Algebraic geometry in varieties of algebras with the given algebra of constants

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To Alexey Ivanovich Kostrikin in occasion of his 70th birthday with very best wishes

Abstract. It was proven in [Pl1] that to each variety of algebras Θ and each algebra $G \in \Theta$ a special Θ-algebraic geometry over G is associated. Classic algebraic geometry is based, by definition, on the variety of associative commutative algebras with identity element over a given field P. This variety we call classic (over P) and denote $Var - P$.

Let, Θ be the variety of associative commutative rings with identity element, and a field P be an algebra in Θ. Consider a new variety $\Theta(P)$, whose objects have the form $h: P \to H$, where $H \in \Theta$ and h is a morphism in Θ. Here, H is an extension of the field P defined by the injection h. The field P plays the role of a constant field. It is easy to see that $Var - P$ coincides with $\Theta(P)$.

One can proceed in the similar way from the arbitrary variety of algebras Θ and distinguish an algebra G to be an algebra of constants. We come to a new variety $\Theta(G)$. Elements of G are additional nullary operations, which may be not included in the signature of the variety Θ.

Algebraic geometry in such $\Theta(G)$ is the subject of this paper. The main results of the paper are the theorems 7, 11, 14.

Contents:

§1. Universal algebraic geometry

1. Basic notions Let Θ be a variety of algebras, X a finite set and $W = W(X)$ the free in Θ algebra over X. Consider an arbitrary algebra $G \in \Theta$. The set $Hom(W,G)$ is viewed as an affine space whose points are homomorphisms.

Take a set of points A in $Hom(W,G)$ and a binary relation T in W. The Galois correspondence between such A and T is defined by the rule

$$A = T' = \{\mu | T \subset \mathrm{Ker}\mu\}$$

$$T = A' = \bigcap_{\mu \in A} \mathrm{Ker}\mu$$

Here, $\mathrm{Ker}\mu$ denotes the kernel of a homomorphism $\mu: W \to G$. This kernel is a congruence in W. Let us look at $\mathrm{Ker}\mu$ from the different point of view.

Given free algebra W, consider all formulas of the type $w \equiv w', w, w' \in W$. For every $G \in \Theta$, each formula of such type can be treated as an equation, which is denoted by $w = w'$. Every solution of an equation $w = w'$ is a homomorphism $\mu: W \to G$ (a point in $Hom(W,G)$), such that $w^\mu = w'^\mu$ in G. Thus, $\mathrm{Ker}\mu$ can be viewed as a collection of all equations $w = w'$, for which μ is a solution. In the notations above we have

$$\mathrm{Ker}\mu = \{\mu\}'.$$

Binary relation T also can be treated as a system of equations. Having the Galois correspondence, one can define the Galois closure.

A closed set A, $A = T'$, is called an *affine algebraic variety over the algebra G*. Closed relation T, $T = A'$, is a G-closed congruence in W. For every A its closure is $A'' = (A')'$, and for every T we have $T'' = (T')'$. If T is a congruence in W, then the universal Hilbert Nullstelensatz [Pl3] gives the relation between T and T''.

2. Lattices and categories of algebraic varieties

Intersection $A \cap B$ of algebraic varieties A and B is also an algebraic variety. Union $A \cup B$ of algebraic varieties A and B is not necessarily an algebraic variety. If $A = T_1'$, $B = T_2'$, then $A \cup B \subset (T_1 \cap T_2)'$. Definition 1 An algebra $G \in \Theta$ is called stable in Θ, if for every $W = W(X)$ and every T_1 and T_2 there is the equality

$$(T_1 \cap T_2)' = T_1' \cup T_2',$$

where $'$ is taken on G, T_1 and T_2 are G-closed relations in W.

This means, that G is stable, if the union of algebraic varieties over G is an algebraic variety.

If G is stable then every affine space $Hom(W, G)$ can be treated as a topological space in Zarisky topology. In this topology the closed sets are algebraic varieties.

For every algebra G and every $W = W(X)$ denote by

$$Alv_G(W)$$

the set of all algebraic varieties in $Hom(W, G)$. The set $Alv_G(W)$ can be considered as a lattice, where the union $A\bar{\cup}B$ of two varieties A and B is defined by

$$A\bar{\cup}B = (A \cup B)''.$$

The lattice $Cl_G(W)$ of all G-closed congruences in W is defined in a similar way. The lattices $Alv_G(W)$ and $Cl_G(W)$ are antiisomorphic. Both of them are distributive if G is stable.

