one can exclude axioms (7) and (8) from the set of axioms of the predicate calculus, replacing the generalization law by the derivation laws of P.Bernays. Suppose that the variable x does not occur freely in the formula u. Then

1. $u \to v$ implies $u \to \forall x v$.
2. $v \to u$ implies $\exists x v \to u$.

As for semantics of the predicate calculus one can repeat the reasoning from Section 3.1.2. In particular, Theorems 3.1.12–3.1.14 hold true. In view of Theorem 3.1.12 we can freely use $\vdash$ for the syntactical/semantical derivability of formulas.

3.3.2 Quantifiers on Boolean algebras

Let B be a Boolean algebra.

Definition 3.3.3. *An existential quantifier on a Boolean algebra B is a map $\exists : B \to B$ subject to the conditions:*

1. $\exists 0 = 0$,
2. $b \leq \exists b$,
3. $\exists(b_1 \wedge \exists b_2) = \exists b_1 \wedge \exists b_2$,

where b_1, b_2 are elements of B, 0 is the zero element of B.

Remark 3.3.4. *The map $\exists : B \to B$ can be also considered as a unary operation on the algebra B.*

The *universal quantifier* $\forall$ is defined dually:

$$\forall b = \neg(\exists(\neg b)).$$

Hence,

$$\exists b = \neg(\forall(\neg b)).$$

The universal quantifier can be characterized as a map $\forall : B \to B$ having dual, with respect to $\exists$ properties:

1. $\forall 1 = 1$,
2. $b \geq \forall b$,
3. $\forall(b_1 \vee \forall b_2) = \forall b_1 \vee \forall b_2$.

Proposition 3.3.5. *Let a, b be elements of a Boolean algebra B. The quantifiers $\forall$ and $\exists$ possess the following properties:*

1. $\exists 1 = 1$; $\forall 0 = 0$.
2. $\exists(\exists a) = \exists a$, $\forall(\forall a) = \forall a$, *i.e., the maps $\exists$ and $\forall$ are idempotent:* $\exists^2 = \exists$, $\forall^2 = \forall$.
3. *If $a \leq b$, then $\exists a \leq \exists b$ and $\forall a \leq \forall b$, i.e., the maps $\exists$ and $\forall$ are monotone.*
4. $\exists(a \vee b) = \exists a \vee \exists b$, *i.e., $\exists$ is distributive over $\vee$.*
5. $\forall(a \wedge b) = \forall a \wedge \forall b$, *i.e., $\forall$ is distributive over $\wedge$.*
6. $\exists(\neg(\exists a)) = \neg(\exists a)$.
7. $\exists(\forall a) = \forall a$; $\forall(\exists a) = \exists a$.

8. $\exists(a \wedge \forall b) = \exists a \wedge \forall b$.

Proof. 1. Since $b \leq \exists b$ for every $b \in B$, we have $1 \leq \exists 1$. Hence $\exists 1 = 1$.

2. Indeed:

$$\exists^2 a = \exists(\exists a) = \exists(1 \wedge \exists a) = \exists 1 \wedge \exists a = 1 \wedge \exists a = \exists a.$$

3. Let $a \leq b$. Then $a \leq \exists b$ and $a = a \wedge \exists b$. Hence

$$\exists a = \exists(a \wedge \exists b) = \exists a \wedge \exists b$$

and $\exists a \leq \exists b$.

4. Since $\exists$ is monotone, we have $\exists a \leq \exists(a \vee b)$ and $\exists b \leq \exists(a \vee b)$. Thus, $\exists a \vee \exists b \leq \exists(a \vee b)$. Furthermore,

$$a \vee b \leq \exists a \vee \exists b, \quad \exists(a \vee b) \leq \exists(\exists a \vee \exists b).$$

Applying once again $\exists$ we get the reverse inequality $\exists(a \vee b) \leq \exists a \vee \exists b$. Thus, $\exists(a \vee b) = \exists a \vee \exists b$.

6. We have $\neg(\exists a) \wedge \exists a = 0$. Thus,

$$0 = \exists 0 = \exists(\neg(\exists a) \wedge \exists a) = \exists(\neg(\exists a)) \wedge \exists a.$$

Note that in a Boolean algebra if $a \wedge b = 0$ then $b \leq \neg a$. Hence, $\exists(\neg(\exists a)) \leq \neg(\exists a)$. On the other hand, since $\exists$ is monotone, we have $\neg(\exists a) \leq \exists(\neg(\exists a))$. So, $\exists(\neg(\exists a)) = \neg(\exists a)$.

7. By the definition of the universal quantifier: $\exists(\forall a) = \exists(\neg(\exists(\neg a)))$. Using property (6) we get

$$\exists(\neg(\exists(\neg a))) = \neg(\exists(\neg a)) = \forall a.$$

Thus, $\exists(\forall a) = \forall a$.

8. Using the definition of the existential quantifier and property (7), we have $\exists(a \wedge \forall b) = \exists(a \wedge \exists(\forall b)) = \exists a \wedge \exists(\forall b) = \exists a \wedge \forall b$.

The corresponding statements (1)–(3), (5), (7) for the universal quantifier can be obtained by the duality. □

3.3.3 Examples of quantifiers on Boolean algebras

In this section we give examples of quantifiers on various Boolean algebras (cf. [HalGiv], [Pl-Datab]).

Example 3.3.6. The identity map on any Boolean algebra is a quantifier (in this case the existential and the universal quantifiers coincide).

Example 3.3.7. Let B be a Boolean algebra, $b \in B$. Then the map $\exists$ defined as follows:

$$\exists 0 = 0, \quad \exists b = 1, \text{ for all } b \neq 0,$$

is an existential quantifier.

The next example follows the exposition in [HalGiv].

Example 3.3.8. Let A be a set and $B = \mathcal{P}(A)$ the Boolean algebra of all subsets of A. Consider the Boolean algebra B^A of all functions from A to B. So, if $a \in A$ and $f \in B^A$, then $f(a)$ is a subset of A. Define the map $\exists : B^A \to B^A$ as follows:

$$(\exists f)(a) = \bigcup_{a_i \in A} f(a_i),$$

for all $a \in A$. The set $\bigcup_{a_i \in A} f(a_i)$ is the union of the values of f for all $a_i \in A$. So, $\exists f$ is a constant function. The map $\exists : B^A \to B^A$ defined in such a way satisfies the conditions from Definition 3.3.3.

This example admits a geometrical interpretation. The Boolean algebra B^A is isomorphic to the Boolean algebra of all subsets of $A \times A$. This isomorphism assigns to each $f \in B^A$ the subset A_f in $A \times A$:

$$A_f = \{(a_1, a_2) \mid a_2 \in f(a_1)\}.$$

So, the set corresponding to the function $\exists f$ is

$$\exists A_f = \{(a_1, a_2) \mid a_2 \in (\exists f)(a_1) = \bigcup_{a_i \in A} f(a_i)\},$$

We can also describe this set as follows:

$$\exists A_f = \{(a_1, a_2) \mid \text{ there is } a_i \in A \text{ such that } a_2 \in f(a_i)\}.$$

Remark 3.3.9. *Let us, at the moment, come back to the algebra B^A. If we associate with a function $f \in B^A$ the statement "a_2 belongs to $f(a_1)$", then the function $\exists f$ correspond to the statement "there is $a_i \in A$ such that a_2 belongs to $f(a_i)$".*

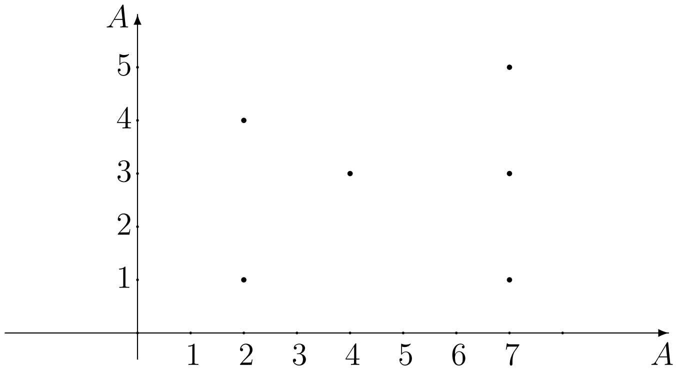
Let us illustrate all above by some pictures. Suppose that A is the set of real numbers. Then $A \times A$ is the Cartesian plane. Let f_1 be the function from B^A defined by:

$$f_1(a) = \begin{cases} \{1, 4\}, \text{ if } a = 2, \\ \{3\}, \text{ if } a = 4, \\ \{1, 3, 5\}, \text{ if } a = 7, \\ \varnothing, \text{ otherwise.} \end{cases}$$

Then the constant function $\exists f_1$ is defined for every $a \in A$ as follows:

$$(\exists f_1)(a) = \{1, 3, 4, 5\}.$$

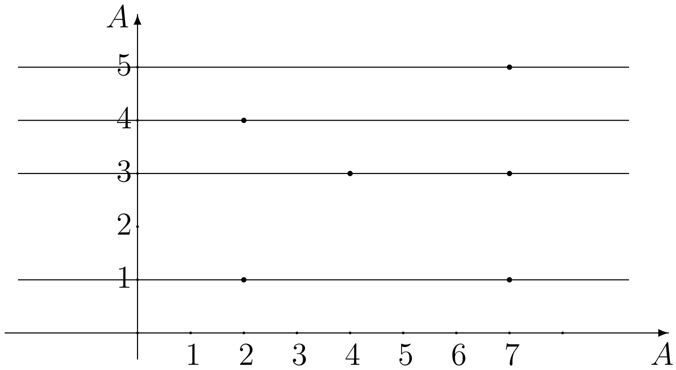
On the plane we have the following situation:



In this case,

$$A_{f_1} = \{(2, 1), (2, 4), (4, 3), (7, 1), (7, 3), (7, 5)\},$$

and the set $\exists A_{f_1}$ consists of all points (a_1, a_2), such that the second coordinate a_2 is one of 1, 3, 4, or 5. So, the set $\exists A_{f_1}$ is the union of four horizontal lines:



Let us give one more similar example. Define f_2 to be the following function from B^A:

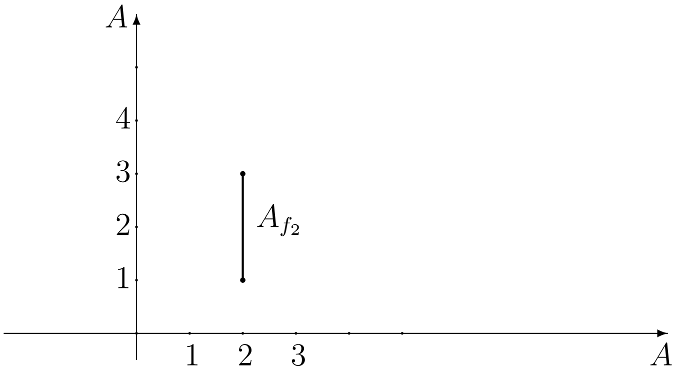
$$f_2(a) = \begin{cases} \{a' \in A \mid 1 \le a' \le 2\}, \text{ if } a = 2, \\ \varnothing, \text{ otherwise.} \end{cases}$$