Algebras G_1 and G_2 are called *equivalent*, if for every $W = W(X)$,

$$Cl_{G_1}(W) = Cl_{G_2}(W).$$

For every variety of algebras Θ denote by Θ^0 the category of all free in Θ algebras $W = W(X)$ with finite X. This is a full subcategory in Θ. Then we can reformulate the definition above, introducing the functors:

$$Alv_G\colon \Theta^0 \to Set, \text{ and } Cl_G\colon \Theta^0 \to Set.$$

Algebras G_1 and G_2 are equivalent if the functors Cl_{G_1} and Cl_{G_2} coincide.

If G_1 and G_2 are equivalent, then $Var\, G_1 = Var\, G_2$.

Algebras G_1 and G_2 are called *(geometrically) similar*, if there exists an isomorphism of categories

$$\varphi\colon (Var\, G_1)^0 \to (Var\, G_2)^0,$$

which induces the isomorphism of the corresponding functors Cl_{G_1} and $Cl_{G_2}\varphi$. If $Var\, G_1 = Var\, G_2$ and $\varphi = 1$, then G_1 and G_2 are equivalent.

Now consider the categories $K_\Theta(G)$ and $\mathbb{C}_\Theta(G)$. The second category is the full subcategory of Θ, whose objects are algebras of the form $W(X)/T$, where T is a G-closed congruence in W. Objects of $K_\Theta(G)$ are algebraic varieties (A, X). Here X shows that A is a variety in the space $Hom(W(X), G)$. Morphisms in $K_\Theta(G)$ have the form

$$(\bar{s}, s)\colon (A, X) \to (B, Y),$$

where

$$s\colon W(Y) \to W(X)$$

is a morphism in the category Θ^0, $\bar{s}\colon A \to B$ is induced by s. We have the following commutative diagram

$$\begin{tikzcd} W(Y) \arrow[r, "s"] \arrow[d, "\mu_Y"'] & W(X) \arrow[d, "\mu_X"] \\ W(Y)/B' \arrow[r, "\bar{s}"] & W(X)/A' \end{tikzcd}$$

Here, μ_Y, μ_X are natural homomorphisms and $\bar{s}$ is a morphism in $\mathbb{C}_\Theta(G)$, which is dual to $\bar{s}\colon A \to B$. The categories $K_\Theta(G)$ and $\mathbb{C}_\Theta(G)$ are dually isomorphic by the transition

$$(A, X) \to W(X)/A'.$$

Algebras G_1 and G_2 are called *categorically equivalent*, if the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are isomorphic. One of the main problems is to study relations between equivalence, similarity and categorical equivalence.

In the classical case for the ground field P and its extension L the corresponding category of algebraic varieties is denoted by $K_P(L)$. See also [Hart], [Scha], [SZ].

3. Some information

The category $K_\Theta(G)$ is a geometrical invariant of the algebra G. This invariant "measures" possibilities of G in respect to solution in it systems of equations of the form $w = w'$. In other words, we can consider the category $K_\Theta(G)$ as a measure of "algebraic closedness" of the algebra G. Algebras G_1 and G_2 have equal possibilities if $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are isomorphic, i.e. G_1 and G_2 are categorically equivalent. It is easy to see that if algebras G_1 and G_2 are equivalent, then they are categorically equivalent, i.e. they have equal possibilities in the sense above. Besides, if G_1 and G_2 are equivalent, then for every $W = W(X)$ the lattices $Alv_{G_1}(W)$ and $Alv_{G_2}(W)$ are isomorphic. In particular, if one of the algebras, say, G_1 is stable, then both the lattices $Alv_{G_1}(W)$ and $Alv_{G_2}(W)$ are distributive. However, we cannot state that if G_1 and G_2 are equivalent, and G_1 is stable, then G_2 is stable (see Section 3).

As we have noticed, equivalence of algebras G_1 and G_2 implies their similarity. Let us go back to isomorphism of the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$. An isomorphism of these categories is called correct isomorphism, if it deals with both the components $\bar{s}$ and s in the definition of categorical morphisms. Not every isomorphism is correct. The following theorem ([Pl2]) takes place.

Categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are correctly isomorphic if and only if the algebras G_1 and G_2 are similar.

In particular, similarity implies categorical equivalence. In some cases the notions of similarity and equivalence coincide.

Let us note, at last, that if G is stable, then the functor Alv_G is a functor from the category Θ^0 to the category of distributive lattices.

§2. Variety of algebras with the given algebra of constants

1. Definitions Fix a variety of algebras Θ. Consider the category $\tilde{\Theta}$, whose objects are morphisms in Θ of the form $h: G \to H$. We call h embedding, while h is not necessarily an injection. Morphisms in $\tilde{\Theta}$ are represented by diagrams

$$\begin{tikzcd} G_1 \arrow[r, "h_1"] \arrow[d, "\sigma"'] & H_1 \arrow[d, "\mu"] \\ G_2 \arrow[r, "h_2"] & H_2 \end{tikzcd}$$

Morphisms in $\tilde{\Theta}$ we consider also as pairs (σ, μ) with pairwise multiplication. This rule follows from the commutativity of the diagrams

$$\begin{tikzcd} G_1 \arrow[r, "h_1"] \arrow[d, "\sigma_1"'] & H_1 \arrow[d, "\mu_1"] \\ G_2 \arrow[r, "h_2"] \arrow[d, "\sigma_2"'] & H_2 \arrow[d, "\mu_2"] \\ G_3 \arrow[r, "h_3"] & H_3 \end{tikzcd}$$

For an arbitrary $G \in \Theta$ consider the full subcategory $\Theta(G)$ in $\tilde{\Theta}$ whose objects are embeddings $h: G \to H$ with the fixed G. The algebra G plays the role of algebra of constants.

The objects of $\Theta(G)$ are called G-algebras or algebras over G. Morphisms in $\Theta(G)$ have the form

$$\begin{tikzcd} G \arrow[r, "h_1"] \arrow[d, "\sigma"'] & H_1 \arrow[d, "\mu"] \\ G \arrow[r, "h_2"] & H_2 \end{tikzcd}$$

where σ is an endomorphism of the constant algebra G.

Morphisms of such form are called semimorphisms of the new category, which is also denoted by $\Theta(G)$. Morphisms in this new $\Theta(G)$ have the form

$$\begin{tikzcd} G \arrow[r, "h_1"] \arrow[dr, "h_2"'] & H_1 \arrow[d, "\mu"] \\ 2 & {} \end{tikzcd}$$

or, what is the same,

$$\begin{tikzcd} G \arrow[r, "h_1"] \arrow[d, "id_G=1"'] & H_1 \arrow[d, "\mu"] \\ G \arrow[r, "h_2"] & H_2 \end{tikzcd}$$

We consider this new $\Theta(G)$ with fixed G.

If the algebra G is defined by generators and relators, the category $\Theta(G)$ can be considered as a variety. Generators are treated as additional nullary operations, while defining relations are added to identities of the variety Θ. The varieties $\Theta(G)$ are different, if G is defined in different way by generators and relators. We will assume that all elements of G are generators.

Given a set of variables X, the free in $\Theta(G)$ algebra $W = W(X)$ is the free in Θ product

$$W = G * W_0(X),$$

where W_0 is the free in Θ algebra over X. The corresponding embedding is

$$i_G : G \to G * W_0.$$

We have also

$$i_{W_0} : W_0 \to G * W_0.$$

For every $\mu_0 : W_0 \to G$ we have $\mu : W \to G$, such that

$$
\begin{tikzcd}
W_0 \arrow[rr, "i_{W_0}"] \arrow[dr, "\mu_0"'] & & G * W_0 \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
G \arrow[rr, "i_G"] \arrow[dr, "1_G"'] & & G * W_0 \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}
$$

In particular, the embedding $h = i_G$ is an injection.

In what follows we proceed from varieties of the type $\Theta(G)$, and consider algebraic geometry in a variety $\Theta(G)$ over a G-algebra H from $\Theta(G)$.