Then the constant function $\exists f_2$ is defined for every $a \in A$ by the rule:

$$(\exists f_2)(a) = \{a' \in A \mid 1 \le a' \le 2\}.$$

On the plane we have:

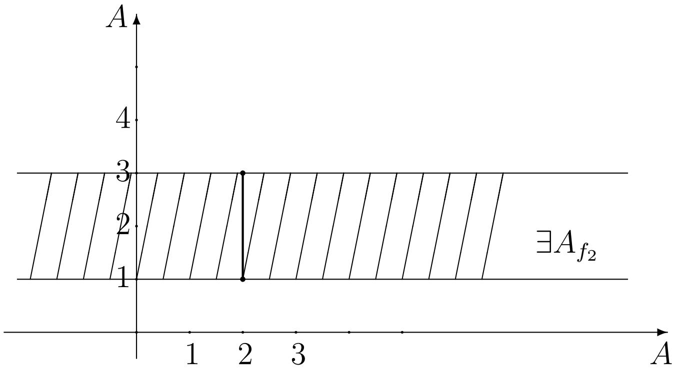
$$A_{f_2} = \{(2, a') \mid 1 \le a' \le 3\},$$



By definition the set $\exists A_{f_2}$ consists of all points (a_1, a_2) such that $a_1 \in A$ and $a_2 \in \bigcup_{a_i \in A} f(a_i)$. In this case

$$\bigcup_{a_i \in A} f(a_i) = \{a' \mid 1 \leq a' \leq 3\}.$$

So, $\exists A_{f_2}$ is the cylinder:



Example 3.3.10. Let A_1 and A_2 be two (not necessarily different) sets. Let $B = \mathcal{P}(A_1 \times A_2)$ be the Boolean algebra of all subsets of the set $A_1 \times A_2$. Let

$$A = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$$

be an element of the algebra B. Define two maps:

$$\exists x_1 : B \to B \text{ and } \exists x_2 : B \to B$$

as follows:

$$\exists x_1 A = \{(a_1, a_2) \mid \text{ there is } a_1' \in A_1 \text{ such that } (a_1', a_2) \in A\},$$

$$\exists x_2 A = \{(a_1, a_2) \mid \text{ there is } a_2' \in A_2 \text{ such that } (a_1, a_2') \in A\}.$$

These maps are existential quantifiers on the Boolean algebra B. Let us check that the map $\exists x_1$ is an existential quantifier. The proof for $\exists x_2$ is similar.

Proposition 3.3.11. *The map $\exists x_1$ defined above is an existential quantifier on the Boolean algebra $B = \mathcal{P}(A_1 \times A_2)$.*

Proof. The zero element of the algebra B is the empty set $\varnothing$. So,

$$\exists x_1 \varnothing = \{(a_1, a_2) \mid \text{ there is } a_1' \in A_1 \text{ such that } (a_1', a_2) \in \varnothing\}.$$

Thus,

$$\exists x_1 \varnothing = \varnothing.$$

From the definition of $\exists x_1$ it follows that, if a point (a_1, a_2) belongs to the set A, then this point belongs to the set $\exists x_1 A$. So,

$$A \subseteq \exists x_1 A$$

for any element $A \in B$. It remains to check that

$$\exists x_1 (A \cap \exists x_1 A') = \exists x_1 A \cap \exists x_1 A',$$

for every $A, A' \in B$. Note that if some point (a_1, a_2) belongs to the set $\exists x_1 A$, then the point (a_1', a_2) lies in $\exists x_1 A$ for every $a_1' \in A_1$. Let a point (a_1, a_2) belong to the set $\exists x_1 (A \cap \exists x_1 A')$. This means that there exists a point (a_1', a_2) in $A \cap \exists x_1 A'$. The point (a_1', a_2) lies in $A \subseteq \exists x_1 A$. So, the set $\exists x_1 A$ contains the point (a_1, a_2). The point (a_1', a_2) lies also in $\exists x_1 A'$, then the point (a_1, a_2) belongs to $\exists x_1 A'$. Thus,

$$(a_1, a_2) \in \exists x_1 A \cap \exists x_1 A'.$$

Now, let (a_1, a_2) lie in $\exists x_1 A \cap \exists x_1 A'$. Since (a_1, a_2) is in $\exists x_1 A$, then there is a point (a_1', a_2) in A, and this point also belongs to $\exists x_1 A'$. So, (a_1', a_2) lies in $A \cap \exists x_1 A' \subseteq \exists x_1 (A \cap \exists x_1 A')$. Hence,

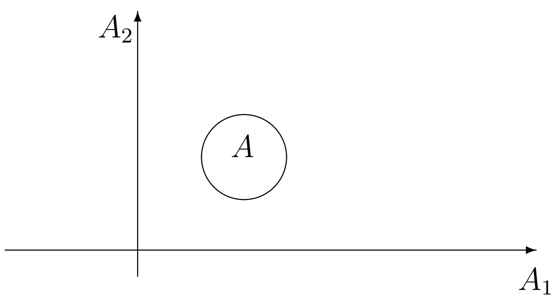
$$(a_1, a_2) \in \exists x_1 (A \cap \exists x_1 A').$$

Thus,

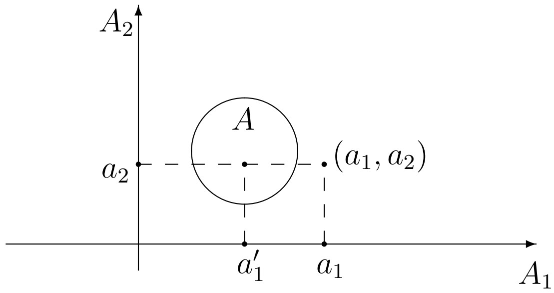
$$\exists x_1 (A \cap \exists x_1 A') = \exists x_1 A \cap \exists x_1 A'.$$

The proposition is proved. $\square$

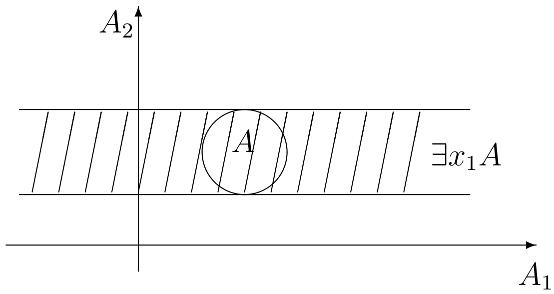
Let us illustrate this example on the plane. Take a set of points A:



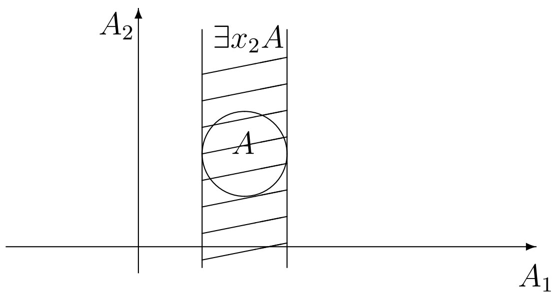
Recall that the set $\exists x_1 A$ consists of all points $(a_1, a_2) \in A_1 \times A_2$ such that the point (a_1', a_2) belongs to A for some $a_1' \in A_1$. This means that a point (a_1, a_2) lies in $\exists x_1 A$ if and only if the horizontal line passing through the point (a_1, a_2) has an intersection with the set A.



So, the set $\exists x_1 A$ is the cylinder:



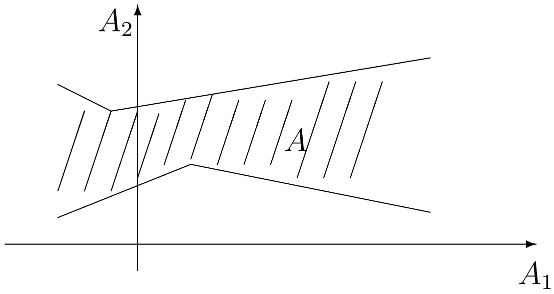
For similar reasons we get that the set $\exists x_2 A$ is the following one:



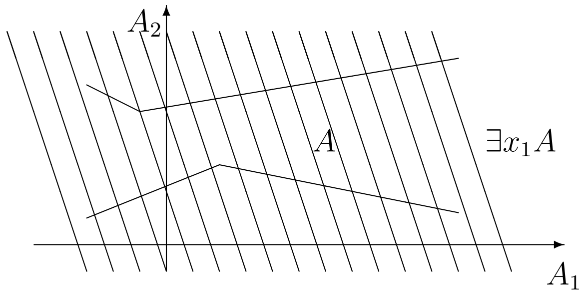
Note that for the given set A, the universal quantifiers $\forall x_1 A$ and $\forall x_2 A$ produce empty sets. Indeed, by definition

$$\forall x_1 A = \neg(\exists x_1(\neg A)).$$

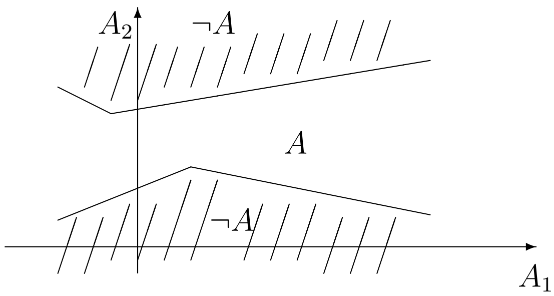
The set $\neg A$ is the set-theoretical complement of A. So, the set $\exists x_1(\neg A)$ is the whole plane $A_1 \times A_2$ and $\neg(\exists x_1(\neg A))$ is the empty set. Recall that $A \subseteq \exists x_1 A$. So, if the set A is "unbounded" then the set $\exists x_1 A$ is also "unbounded". For instance, let A be the following set:



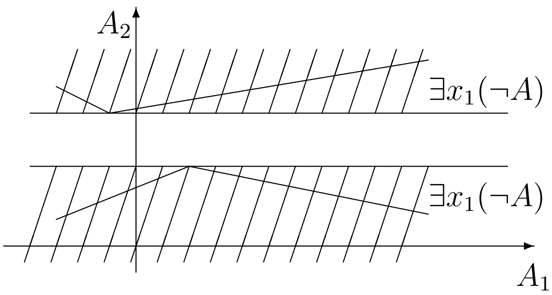
Then the sets $\exists x_1 A$ and $\exists x_2 A$ are the whole plane $A_1 \times A_2$.