2. Additional remarks on varieties $\Theta(G)$

Let the homomorphism

$$
\begin{tikzcd}
G \arrow[r, "h_1"] \arrow[dr, "h_2"'] & H_1 \arrow[d, "\mu"] \\
& H_2
\end{tikzcd}
$$

be given and $\mathrm{Ker}\mu = T$ be its kernel in H_1. Consider the homomorphism $h_0 = h_1\mu_0 \colon G \to H_1/T$ defined by

$$G \xrightarrow{h_1} H_1 \xrightarrow{\mu_0} H_1/T.$$

Such h_0 is considered as a factor embedding by the congruence T, which assumed to be a congruence of G-algebra H_1. We have

$$
\begin{tikzcd}
G \arrow[r, "h_1"] \arrow[dr, "h_2"'] & H_1 \arrow[r, "\mu_0"] \arrow[d, "\mu"] & H_1/T \arrow[dl, "\mu_1"] \\
& H_2 &
\end{tikzcd}
$$

with the injection μ_1. This gives

$$
\begin{tikzcd}
G \arrow[rr, "h_0"] \arrow[dr, "h_2"'] & & H_1/T \arrow[dl, "\mu_1"] \\
& H_2 &
\end{tikzcd}
$$

Thus, congruences of algebras H in Θ are, at the same time, congruences of G-algebras H. It is easy to see, that a subalgebra of G-algebra H with the embedding $h \colon G \longrightarrow H$ is the subalgebra H_1 in H, which contains $Im\, h$. Correspondingly one can define

$$h_1 = h \colon G \longrightarrow H_1.$$

Cartesian and free products in $\Theta(G)$ are constructed naturally. Namely, fix a set I. For every $\alpha \in I$ there is G-algebra $h_\alpha : G \to H_\alpha$, $H_\alpha \in \Theta$. Let $H = \prod\limits_\alpha H_\alpha$ be the Cartesian product in Θ and $\pi_\alpha : H \to H_\alpha$ be the corresponding projections. Define $h: G \to H$ by the rule $g^h(\alpha) = g^{h_\alpha} = g^{h\pi_\alpha}$, $g \in G$. We have the commutative diagram

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "h_\alpha"'] & H \arrow[d, "\pi_\alpha"] \\ & H_\alpha \end{tikzcd}$$

and π_α are morphisms in $\Theta(G)$. Check that we obtained a Cartesian product in the category $\Theta(G)$. Take a G-algebra $f: G \to F$ and let the homomorphisms

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "h_\alpha"'] & F \arrow[d, "\nu_\alpha"] \\ & H_\alpha \end{tikzcd}$$

be given. The following diagram

$$\begin{tikzcd} F \arrow[r, "\nu"] \arrow[dr, "\nu_\alpha"'] & H \arrow[d, "\pi_\alpha"] \\ & H_\alpha \end{tikzcd}$$

takes place in Θ. We want to make this diagram commutative in $\Theta(G)$. It suffices to consider the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\nu"] \arrow[dr] & H \arrow[d, "\pi_\alpha"] \\ G \arrow[u, "f"] \arrow[r, "h_\alpha"'] \arrow[ur] & H_\alpha \end{tikzcd}$$

where $h: G \to H$, $\nu_\alpha : F \to H_\alpha$. Now suppose G_0 is a subalgebra in G. Then the free algebras in $\Theta(G_0)$ and $\Theta(G)$ are $G_0 * W_0(X)$ and $G * W_0(X)$ respectively. The second one can be presented as an amalgamated product $G \underset{G_0}{*} (G_0 * W_0(X))$. In this situation the part G_0 of constants from G is already included in the signature of the variety $\Theta(G_0)$.

3. Examples

Let Θ be the variety of all commutative and associative rings with unit and P be a field. Consider $\Theta(P)$ and $Var - P$. If H is an algebra over the field P, 1 is its unit, then for every $\lambda \in P$ take $\lambda^h = \lambda \cdot 1$. This gives embedding $h: P \longrightarrow H$, where H is considered as $H \in \Theta$. Let now $h: P \longrightarrow H$ be an embedding of rings. For every $\lambda \in P$ and every

$a \in H$ define $\lambda a = \lambda^h \cdot a$. If $a = 1$ then $\lambda 1 = \lambda^h \cdot 1 = \lambda^h$. Note that h is an injection, since 1 is introduced in the signature. One can check that such λa defines a structure of P-algebra on H.

Let now $\mu : H_1 \to H_2$ be a homomorphism of P-algebras. For every $\lambda \in P$ and $a \in H$ we have $(\lambda a)^\mu = \lambda a^\mu$. For $a = 1$ we have $(\lambda \cdot 1)^\mu = \lambda \cdot 1^\mu$, $\lambda^{h_1 \mu} = \lambda^{h_2}$, and thus

$$\begin{tikzcd} P \arrow[r, "h_1"] \arrow[dr, "h_2"'] & H_1 \arrow[d, "\mu"] \\ & H_2 \end{tikzcd}$$

If, on the other hand, the diagram above holds, then $\lambda^{h_1 \mu} = \lambda^{h_2}, \lambda \in P$, and for every $a \in H_1$ we have

$$(\lambda a)^\mu = (\lambda^{h_1} \cdot a)^\mu = \lambda^{h_1 \mu} \cdot a^\mu = \lambda^{h_2} \cdot a^\mu = \lambda a^\mu.$$

Thus, the varieties $\Theta(P)$ and $Var - P$ can be identified.

Let now $\mathbb{Z}[X]$ be the ring of polynomials over $\mathbb{Z}$, which is the free ring in Θ, and $P[X]$ be the algebra of polynomials over P, which is free in $Var - P$. Consider the presentation $P[X] = P * \mathbb{Z}[X]$. We have embeddings $i_G : P \to P * \mathbb{Z}[X]$ and $i_{\mathbb{Z}[X]} = \pi \colon \mathbb{Z}[X] \to P * \mathbb{Z}[X]$. The first one is always injection. If $char P = 0$ then π is injection too. If $char P = p$ then $Ker \pi$ is the ideal I in $\mathbb{Z}[X]$, consisting of all polynomials with the coefficients, dividing p. In this case the free product $P * \mathbb{Z}[X]$ is presented as $P * \mathbb{Z}_p[X]$.

Let us give one remark about semimorphisms in this example.

Consider the commutative diagram

$$\begin{tikzcd} P \arrow[r, "h_1"] \arrow[d, "\sigma"'] & H_1 \arrow[d, "\mu"] \\ P \arrow[r, "h_2"] & H_2 \end{tikzcd}$$

then, for $\lambda \in P$ we have $\lambda^{h_1, \mu} = \lambda^{\sigma h_2}$. Let now $a \in H_1$. Then $(\lambda a)^\mu = (\lambda^{h_1} a)^\mu = \lambda^{h_1, \mu} \cdot a^\mu = \lambda^{\sigma h_2} \cdot a^\mu = \lambda^\sigma a^\mu$. This means that $(\lambda a)^\mu = \lambda^\sigma a^\mu$.

We can also consider the situation when Θ consists of not necessarily commutative rings. In this case in order to get the variety of P-algebras, also not necessarily commutative, we have to take embeddings $h \colon P \to H$, such that $Im(h)$ lies in the center of H. If this condition is not fulfilled, then we get another variety over P with $\lambda(ab) = (\lambda(a))b$ but not necessarily $a(\lambda b)$. In this way one can consider algebras over skew fields.

The notion of G-algebra has been considered in [BMR] and [Pl1,2,3].

§3. Stable algebras

1. Multioperator groups

Every field P is stable in the variety $Var-P$. On the other hand, commutative groups are not stable. Property being stable is connected with the idea of anticommutativity. The last notion can be well formulated for multioperator groups (Ω-groups).

Recall that multioperator group is a group, not necessarily commutative, with the operation written additively, which is endowed, possibly, with some additional signature Ω. Rings, groups and Lie algebras are Ω-groups.

In every Ω-group its zero element is an Ω-subgroup. Congruences of Ω-groups are realized by ideals. In rings these are usual ideals, in groups we get normal subgroups, etc.

The notion of multioperator group is discussed in detail in [Hig], [Ku], [Pl7].

Let G be an Ω-group. If $a, b \in G$ then $[a, b] = -a - b + a + b$. Let w be a n-ary operation from Ω. Then define

$$[a_1, \dots, a_n;\, b_1, \dots,\, b_n;\, w] =$$

$$= -a_1 \dots a_n w - b_1 \dots b_n w + (a_1 + b_1) \dots (a_n + b_n) w,$$

where $a_i, b_i \in G$. If such a commutator is identically zero in G, this means that addition commutes with the operation w.

Ω-group is called abelian if its additive group is abelian and addition commutes with every $w \in \Omega$.