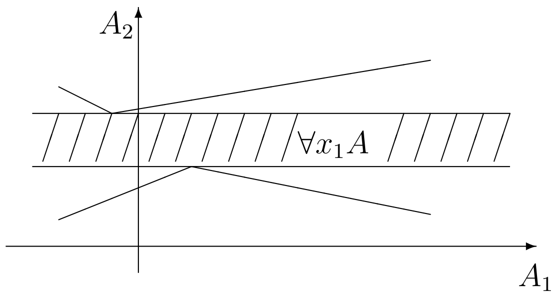
Let us consider the universal quantifiers $\forall x_1 A$ and $\forall x_2 A$ for this particular set A. The set $\neg A$ is the following one:



The set $\exists x_1(\neg A)$ is:



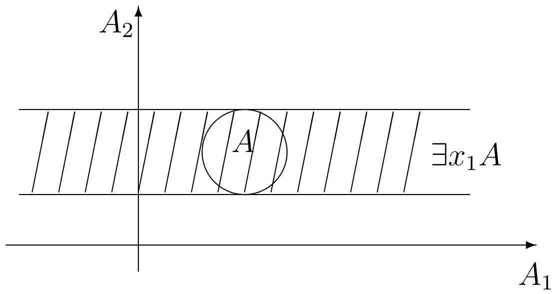
Finally, the set $\forall x_1 A = \neg(\exists x_1(\neg A))$ is the cylinder:



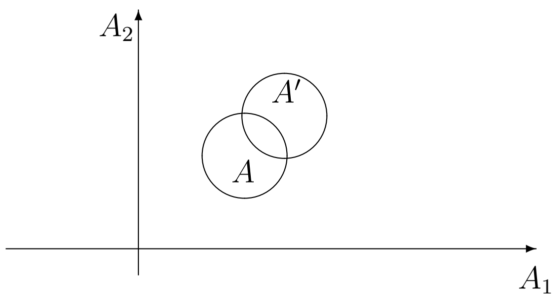
One can show that the set $\forall x_2 A$ is empty, since $\forall x A$ consists of the points $(a, b) \in A$ such that all (a', b) also belong to A. Thus, $\forall x A$ is the largest cylinder in the direction of x among those lying in A.

We use Example 3.3.10 in order to illustrate on the plane the definition and some properties of quantifiers.

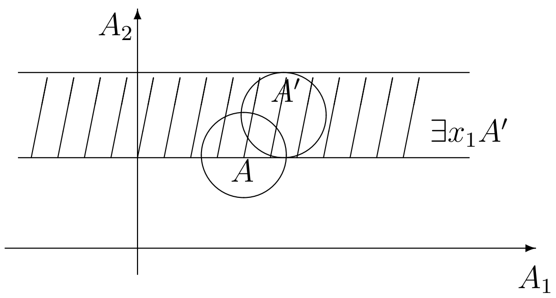
The pictures above depict first two conditions of Definition 3.3.3: $\exists \varnothing = \varnothing$ and $A \subseteq \exists x_i A$, $i = 1, 2$. For instance, for $i = 1$:



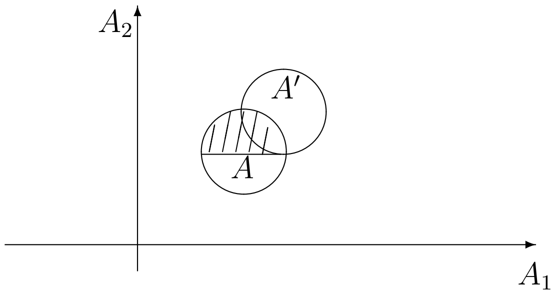
The third condition of Definition 3.3.3 is $\exists(b_1 \wedge \exists b_2) = \exists b_1 \wedge \exists b_2$. Take, for example, sets A and A' as follows:



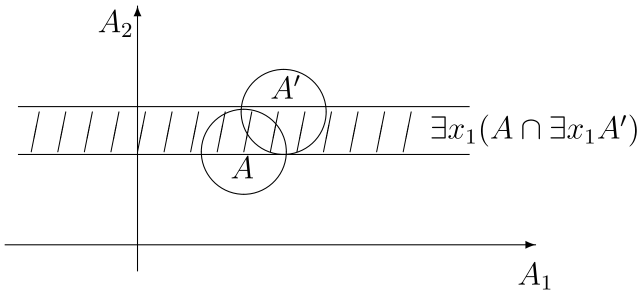
and exhibit the equality $\exists x_1(A \cap \exists x_1 A') = \exists x_1 A \cap \exists x_1 A'$. Indeed, $\exists x_1 A'$ is



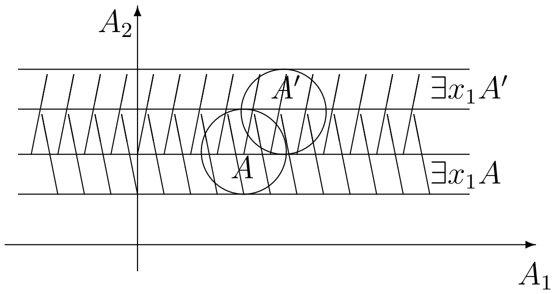
Then $A \cap \exists x_1 A'$ is the following set



and $\exists x_1 (A \cap \exists x_1 A')$ is



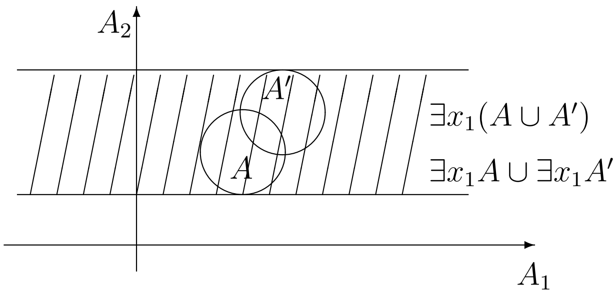
On the other hand, the set $\exists x_1 A \cap \exists x_1 A'$ looks as follows



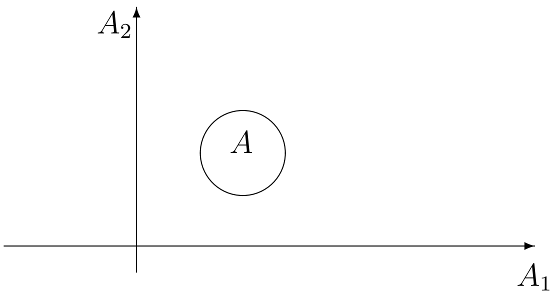
Hence, the set $A \cap \exists x_1 A'$ coincides with $\exists x_1 A \cap \exists x_1 A'$.

Let us illustrate properties (4), (6), (7) and (8) of quantifiers (Proposition 3.3.5). It is easy to see the other properties using the previous pictures.

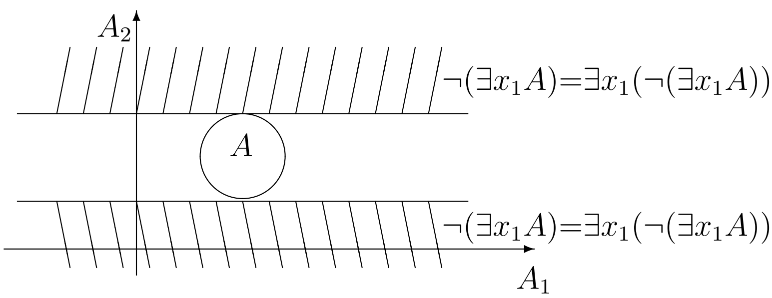
Property (4): $\exists x_1(A \cup A') = \exists x_1 A \cup \exists x_1 A'$. Let A, A' be the sets as before. Then $\exists x_1(A \cup A')$ and $\exists x_1 A \cup \exists x_1 A'$ give rise to the following picture:



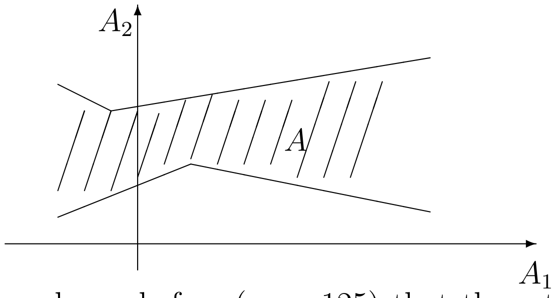
Property (6): $\exists x_1(\neg(\exists x_1 A)) = \neg(\exists x_1 A)$. Let A be a set



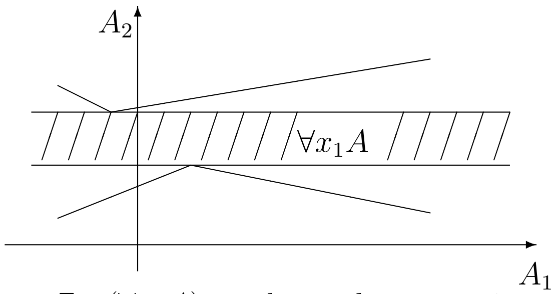
Then the sets $\neg(\exists x_1 A)$ and $\exists x_1(\neg(\exists x_1 A))$ coincide



Property (7a): $\exists x_1(\forall x_1 A) = \forall x_1 A$. Let A be a set



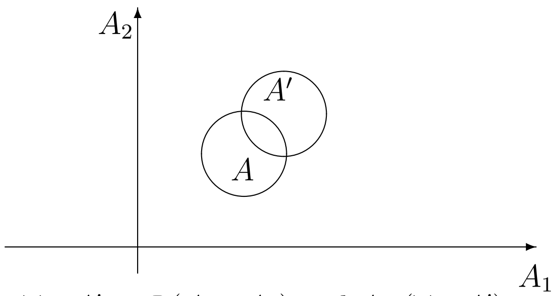
It was shown before (page 125) that the set $\forall x_1 A$ is the following one:



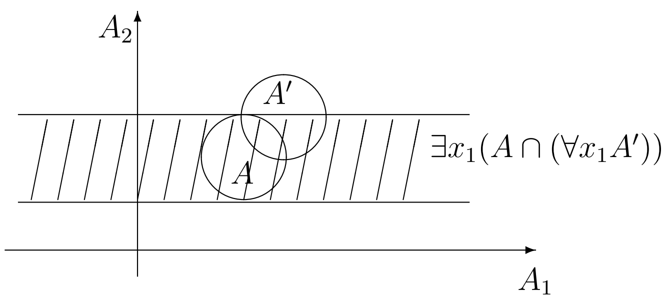
The set $\exists x_1(\forall x_1 A)$ produces the same picture.

Property (7b): $\forall x_1(\exists x_1 A) = \exists x_1 A$. For the same set A the set $\exists x_1 A$ is the whole algebra $\mathcal{P}(A_1 \times A_2)$. Then $\forall 1 = 1$ implies that both $\forall x_1(\exists x_1 A)$ and $\exists x_1 A$ coincide with $\mathcal{P}(A_1 \times A_2)$.

Property (8): $\exists x_1(A \cap (\forall x_1 A')) = \exists x_1 A \cap \forall x_1 A'$. Let A, A' be the following sets.



Then $\forall x_1 A' = \mathcal{P}(A_1 \times A_2)$ and $A \cap (\forall x_1 A') = A$. So, $\exists x_1(A \cap (\forall x_1 A'))$ is the following one:



This is exactly the intersection of $\mathcal{P}(A_1 \times A_2)$ and the cylinder $\exists x_1 A$.

Other configurations of the sets A and A' are left as an exercise.

3.3.4 Quantifier algebras

Now we continue the way:

$$Boolean\ algebras \Rightarrow Monadic\ algebras \Rightarrow$$

$$\Rightarrow Quantifier\ algebras \Rightarrow Polyadic\ algebras,$$

which leads to an algebraization of the predicate calculus. First, we dwell on concepts of monadic and, especially, quantifier algebras which are situated between Boolean algebras and polyadic algebras.

Definition 3.3.12. *A monadic algebra is a pair $(B, \exists)$, where B is a Boolean algebra and $\exists$ is a quantifier on B.*

In other words, a monadic algebra is a Boolean algebra with a single additional operation $\exists$. Examples from Subsection 3.3.2 of Boolean algebras equipped with quantifiers provide examples of monadic algebras. For more details and for properties of monadic algebras see [Halm], [HalGiv].

Definition 3.3.13. *Let X be a set. A Boolean algebra B is a quantifier X-algebra if a quantifier $\exists(Y)\colon B \to B$ is defined for every subset $Y \subset X$, and the following conditions hold:*

1 . $\exists(\varnothing) = I_B$, the identity function on B,

2 . $\exists(X_1 \bigcup X_2) = \exists(X_1)\exists(X_2)$, where X_1, X_2 are subsets of X.

If we restrict ourselves to finite non-trivial subsets Y of X, then a Boolean algebra B is a *quantifier X-algebra* if a quantifier $\exists x\colon B \to B$ is defined for every variable $x \in X$, and

$$\exists x \exists y = \exists y \exists x,$$

for every $x, y \in X$. Indeed, condition (2) from Definition 3.3.13 implies commutativity of quantifiers, and, conversely, one can define $\exists(Y) = \exists y_1 \dots \exists y_k$, where $Y = \{y_1, \dots, y_k\}$.

Remark 3.3.14. *The quantifier algebras defined here are a version of diagonal-free cylindric algebras of Tarski (see [HMT]).*

The next example generalizes Example 3.3.10.

Example 3.3.15. Let D be the Cartesian product of the sets $D_\alpha, \alpha \in I$. Consider the Boolean algebra $\mathcal{P}(D)$ of all subsets of D. Let J be a subset of I. Define the quantifier $\exists(J)$ as follows: for any subset $A \subset D$, an element a belongs to the set $\exists(J)A$ if and only if there is an element $a' \in A$ such that $a(\alpha) = a'(\alpha)$ for all α outside J. Correspondingly, $b \in \forall(J)A$ means that every a such that $b(\alpha) = a(\alpha)$ outside J belongs to A.

Correspondingly, $a \in \forall(J)A$ means that $a \in A$ and every a' such that $a(\alpha) = a'(\alpha)$ outside J, also belongs to A.

In order to check that $\exists(J)$ is, indeed, a quantifier we need to verify that it satisfies conditions (1)–(3) from the definition of an existential quantifier. Since conditions (1)–(2) are clear, only the third condition should be verified. Clearly, $A \subset B$ implies $\exists(J)A \subset \exists(J)B$, and $\exists(J)^2 = \exists(J)$. Therefore,

$$\exists(J)(A \cap \exists(J)B) \subset \exists(J)A \cap \exists(J)B.$$

To prove the converse, assume that $a \in \exists(J)A \cap \exists(J)B$. Let $b \in A$ and $a(\alpha) = b(\alpha)$ outside J. Since $a \in \exists(J)B$, we conclude that

$$b \in \exists(J)(\exists(J)B) = \exists(J)B.$$

Hence, $b \in A \cap \exists(J)B$, and $a \in \exists(J)(A \cap \exists(J)B)$.

The Boolean algebra $\mathcal{P}(D)$ considered together with quantifiers $\exists(J)$ satisfies conditions (1)–(2) from Definition 3.3.13 (see [Pl-Datab]). Thus, $\mathcal{P}(D)$ is a quantifier I-algebra.

3.3.5 Polyadic algebras

Quantifier algebras do not yet represent the predicate calculus. The main obstruction is as follows: in terms of Boolean operations and quantifiers we cannot describe the transformations of individual variables. In other words, we have no way to convert, for example, a sentence $p(x_1, x_2, x_3)$ into $p(x_2, x_1, x_3)$.

This problem did not exist when we treated Boolean algebra as an algebraization of propositional calculus. In this case any substitution of variables respects logical connectives and parentheses. Hence, substitutions of variables in the propositions sentences are realized by homomorphisms of Boolean algebras.

This is not true for quantifier algebras because interaction of the variables substitutions with quantifiers in a predicate calculus is subject to more complex rules (see Subsection 3.3.1).

There are various ways to solve this problem. One of them is to consider quantifier algebras together with their endomorphisms and to impose additional axioms on these endomorphisms which imitate the substitution rules for the predicate calculus. In other words the idea is to postulate the laws of transformation of variables as the axioms of the corresponding algebra.

Using this idea we come up with the notion of a polyadic algebra introduced by P. Halmos [Halm]. Another important approach was considered by A. Tarski who introduced cylindric algebras [HMT]. These algebras correspond to predicate calculus with equality.

Let $X = \{x_\alpha, \alpha \in I\}$ be a (finite or infinite) set and S be the semigroup of all transformations of X. We can consider I as a copy of the set X.

Definition 3.3.16. *A polyadic X-algebra B is a quantifier X-algebra considered together with a representation of the semigroup S as a semigroup of Boolean endomorphisms subject to the following conditions:*

1. *the unit s_{id} of S acts trivially on B;*
2. $s_1\exists(J)a = s_2\exists(J)a$, *if $s_1, s_2 \in S$ satisfy $s_1(x) = s_2(x)$ for all $x \in X \setminus J$;*
3. $\exists(J)sa = s\exists(s^{-1}J)a$, *if $s \in S$ never maps two distinct elements of X onto the same element of J, (i.e., if $sx_1 = sx_2 \in J$ then $x_1 = x_2$).*

Here $a \in B$, $J \subset X$, and $s^{-1}J$ is the full inverse image of J under s.

Reformulating this definition of a polyadic algebra in terms of an algebra having the signature of operations and a set of identities we have:

Definition 3.3.17. *A polyadic X-algebra B is an algebra in the signature*

$$\Omega = \{\wedge,\ \vee,\ \neg,\ 0,\ 1,\ s_\sigma,\ \exists(J)\},$$

where $\sigma : X \to X$, $J \subset X$. The algebra B in the signature

$$\{\wedge,\ \vee,\ \neg,\ 0,\ 1\}$$

is a Boolean algebra and s_σ, $\exists(J)$ are unary operations, which interact as follows:

1. $s_\sigma(a \vee b) = s_\sigma a \vee s_\sigma b$,
2. $s_\sigma(\neg a) = \neg(s_\sigma a)$,
3. $(s_\sigma s_\tau)a = s_{\sigma\tau}a$,

4. $s_{id} = id_B$*,*

5. $\exists(J)0 = 0$*,*

6. $a \leq \exists(J)a$*,*

7. $\exists(J)(a \wedge \exists(J)b) = \exists(J)a \wedge \exists(J)b$*,*

8. $\exists(\varnothing) = I_B$*, the identity function on* B*,*

9. $\exists(J_1 \cup J_2) = \exists(J_1)\exists(J_2)$*, where* J_1*,* J_2 *are subsets of* X*,*

10. $s_\sigma \exists(J)a = s_\tau \exists(J)a$*, if* σ, τ *satisfy* $\sigma(x) = \tau(x)$ *for* $x \in X \setminus J$*,*

11. $\exists(J)s_\sigma a = s_\sigma \exists(\sigma^{-1}J)a$*, if* σ *is injective on* $\sigma^{-1}J$*,*

for all $a, b \in B$*;* $\sigma, \tau : X \to X$*;* $J, J_1, J_2 \subset X$*.*

Identities (1)–(4) demonstrate that $S = \{s_\sigma\}$, $\sigma : X \to X$ is the semigroup of endomorphisms of the Boolean algebra B, identities (5)–(7) came from the definition of a quantifier $\exists$, (8)-(9) are extracted from the definition of a quantifier algebra, and, finally, (10)–(11) control interaction of transformations with quantifiers.

This set of identities determines the variety of polyadic X-algebras.

Conditions (10)–(11) which look, at a first glance, complicated play a special role. In fact, they are related to substitution rules in a predicate calculus (cf. Subsection 3.3.1) and were invented by P.Halmos after numerous experiments.

To make them transparent, suppose J consists of a single element $\{x_\alpha\}$. Let $\sigma = \sigma_\alpha^\beta : X \to X$ take x_α to x_β and leave all other elements of X unchanged. Since s_σ and s_{id} coincide outside $J = \{x_\alpha\}$, the corresponding s_σ acting on B possesses the property

$$s_\sigma \exists(J) = \exists(J).$$

Quoting [Halm], one can say that *"this equation corresponds to a familiar fact, that once a variable has been quantified, the replacement of that variable by another one has no further effect"*.

Note that $\sigma^{-1}J = \varnothing$ for σ and J as above. From condition (11) it follows that

$$\exists(J)s_\sigma = s_\sigma.$$

This equation corresponds to the fact, that *"once a variable has been replaced by another one, a quantification on the replaced variable has no further effect" ([Halm]).*

Remark 3.3.18. *Since it is customary for Boolean algebras, we will use* $+$ *for* $\vee$*,* $\cdot$ *for* $\wedge$ *and* $\bar{a}$ *for* $\neg a$ *in the case of polyadic algebras without a special notice.*

The cardinal number of X is called *the degree* of the algebra B. Each element of X is called a variable of B. If X is the empty set, than $S = \varnothing$. So, Boolean algebras are polyadic algebras of degree 0. If X is one-element set, then there is only one transformation, namely, the identity transformation. The polyadic algebras of degree 1 are monadic algebras (see [Halm] for details).

Let b be an element of a polyadic X-algebra B.

Definition 3.3.19. *An element b is independent of a subset $J \subset X$ if $\exists(J)b = b$.*

Remark 3.3.20. *The concept of independence corresponds to the logical notion of bounded variables.*

Definition 3.3.21. *The set J is a support of b if b is independent of $J' = X \setminus J$.*

It is easy to see that all supports of the given element b in B constitute a filter in the power algebra $\mathcal{P}(X)$. If $b \in B$ has a finite support, then it has a minimal finite support $\Delta(b)$. It is exactly the set of elements $x \in X$ such that $\exists x b \neq b$. Now we define a class of polyadic algebras which is most relevant to the aims of the algebraization of the predicate calculus.

Definition 3.3.22. *A polyadic X-algebra is called locally finite if each of its elements has a finite support.*

Since in a logical calculus the set of variables is assumed to be infinite while each formula depends only on finite number of variables, the polyadic algebras in question should be locally finite polyadic algebras of infinite degree.

In view of Definition 3.3.13 in locally finite polyadic algebras we can restrict ourselves to the quantifiers of the form $\exists x$, where x is a variable in X.

3.3.6 Examples of polyadic algebras

Example 3.3.23. Polyadic power algebra $\mathcal{P}(D)$.

Let again D be the Cartesian product of the sets $D_\alpha, \alpha \in I$. The Boolean power algebra $\mathcal{P}(D)$ of all subsets of D is a quantifier I-algebra (see Example 3.3.15).

Let S be the semigroup of all transformations of the set I. First of all, define the action of S on D as follows:

$$(as)(\alpha) = a(s\alpha),$$

$a \in D$, $s \in S$, $\alpha \in I$. Now we specify the representation of the semigroup S as a semigroup of Boolean endomorphisms of $\mathcal{P}(D)$. For each $s \in S$ the corresponding endomorphism $s \in End(\mathcal{P}(D))$ is defined as:

$$sA = \{a \in D \mid as \in A\},$$

where $A \subset \mathcal{P}(D)$. Then

Proposition 3.3.24. *The quantifier algebra $\mathcal{P}(D)$ with the given representation of the semigroup S is a polyadic algebra.*

Proof. We shall check three conditions of Definition 3.3.16. Condition (1) is obvious.

Let us verify condition (2). Suppose that s_1 and s_2 agree outside J, and choose $A \subset D$. We need to verify that

$$s_1\exists(J)A = s_2\exists(J)A.$$

Let $a \in s_1\exists(J)A$, i.e., $as_1 = b \in \exists(J)A$. Take $c \in A$ such that $b(\alpha) = c(\alpha)$ outside J. We have

$$b(\alpha) = as_1(\alpha) = a(s_1\alpha) = a(s_2\alpha) = as_2(\alpha),$$

where $\alpha \in \bar{J}$, $\bar{J}$ is the complement of J. So, $c(\alpha) = as_2(\alpha)$ outside J, $as_2 \in \exists(J)A$, and $a \in s_2\exists(J)A$. The inverse inclusion holds for the same reasons.

Applying condition (2) for the case $J = I$, we get two properties: if A is empty then $\exists(I)A = \varnothing$, and if $A \neq \varnothing$ then $\exists(I)A = D$.

We move to condition (3). Let $a \in \exists(J)sA$, $\quad b \in sA$ with $a(\alpha) = b(\alpha)$ outside J, so that $bs = c \in A$. For α out of $s^{-1}J$, s_α does not belong to J, and then

$$as(\alpha) = a(s\alpha) = b(s\alpha) = c(\alpha).$$

It follows that $as \in \exists(s^{-1}J)A$ and $a \in s\exists(s^{-1}J)A$. We notice that the inclusion $\exists(J)sA \subset s\exists(s^{-1}J)A$ does not presuppose any restrictions on s. The restriction mentioned in the condition will be used to prove the converse conclusion.

Assume that $a \in s\exists(s^{-1}J)A$, i.e., $as \in \exists(s^{-1}J)A$, and select $b \in A$ such that $as(\alpha) = b(\alpha)$ outside $s^{-1}J$. We construct an element c as follows: $c(\alpha) = a(\alpha)$ outside J, $\quad c(\alpha) = b(s^{-1}\alpha)$ if $\alpha \in s(s^{-1}J)$ $\quad$ ($s(s^{-1}J)$ is a part of J, so $s^{-1}\alpha$ makes sense here by condition (3)); on the rest c may be defined arbitrary. Now if $\beta \in s^{-1}J$, then

$$cs(\beta) = c(s\beta) = b(s^{-1}s\beta) = b(\beta),$$

and

$$cs(\beta) = c(s\beta) = a(s\beta) = as(\beta) = b(\beta),$$

otherwise. Consequently, $b = cs \in A$ and $c \in sA$. Since a coincides with c outside J, we conclude that $a \in \exists(J)sA$. □

The polyadic algebra $\mathcal{P}(D)$, where D is the Cartesian product of the sets $D_\alpha, \alpha \in I$, can be treated as the algebra $\mathbf{2}^D = Fun(D, \mathbf{2})$ of characteristic functions on D. In the next example we replace the Boolean algebra $\mathbf{2}$ by an arbitrary Boolean algebra B.

Example 3.3.25. Functional polyadic algebra.

Consider the set $Fun(D, B)$ of all functions from the Cartesian product $D = \prod D_\alpha$ onto B, where B is a Boolean algebra, $\alpha \in I$. For the sake of simplicity assume that B is a finite algebra or, more generally, a complete Boolean algebra, i.e., an algebra which contains supremum (arbitrary join) and infimum (arbitrary meet) of any subset of elements from B.

The algebra $Fun(D, B)$ is a Boolean algebra (cf. Example 1.1.10). Define, first, the action of the semigroup S of all transformation of the set I on $Fun(D, B)$. Let $f \in Fun(D, B)$, $s \in S$, $a \in D$. The action of S on $Fun(D, B)$ is defined as follows:

$$(sf)(a) = f(as),$$

where $as(\alpha) = a(s\alpha)$, for $\alpha \in I$.

Now we define quantifiers on $Fun(D, B)$. Let J be a subset of I. Let J_* be the binary relation on D determined by the rule:

$$aJ_*b \Leftrightarrow a(\alpha) = b(\alpha) \text{ whenever } \alpha \in I \setminus J,$$

for all $a, b \in D$. The quantifier $\exists(J)f$ is defined by

$$\exists(J)f(a) = \bigvee\{f(b) \mid aJ_*b\}.$$

Since B is a complete Boolean algebra the latter formula makes sense. One can check that $\exists(J)$ is indeed an existential quantifier and that all axioms of a polyadic algebra are fulfilled for $Fun(D, B)$.

In the case when B is an arbitrary Boolean algebra the algebra $Fun(D, B)$ is not necessarily a polyadic algebra. However, it contains a maximal Boolean subalgebra which is invariant with respect to the action of the semigroup S and with the action of $\exists(J)$. Hence, in general case the functional polyadic algebra is defined as follows.

Definition 3.3.26. *A functional polyadic I-algebra is a Boolean subalgebra $Fun'(D, B)$ of $Fun(D, B)$ such that*

1. *$sf \in Fun'(D,B)$ whenever $f \in Fun'(D,B)$, $s \in S$,*
2. *$\exists(J)f$ exists and belongs to $Fun'(D,B)$ whenever $f \in Fun'(D,B)$, $J \subset I$.*

3.3.7 QA_I-algebras

In this section we follow the approach of C.Pinter [Pi2], where a transparent set of axioms for locally finite polyadic X-algebras is introduced. Pinter defined a class of QA_I-algebras which is close to polyadic algebras. We start from the definition of a QA_I-algebra and then list some properties of these algebras.

Definition 3.3.27 ([Pi2]). *A QA_I-algebra B of degree I is an algebra in the signature*

$$\Omega = \{\wedge,\ \vee,\ \neg,\ 0,\ 1,\ s_\lambda^\kappa,\ \exists\alpha\},\ \alpha, \lambda, \kappa \in I,$$

where B in the signature

$$\{\wedge,\ \vee,\ \neg,\ 0,\ 1\}$$

is a Boolean algebra and s_λ^κ, $\exists\alpha$ are unary operations, which interact as follows:

1. *$s_\lambda^\kappa(\neg a) = \neg(s_\lambda^\kappa a)$,*
2. *$s_\lambda^\kappa(a \vee b) = s_\lambda^\kappa a \vee s_\lambda^\kappa b$,*
3. *$s_\kappa^\kappa = id_B$,*
4. *$(s_\lambda^\kappa s_k^\mu) = (s_\lambda^\kappa s_\lambda^\mu)$*
5. *$\exists\alpha(a \vee b) = \exists\alpha a \vee \exists\alpha b$*
6. *$a \leq \exists\alpha a$,*
7. *$s_\lambda^\kappa \exists\kappa = \exists\kappa$,*
8. *$\exists\kappa s_\lambda^\kappa = s_\lambda^\kappa$, $\kappa \neq \lambda$,*
9. *$s_\lambda^\kappa \exists\mu = \exists\mu s_\lambda^\kappa$, $\mu \neq \kappa, \lambda$.*

for all $a, b \in B$; $\mu, \kappa, \lambda, \alpha \in I$.

In a polyadic algebra we use an extended notion for the operations s_λ^κ and $\exists\alpha$. Instead of quantifying over a single variable, as in a QA-algebra, in a polyadic algebra we may quantify over an arbitrary set of variables. Similarly, the simultaneous substitution of arbitrarily many variables is permitted.

Remark 3.3.28. *Every polyadic I-algebra is a QA_I-algebra. A polyadic algebra becomes a QA-algebra after removing some of its operations.*

Lemma 3.3.29. *Let B be an arbitrary QA_I-algebra. Then the following statements hold for all $a, b \in B$ and all $\kappa, \lambda, \mu \in I$.*

(i) $\exists\kappa 0 = 0$,

(ii) $\exists\kappa(a \wedge \exists\kappa b) = \exists\kappa a \wedge \exists\kappa b$,

(iii) $\exists\kappa\exists\lambda = \exists\lambda\exists\kappa$,

(iv) $s^\kappa_\lambda s^\kappa_\mu = s^\kappa_\mu$, if $\mu \neq \kappa$,

(v) $s^\kappa_\lambda s^\mu_\nu = s^\mu_\nu s^\kappa_\lambda$, if $\kappa \neq \nu, \mu$ and $\mu \neq \lambda$.

Proof. We start with the following statements

(a) $a \leq b$ implies $s^\kappa_\lambda a \leq s^\kappa_\lambda b$,

(b) $a \leq b$ implies $\exists\kappa a \leq \exists\kappa b$,

(c) $s^\kappa_\lambda a \leq \exists\kappa a$.

(d) $\exists\kappa a$ is the least element of the set $\{b \in \text{range } s^\kappa_\lambda \mid b \geq a\}$, if $\kappa \neq \lambda$.

Since $a \leq b$ means $a \vee b = b$, item (a) is an immediate consequence of axiom (2) of Definition 3.3.27; (b) is an immediate consequence of axiom (5) of Definition 3.3.27. Finally, (c) follows from (a) and axioms (6-7) of the same definition.

The statement (d) is true, since by (7), $\exists a$ belongs to the range s^κ_λ and by (6), $a \leq \exists\kappa a$. If b lies in range s^κ_λ, then for some $c \in B$ we have $b = s^\kappa_\lambda c = \exists\kappa s^\kappa_\lambda c = \exists\kappa b$. Thus, if b in range s^κ_λ and $b \geq a$, then by (2), $\exists\kappa \leq \exists\kappa b = b$.

It follows from axiom (4) and Theorems 4-5 from [Halm] (page 45), that $\exists\kappa$ is a quantifier in the sense of polyadic algebras, and, hence, it satisfies (i) and (ii).

Using axioms (7)-(9) from Definition 3.3.27 repeatedly, for any $\mu \neq \lambda, \kappa$, we have

$$\exists\lambda\exists\kappa\exists\lambda a = \exists\lambda\exists\kappa s^\lambda_\mu \exists\lambda a = \exists\lambda\exists s^\lambda_\mu \kappa\exists\lambda a =$$
$$s^\lambda_\mu \exists\kappa\exists\lambda a = \exists\kappa s^\lambda_\mu \exists\lambda a = \exists\kappa\exists\lambda a.$$

Now by axiom (6) and by (b),

$$\exists\lambda\exists\kappa a \leq \exists\lambda\exists\kappa a\exists\lambda a = \exists\kappa\exists\lambda a.$$

Symmetrically, $\exists\kappa\exists\lambda a \leq \exists\lambda\exists\kappa a$. Hence, $\exists\kappa\exists\lambda a = \exists\lambda\exists\kappa a$, and (iii) is proved.

Item (iv) follows from axioms (7) and (8). Indeed, if $\mu \neq \kappa$, then

$$s^{\kappa}_{\lambda}s^{\kappa}_{\mu} = s^{\kappa}_{\lambda}\exists\kappa s^{\kappa}_{\mu} = \exists\kappa s^{\kappa}_{\mu} = s^{\kappa}_{\mu}.$$

Letting $\mu = \lambda$ in (iv) we have

$$s^{\kappa}_{\lambda}s^{\kappa}_{\lambda} = s^{\kappa}_{\lambda}.$$

We omit the proof of (v), referring to the original paper [Pi2]. $\square$

Let B is a QA_I-algebra, and $a \in B$.

Definition 3.3.30. *The support Δa of a is the set of all $\kappa \in I$, such that $\exists\kappa a \neq a$.*

In view of axioms (7) and (8), the set Δa is also the set of all $\kappa \in I$, such that $s^{\kappa}_{\lambda}a \neq a$.

Definition 3.3.31. *A QA_I-algebra B is called locally finite if the support Δa is finite for every $a \in B$.*

As was mentioned in Remark 3.3.28, every I-polyadic algebra is a QA_I-algebra. The converse is true for locally finite QA-algebras of infinite degree.

Theorem 3.3.32. *A locally finite QA_I-algebra of infinite degree is a locally finite polyadic I-algebra.*

Proof. We preface the formal reasoning with a few general observations. Locally finite polyadic X-algebras possess a lot of good properties. One of them is an ability to replace the operations of the general type $\exists(J)$ and s_τ, where J is an arbitrary subset in X and τ is an arbitrary transformation of the set X, by "local" operations $\exists\alpha$ and s^{k}_{λ}. Their prototype is the quantification along a single variable and substitutions of a single variable by another one.

Now our aim is to equip an arbitrary QA_I-algebra with the structure of a locally finite polyadic algebra.

Let B be a locally finite QA_I-algebra and let $J \subset I$. Define $\exists(J)a$ by

$$\exists(J)a = \exists\kappa_1 \ldots \exists\kappa_n a,$$

where $\{\kappa_1, \ldots, \kappa_n\} = J \cap \Delta a$. Since all $\exists\kappa_i$ commute and the product of two commuting quantifiers is again a quantifier, $\exists(J)$ is unambiguously defined and is a quantifier.

A map $\sigma : I \to I$ is called *replacement* if $\sigma(\kappa) = \lambda$ and $\sigma(\mu) = \mu$ for every $\mu \neq \kappa$. This map is denoted (κ/λ).

If J is finite, then for any $\sigma : I \to I$ the restriction of σ to J can be represented as a product of replacements (κ_i/λ_i). So, we define the operation s_σ by

$$s_\sigma a = s_{\lambda_1}^{\kappa_1} \dots s_{\lambda_n}^{\kappa_n} a,$$

where all pairs (κ_i, λ_i) correspond to the restriction of σ to Δa, $a \in B$. In view of (iv) and (v) from Lemma 3.3.29, this definition does not depend on the order of replacements. One can show that this definition is indeed unambiguous in the course of the choice of replacements (see [Gal], [Halm] for the details).

Since $\exists(J)$ is a quantifier, axioms (5)-(7) from Definition 3.3.17 of a polyadic algebra are fulfilled. Axiom (7) of Definition 3.3.27 guarantees that (8) of Definition 3.3.17 holds.

Now let J_1, J_2 be subsets of I. We shall check that $\exists(J_1 \cup J_2) = \exists(J_1)\exists(J_2)$. Indeed,

$$\exists(J_1 \cup J_2)a = \exists\alpha_1 \dots \exists\alpha_k a,$$

where $\{\alpha_1, \dots, \alpha_k\} = (J_1 \cup J_2) \cap \Delta a$, $a \in B$. Rewriting $\{\alpha_1, \dots, \alpha_k\} = (J_1 \cup \Delta a) \cap (J_2 \cup \Delta a)$ and using permutable quantifiers $\exists\alpha_t$, $t = 1, \dots, k$, we have $\exists(J_1 \cup J_2) = \exists(J_1)\exists(J_2)$.

Straightforward check shows that axioms (1)-(4) of a polyadic algebra, which regulate the action of s_σ, follow from the corresponding axioms for the operations s_λ^κ.

It remains to verify the technical axioms (10)-(11) of a polyadic algebra (see Def. 3.3.17). The proof of these axioms is a bit long although the meaning is clear and explained after the definition of the polyadic algebra. In order not to overload the exposition with technicalities we omit it here, and note for the reader that the formal proof is contained in [Gal], Corollary 2 and Lemma 9.

The key point of the proof in [Gal] is Lemma 7 which enables one to study the effect of a finite transformation σ on a finite set J by examining the image of each element λ separately. The method is commonly used in mathematical logic; i.e., mapping the element λ first into another element μ far from the scene of the action, and then mapping μ into $\sigma(\lambda)$. $\square$

3.3.8 Algebraization of the predicate calculus

Let an infinite set of variables $X = \{x_\alpha,\ \alpha \in I\}$ be fixed. The set of formulas, axioms and derivation rules of the predicate calculus over X is given in Subsection 3.3.1. Denote the set of formulas by $\mathfrak{L}_P(X)$.

We view $\mathfrak{L}_P(X)$ as the absolutely free algebra over atomic formulas with respect to signature $\{\vee,\ \neg,\ \exists x_\alpha\}$. Since atomic formulas are just variables, $\mathfrak{L}_P(X)$ is the absolutely free algebra over X.

For the aims of applications it is convenient to treat $\mathfrak{L}_P(X)$ as an algebra in the extended signature $\{\vee,\ \wedge,\ \neg,\ \exists x_\alpha,\ \forall x_\alpha\}$.

As usual, denote by τ the Lindenbaum-Tarski equivalence on the set of formulas $\mathfrak{L}_P(X)$. So, $u\tau v$ if and only if

$$\vdash (u \to v) \wedge (v \to u).$$

It was mentioned in Proposition 3.1.15 that the equivalence τ is a congruence for every first-order theory. This fact is a key point first used by Tarski ([Ta], [Ta1]) to establish an algebraic relation between propositional calculus and Boolean algebras (see also [FJP] and references therein).

Let us check the implications:

1. $(u_1\tau v_1) \wedge (u_2\tau v_2) \vdash (u_1 \vee u_2)\tau(v_1 \vee v_2)$,
2. $u\tau v \vdash (\neg u)\tau(\neg v)$,
3. $u\tau v \vdash (\exists x_\alpha u)\tau(\exists x_\alpha v)$,

Items (1)-(2) have been checked in Proposition 3.2.18. Suppose now $u\tau v$, that is $\vdash (u \to v) \wedge (v \to u)$. So, $\vdash (u \to v)$. Since v implies $\exists x_\alpha v$ we have $\vdash (u \to \exists x_\alpha v)$. By generalization rule $\vdash \forall x_\alpha(u \to \exists x_\alpha v)$. The latter implies $\exists x_\alpha u \to \exists x_\alpha v$. Since we have symmetrically $\vdash (v \to u)$, the property $u\tau v \vdash (\exists x_\alpha u)\tau(\exists x_\alpha v)$ follows.

So, on the base of $\mathfrak{L}_P(X)$ one can construct the quotient algebra $\mathfrak{L}_P(X)/\tau$ which can be viewed as the algebra of formulas.

The algebra $\mathfrak{L}_P(X)/\tau$ possesses the following property.

Theorem 3.3.33. *The algebra $\mathfrak{L}_P(X)/\tau$ is a locally finite X-polyadic algebra.*

Proof. Let us show that the algebra $\mathfrak{L}_P(X)/\tau$ has a structure of QA_X-algebra.

Proposition 3.2.19 implies that $\mathfrak{L}_P(X)/\tau$ is a Boolean algebra.

We shall define the operations $\exists\alpha$ and s^κ_λ. Given $\alpha \in I$ and $u \in \mathfrak{L}_P(X)$, we set

$$\exists\alpha u = \exists x_\alpha u,$$

and

$$s^\kappa_\lambda u = \begin{cases} u[x_\kappa/x_\lambda], & \text{if the substitution } [x_\kappa/x_\lambda] \text{ is proper,} \\ u, & \text{otherwise.} \end{cases}$$

Define the action of $\exists x_\alpha$ and s^κ_λ on the Boolean algebra $\mathfrak{L}_P(X)/\tau$ by

$$\exists x_\alpha [u]_\tau = [\exists x_\alpha u]_\tau,$$

$$s^\kappa_\lambda [u]_\tau = [s^\kappa_\lambda u]_\tau,$$

where $[\ \ ]_\tau$ stands for the τ-equivalence class. The first definition is correct. Indeed, suppose $u_1 \tau u_2$. If $u_1 \to u_2$, then $u_1 \to \exists x_\alpha u_2$ since $u_2 \to \exists x_\alpha u_2$ is an axiom (see axiom (6) on page 115). Then, by the second rule of Bernays: $\exists x_\alpha u_1 \to \exists x_\alpha u_2$. By symmetry, $\exists x_\alpha u_1 \tau \exists x_\alpha u_2$. The second definition is also correct, since $u_1 \tau u_2$ implies $su_1 \tau su_2$.

So, we shall check that axioms (1)–(9) from Definition 3.3.27 are fulfilled on $\mathfrak{L}_P(X)/\tau$.

The substitution rules (1)–(3) from Subsection 3.3.1 give rise to axioms (1)–(4) from Definition 3.3.27. Axiom (5) is the logical equivalence of formulas (4) in Proposition 3.3.2.

Now, we shall check that in $\mathfrak{L}_P(X)/\tau$ we have $u \leq \exists x_\alpha u$, i.e., $(u \wedge \exists x_\alpha u)\tau u$. But $((u \wedge \exists x_\alpha u) \to u) \wedge (u \to (u \wedge \exists x_\alpha u))$ holds in view of the axiom $u \to \exists x u$. Thus, axiom (6) holds true.

As it was said before, axiom (8) expresses the well-known in the first-order logic fact that once a variable has been replaced by another one, a quantification on the replaced variable has no further effect. Analogously, axiom (9) means that the replacement of a quantified variable by another one changes nothing. Both of these substitution operations follow from rules (4) and (5) postulated in Subsection 3.3.1. Thus, $\mathfrak{L}_P(X)/\tau$ is a QA_X-algebra.

By the definition of a first-order formula each $u \in \mathfrak{L}_P(X)$ has a finite support and as we have seen $u \in \mathfrak{L}_P(X)/\tau$ is a locally finite QA_X-algebra.

According to Theorem 3.3.32, $\mathfrak{L}_P(X)/\tau$ is a locally finite polyadic algebra. $\square$

3.3.9 Ideals and filters of polyadic algebras

Let H_1 and H_2 be polyadic X-algebras with the acting semigroup S. We will consider H_1 and H_2 as Boolean algebras in the signature $\{\cdot,\ +,\ ^-,\ 0,\ 1\}$.

A homomorphism of polyadic algebras should respect all operations. Thus, a homomorphism $\mu : H_1 \to H_2$ is a Boolean homomorphism, which additionally is compatible with quantifiers and transformations:

$$\mu(\exists(J)a) = \exists(J)\mu(a), \quad a \in H,\ J \subset X;$$

$$\mu(sa) = s\mu(a), \quad a \in H, \ s \in S.$$

Let δ be the congruence of H determined by this homomorphism. Just as in the case of Boolean algebras, we consider the equivalence classes of 0 and 1, respectively.

Definition 3.3.34. *A subset U of a polyadic X-algebra H is called a polyadic ideal if*

- *U is an ideal of the Boolean algebra H;*
- *$J \subseteq X$ and $a \in U$, then $\exists(J)a \in U$;*
- *if $a \in U$ and $s \in S$, then $sa \in U$.*

Dually,

Definition 3.3.35. *A subset F of a polyadic X-algebra H is called* a polyadic filter *if*

- *F is a filter of the Boolean algebra H;*
- *if $J \subseteq X$ and $a \in F$, then $\forall(J)a \in F$;*
- *$a \in F$ and $s \in S$, then $sa \in F$.*

Straightforward check shows that the δ-equivalence class of 0 is an ideal, and the class of 1 is a filter.

Moreover, polyadic ideals (correspondingly, filters) are in one-to-one correspondence with congruences of polyadic algebras. Indeed, every ideal U of H is also a Boolean ideal. It defines a Boolean congruence δ:

$$a\delta b \Leftrightarrow a\bar{b} + \bar{a}b \in U.$$

Let $a\delta b$, $a, b \in H$. We have

$$(sa)(\overline{sb}) + (\overline{sa})(sb) = s(a\bar{b} + \bar{a}b) \in U$$

providing $a\bar{b} + \bar{a}b \in U$. Thus, $a\delta b$ implies $(sa)\delta(sb)$.

It remains to check that $a\delta b$ yields $\exists(J)a\delta\exists(J)b$. Indeed,

$$\exists(J)a \cdot \overline{\exists(J)b} + \overline{\exists(J)a} \cdot \exists(J)b = \exists(J)a \cdot \forall(J)\bar{b} + \forall(J)\bar{a} \cdot \exists(J)b.$$

Using the equality $\exists(a\forall b) = \exists a \cdot \forall b$ (see Proposition 3.3.5), we obtain

$$\exists(J)a \cdot \forall(J)\bar{b} + \forall(J)\bar{a} \cdot \exists(J)b = \exists(J)(a \cdot \forall(J)\bar{b}) + \exists(J)(\forall(J)\bar{a} \cdot b).$$

Since $\forall(J)\bar{b} \leq \bar{b}$ then $a \cdot \forall(J)\bar{b} \leq a \cdot \bar{b}$. The map $\exists$ is monotonic, so

$$\exists(J)(a \cdot \forall(J)\bar{b}) + \exists(J)(\forall(J)\bar{a} \cdot b) \leq \exists(J)(a\bar{b}) + \exists(J)(\bar{a}b).$$

Thus,

$$\exists(J)(a\bar{b}) + \exists(J)(\bar{a}b) = \exists(J)(a\bar{b} + \bar{a}b) \in U$$

by properties of quantifiers (see Proposition 3.3.5).

So, all congruences on polyadic algebras and, correspondingly, the kernels of homomorphisms are represented by the ideals U or filters $F = \overline{U}$.

Proposition 3.3.36. *A subset U of a polyadic X-algebra H is a polyadic ideal if and only if U is an ideal of the Boolean algebra H and $\exists(X)a \in U$ for every $a \in U$. A subset F is a polyadic filter of H if and only if F is a Boolean filter of H and $\forall(X)a \in F$ whenever $a \in F$.*

Proof. The latter assertion easily follows from the first one if we take into account the connection between ideals and filters. We shall prove the first statement.

If U is a polyadic ideal then by definition U is a Boolean ideal and $\exists(X)a$ belongs to U for all $a \in U$.

Now we assume that U is an ideal of the Boolean algebra H and that $\exists(X)a \in U$ for every $a \in U$. If $J \subseteq X$, then

$$\exists(J)a \leq \exists(X)(\exists(J)a) = \exists(X \cup J)a = \exists(X)a.$$

So, U contains $\exists(J)a$ for every $J \subseteq X$.

Now we verify that $sa \in U$, if $a \in U$ and $s \in S$. Since $a \leq \exists(X)a$ and s is a Boolean homomorphism, then $sa \leq s\exists(X)a$. Furthermore, according to axioms (1) and (2) from Definition 3.3.16,

$$s\exists(X)a = s_{id}\exists(X)a = \exists(X)a,$$

since $s = s_{id}$ outside X. Hence, $sa \leq \exists(X)a$ and $sa \in U$. □

Now we look for the rule which describes the polyadic filter $F(T)$ generated by a set of elements T in H.

Let H be a polyadic X-algebra and T be a subset in H. Let T_1 be the set of elements of the form $\forall(X)a$, where $a \in T$. Denote by $F_b(T_1)$ the Boolean filter in H over T_1.

Proposition 3.3.37. $F_b(T_1)$ *is a polyadic filter and it coincides with the polyadic filter $F(T)$.*

Proof. Let us check that $F_b(T_1)$ is a polyadic filter in H. Since $F_b(T_1)$ is a Boolean filter generated by T_1, by Proposition 3.2.11 every element of $F_b(T_1)$ has the form

$$\forall(X)a_1 \ldots \forall(X)a_n + b,$$

where $a_i \in T$, $b \in H$. Now

$$\forall(X)(\forall(X)a_1 \dots \forall(X)a_n + b) = \forall(X)a_1 \dots \forall(X)a_n + \forall(X)b.$$

So, $F_b(T_1)$ is closed under $\forall(X)$ and is a polyadic filter by Proposition 3.3.36.

Show that $F_b(T_1) = F(T)$. Since $\forall(X)a \leq a$, the filter $F_b(T_1)$ contains T and $F(T) \subseteq F_b(T_1)$. On the other hand, by Definition 3.3.35, $T_1 \subseteq F(T)$, thus $F_b(T_1) \subseteq F(T)$. $\square$

Analogously, denote by T_2 the set of elements of the form $\exists(X)a$, where $a \in T$, and by $U_b(T_2)$ the Boolean filter in H over T_2. Then $U_b(T_2)$ is a polyadic ideal which coincides with the polyadic ideal $U(T)$ generated by T.

Define an element $a \in H$ to be *derivable from the set* T if there exists a sequence $a_0, a_1, \dots, a_i, \dots a_n = a$ such that $a_0 \in T$, and either $a_i \in T$, or $a_i = \forall(J)a_k$, $k < i$, $J \subset X$, or there exist a_k, a_l, $k, l < i$ with $a_l = a_k \to a_i$.

A straightforward computation shows that $F(T)$ is exactly the set of all elements of H derivable from T. Hence, the problems of derivability in the predicate calculus are translated in the corresponding polyadic algebra to the problems about the filters generated by some set of elements.

A polyadic ideal U of a polyadic algebra H is said to be *maximal* if U is distinct from H and is included in no other proper polyadic ideal of this algebra. The ideal U is maximal if and only if the quotient algebra H/U is simple. A polyadic filter F of H is *a maximal filter (an ultrafilter)* if F is distinct from H and is included in no other proper polyadic filter of this algebra.

By Proposition 3.2.7, a Boolean ideal (filter) is maximal if it contains either an element of the algebra H or its negation. The conclusion of Proposition 3.2.7 remains true for polyadic ideals and filters with the following modification. An element a of a polyadic X-algebra is said to be *closed* if $\exists(X)a = a$.

Proposition 3.3.38 ([Halm]). *The set B of closed elements in a polyadic algebra H is a Boolean subalgebra. The polyadic ideal U of H is maximal if and only if the Boolean ideal $B \bigcap U$ is maximal in H.* $\square$

So, the conclusion of Proposition 3.2.7 remains true for polyadic ideals if we replace an arbitrary $a \in H$ by a closed a. The similar result holds for the maximal filters.

3.3.10 Some facts on structure of polyadic algebras

Definition 3.3.39. *A polyadic algebra is called simple if it does not have non-trivial polyadic ideals.*

According to Theorem 3.2.8 there is only one simple Boolean algebra, the two element algebra $\mathbf{2} = \{0, 1\}$. This is not the case for polyadic algebras.

Theorem 3.3.40 ([Halm],[DM], [Pl-Datab])**.** *A polyadic algebra H is simple if and only if H is isomorphic to a polyadic power algebra $\mathcal{P}(D)$ or its subalgebra.* □

Definition 3.3.41. *A polyadic algebra is called semisimple if the intersection of all its maximal polyadic ideals is zero.*

The next theorem is an analogue of Theorem 3.2.9 for Boolean algebras [Halm].

Theorem 3.3.42. *Every polyadic algebra is semisimple.* □

Theorem 3.3.42 and Theorem 3.3.40 describe the structure of an arbitrary polyadic algebra. They provide a polyadic counter-part of the P.M.Stone's Boolean structural theorem 1.1.12.

3.3.11 Consistency and compactness in polyadic algebras

In accordance with the notion of consistency for first-order theories we call a subset T of H *consistent* if it generates a proper polyadic filter $F(T)$. In other words T is consistent if for every $a \in H$ which is derivable from T, the element $\bar{a}$ is not derivable, because otherwise T contains zero and coincides with H. In view of duality, one can state that T is consistent if it generates a proper polyadic ideal in H.

This definition of consistency is none ether then an algebraic counterpart of the syntactical consistency, which basically claims that in a consistent theory no two contradictory statements are both derivable.

A subset T of H is called *complete* if it generates either the whole H or an ultrafilter $F(T)$ (or a maximal ideal $U(T)$). This notion is an algebraic counterpart of the notion of a complete theory in logic. If $F(T)$ is the whole H, then the set T and, respectively, the theory T, is inconsistent. For consistent theories we have

Proposition 3.3.43. *A consistent set T is complete if and only if the quotient polyadic algebra $H/U(T)$ is simple.* □

In view of Proposition 3.3.38, a consistent set T is complete if and only if for each closed $a \in H$, either a or $\bar{a}$ belongs to $F(T)$. This conclusion fully agrees with the well-known fact that in a first-order theory for each sentence u, either u or its negation is checkable.

We finish this section with one more quote from P.Halmos: *" We see thus that the celebrated Gödel's incompleteness theorem asserts that certain important polyadic logics are either incomplete or inconsistent. In other words, if (H, U) is one of those logics, then either the ideal U is very large $(H = U)$, or else it is rather small (non-maximal)"*. Here H is a polyadic algebra and U is an ideal in H.

3.3.12 Polyadic algebras with equality and cylindric algebras

Up to now we dealt with predicate calculus without relation symbols. Once we intend to apply algebraic logic to geometric and algebraic problems, we need to build a version of the logical calculus with an equality predicate. In particular, the logic with equalities is indispensable for the study of solutions of systems of equations. Thus, we are interested in algebraizations of the first-order logic with equality predicate.

We use for the equality predicate the symbol $\equiv$, which leaves among the symbols of relations of a first-order logic. Note that we can also use the equality symbol $E(\ ,\ )$ instead of $\equiv$, whenever this notation is more convenient and underlines the binary nature of this predicate. The equality predicate expresses a binary relation which satisfies some set of axioms. We assume that in each interpretation this symbol is interpreted as a coincidence of elements, which means that we consider the so-called normal models.

Under equality in a first-order logic we mean a binary predicate $\equiv$ which satisfies the following scheme of axioms:

1. Reflexivity: $\forall x(x \equiv x)$, where x is a variable.
2. Substitution law:

$$(x \equiv y) \to (u(\dots, x, \dots) \to u(\dots, y, \dots)),$$

   where x, y are variables, and $u(\dots, y, \dots)$ is a formula obtained by replacing any number of free occurrences of x in u with y, such that all these remain free occurrences of y.

These axioms imply the well-known properties of the equality predicate: *symmetry* and *transitivity*.

Now we can add symbol $\equiv$ to the set of relations, axioms (1)-(2) to the set of axioms from Subsection 3.3.1, and consider the absolutely free algebra over elements of the form $x_j \equiv x_j$ in the signature $\neg$, $\vee$, $\exists$.

Thus, let $X = \{x_\alpha,\ \alpha \in I\}$ be a set and $\mathfrak{L}(X)$ be an absolutely free algebra with free generators of the form $x_\alpha \equiv x_\beta$, $x_\alpha, x_\beta \in X$ with operations $\vee$, $\neg$, $\exists x_\alpha$. As before, one can define the Lindenbaum-Tarski equivalence relation τ on $\mathfrak{L}(X)$ (see Subsection 3.1.3).

The quotient algebra $\mathfrak{L}(X)/\tau$ is *a cylindric algebra* (see [HMT] for details and for the axiomatic definition of a cylindric algebra).

Shortly speaking, a cylindric X-algebra is a Boolean algebra equipped with the commuting quantifiers of the form $\exists\alpha$, $\alpha \in I$ and with the special elements (equalities) of the form $e(\lambda, \kappa)$. In the Lindenbaum-Tarski algebraization process the distinguished elements $e(\lambda, \kappa)$ are taken to be the equivalence classes of the formulas $x_\kappa \equiv x_\lambda$. The elements $e(\lambda, \kappa)$ satisfy the following axioms:

1. $e(\kappa, \kappa) = 1$.
2. $e(\lambda, \mu) = \exists\kappa(e(\lambda, \kappa) \wedge e(\kappa, \mu))$ if $\kappa \neq \lambda, \mu; a \in B$.
3. $\exists\kappa(e(\kappa, \lambda) \wedge a) \wedge \exists\kappa(e(\kappa, \lambda) \wedge \bar{a}) = 0$ if $\kappa \neq \lambda; a \in B$.

Now we return to polyadic algebras. One can define also the equality predicate on a polyadic algebra.

Let H be a polyadic X-algebra and S be the semigroup of all transformations of X. A predicate of a polyadic algebra is defined as follows (cf. [Halm]).

Definition 3.3.44. *An n-ary predicate of a polyadic X-algebra H is a function P from X^n into H such that if $(x_{\alpha_1}, \ldots, x_{\alpha_n}) \in X^n$ and $s \in S$, then*

$$sP(x_{\alpha_1}, \ldots, x_{\alpha_n}) = P(sx_{\alpha_1}, \ldots, sx_{\alpha_n}).$$

Definition 3.3.45. *A binary predicate E is reflexive if $E(x_\lambda, x_\lambda) = 1$ for every $x_\lambda \in X$.*

Definition 3.3.46. *A binary predicate E of H is substitutive if for every $x_\kappa, x_\lambda \in X$ and $a \in H$*

$$a \wedge E(x_\kappa, x_\lambda) \leq s^\kappa_\lambda a,$$

where s^κ_λ is a transformation on X such that $s^\kappa_\lambda x_\lambda = x_\kappa$, $s^\kappa_\lambda x_\mu = x_\mu$ for $\mu \neq \lambda$.

Definition 3.3.47. *An equality for a polyadic algebra H is a reflexive and substitutive binary predicate of H.*

Remark 3.3.48. *We reserve a free use of the notation $\exists x_\alpha = \exists\alpha$ and $E(x_\alpha, x_\beta) = E(\alpha, \beta)$, where x_α, $x_\beta \in X$, α, $\beta \in I$ in the appropriate places in the sequel.*

Not every polyadic X-algebra can be equipped with an equality. For examples of polyadic algebras with equality and without equality see [Halm]. However, if a polyadic algebra has an equality then it is unique.

Definition 3.3.49. *A polyadic X-algebra with equality is a polyadic X-algebra equipped with an equality predicate.*

The equality in a polyadic algebra enjoys a number of useful properties. The most known of them are

1. Symmetry: $E(x_\alpha, x_\beta) = E(x_\beta, x_\alpha)$.
2. Transitivity: $E(x_\alpha, x_\beta) \wedge E(x_\beta, x_\kappa) \leq E(x_\alpha, x_\kappa)$.

Let us quote some other properties of the equality in polyadic algebras (for proofs see [Halm], [Pi2], [Pl-Datab]).

Proposition 3.3.50. *Let B be a polyadic X-algebra with equality. Then*

i. $a \wedge E(x_\kappa, x_\lambda) = s_\lambda^\kappa a \wedge E(x_\kappa, x_\lambda)$.

ii. $\exists x_\kappa E(x_\kappa, x_\lambda) = 1$.

iii. $s_\lambda^\kappa a = \exists x_\kappa (a \wedge E(x_\kappa, x_\lambda))$ *if* $\kappa \neq \lambda$.

iv. $E(x_\kappa, x_\lambda) = \min\{a \mid s_\lambda^\kappa a = 1\}$.

v. $s_\lambda^\kappa E(x_\kappa, x_\lambda) = 1$.

vi. $s_\nu^\kappa E(x_\lambda, x_\mu) = E(x_\lambda, x_\mu)$ *if* $\kappa \neq \lambda, \mu$.

vii. $s_\mu^\kappa E(x_\kappa, x_\lambda) = E(x_\mu, x_\lambda)$ *if* $\kappa \neq \lambda, \mu$.

viii. $\exists x_\kappa (E(x_\lambda, x_\kappa)) \wedge E(x_\kappa, x_\mu) = E(x_\lambda, x_\mu)$ *if* $\kappa \neq \lambda, \mu$,

where $a \in B$, $\kappa, \lambda, \mu, \nu \in I$.

Any element of the form $E(x_\alpha, x_\beta)$ may be treated as an additional nullary operation on a polyadic algebra, and then the homomorphisms of polyadic algebras with equality predicate must be compatible with them. This means that if H and H' are two polyadic X-algebras with equality predicates E and E', respectively, then a homomorphism $\mu\colon H \to H'$ is subject to condition: $E(x_\alpha, x_\beta)^\mu = E'(x_\alpha, x_\beta)$ for any $x_\alpha, x_\beta \in X$. Similarly, subalgebras must be closed with respect to these elements: if H' is a subalgebra in H, then $E(x_\alpha, x_\beta)$ always must belong to H'.

Let us turn to examples.

Example 3.3.51. We begin with algebras of subsets of Cartesian products, i.e., algebras of the form $\mathcal{P}(D)$, where D is the Cartesian product of the sets D_α, α runs the set I. As we know, $\mathcal{P}(D)$ is a polyadic I-algebra.

The equality predicate E on $\mathcal{P}(D)$ is defined by the condition

$$E(\alpha, \beta) = D_{\alpha\beta},$$

where $D_{\alpha\beta}$ is a diagonal, namely, the set of those $a \in D$ such that $a(\alpha) = a(\beta)$ for all $\alpha, \beta \in I$. The predicate E satisfies the condition to be a predicate on a polyadic algebra and it is reflexive.

Let us check that it is substitutive. Indeed, if A is a subset of D and $a \in A \cap D_{\alpha\beta}$, then $as^\alpha_\beta = a$ and $a \in s^\alpha_\beta A$, where s^α_β is a transformation on I such that $s^\alpha_\beta \beta = \alpha$, $s^\alpha_\beta \mu = \mu$ for $\mu \neq \beta$. Thus, $A \cap D_{\alpha\beta} \subset s^\alpha_\beta A$.

So, $E = D_{\alpha\beta}$ is an equality predicate on the algebra $\mathcal{P}(D)$.

Example 3.3.52. We have defined already the algebra of a first-order calculus with equality predicate $\mathfrak{L}(X)/\tau$ (see page 147). One can consider this algebra as a polyadic X-algebra. Then

$$E(x_\alpha, x_\beta) = [x_\alpha \equiv x_\beta]_\tau,$$

where $[\ \ ]_\tau$ is a τ-equivalence class.

It is intuitively clear that polyadic algebras with equality and cylindric algebras express the same essence, and that there should be a passage back and forth between them. This is indeed the case, if we confine ourselves with the locally finite algebras over an infinite X. The passage is explicit and gives rise to an appropriate construction according to particular needs and, in a sense, according to a particular taste. Throughout the book we use polyadic algebras with equalities as an instrument which allows us to make algebraic and logical geometry more transparent.

Pursuing this goal, in Part II we will generalize polyadic algebras with equality up to multi-sorted Halmos algebras.

Remark 3.3.53. *Two last examples give us semantical and syntactical approaches.*

These approaches are connected to the value homomorphism

$$Val : \mathfrak{L}(X)/\tau \to \mathcal{P}(D),$$

defined as follows:

$$Val([x_\alpha \equiv x_\beta]_\tau) = D_{\alpha\beta}.$$

Since the algebra $\mathfrak{L}(X)$ *is generated by the elements of the form* $x_\alpha \equiv x_\beta$*, we can define the image of all elements from* $\mathfrak{L}(X)/\tau$*.*

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