Abelian groups and Lie algebras are abelian in the usual sense, while for associative rings this notion means that product of any two elements is zero.

Ideal H of an Ω-group G is defined as an Ω-subgroup, such that $[a, b] \in H$ for $a \in H$, $b \in G$, and for every n-ary operation $w \in \Omega$ and for every $a_1, \dots, a_n \in H$, $b_1, \dots, b_n \in G$ we have

$$[a_1, \dots, a_n; b_1,\ \dots, b_n;\ w] \in H.$$

2. Anticommutative Ω-groups Definition 2 An Ω-group G is called anticommutative, if the following conditions are fulfilled

1. G does not have nontrivial abelian ideals

2. Every two non-trivial ideals H_1 and H_2 has non-trivial intersection.

Consider this notion for groups, rings and Lie algebras. Theorem 1 [Pl6] A group G is anticommutative if and only if for every two its elements a and $b, (a, b \neq 1)$ there exists c_1, c_2 such that $c_1^{-1}ac_1$ and $c_2^{-1}bc_2$ does not commute. Theorem 2 An associative ring G is anticommutative if and only if for every nonzero elements $a, b \in G$ there exist c_1 and c_1', c_2 and c_2', such that

$$c_1ac_1' \cdot c_2bc_2' \neq 0 \text{ or}$$

$$c_2bc_2' \cdot c_1ac_1' \neq 0.$$

Proof Assume that the condition on every a and b in G holds. Show that G is anticommutative.

Let H be an abelian ideal in G. This means that $ab = 0$ for every $a, b \in H$. Since H is an ideal, elements c_1ac_1', and c_2bc_2' belong to H. Therefore $c_1ac'c_2bc_2' = 0$, $c_2bc_2'c_1ac_1' = 0$ for every nonzero $a, b \in H$, $c_1, c_1', c_2, c_2' \in G$. We get the contradiction, thus H cannot be abelian.

Let now $H_1 \neq 0$, $H_2 \neq 0$, but $H_1 \cap H_2 = 0$. Take $a \in H_1, b \in H_2, a, b \neq 0$. Then $a' = c_1ac_1' \in H_1$, $b' = c_2bc_2' \in H_2$, $a'b' = b'a' = 0$. Contradiction, and $H_1 \cap H_2 \neq 0$.

Conversely, let the ring G be an anticommutative Ω-group. Take $a, b \in G$, $a, b \neq 0$, and let H_1, H_2 be the ideals, generated by a and b respectively.

Every element from H_1 is a sum of elements of the kind $a' = c_1ac_1'$, and every element from H_2 is a sum of elements of the kind $b' = c_2bc_2'$. If all a' and b' pairwise "commute" then commute the elements $h_1 \in H_1$ and $h_2 \in H_2$, $h_1h_2 = h_2h_1 = 0$. Then nontrivial intersection $H_1 \cap H_2$ is abelian, which contradicts the condition. Therefore, there exist a' and b' such that $a'b' \neq 0$ or $b'a' \neq 0$. The theorem is proved.

Theorem 3 A Lie algebra G is anticommutative if and only if for every $a, b \in G, a, b \neq 0$ there exist $a' = [a, c_1, \ldots, c_n]$ and $b' = [b, c_1', \ldots, c_m']$, such that $[a', b'] \neq 0$. Proof In the statement $[a, b]$ denotes Lie commutator of a and b, and $[[a, c_1], \ldots, c_n]$ is denoted by $[a, c_1, \ldots, c_n]$.

Let all elements a, b satisfy the condition and let H be an abelian ideal in G, $a, b \in H$, $a, b \neq 0$. All a' and b' lie in H, $[a', b'] = 0$. Contradiction. Let H_1, H_2 be non-trivial ideals in G, and $H_1 \cap H_2 = 0$. Take $a \in H_1, b \in H_2$. All a' lie in H_1, all b' lie in H_2. Then $[a', b'] = 0$. Contradiction, $H_1 \cap H_2 \neq 0$. Conversely, let G be anticommutative;

$a, b \in G, a, b \neq 0$, and let H_1 and H_2 be ideals, generated by a and b respectively. H_1 consists of all linear combinations of all a', H_2 consists of all linear combinations of all b'. If always $[a', b'] = 0$, then $H = H_1 \cap H_2$ is non-trivial abelian ideal. Therefore, there exist a' and b', such that $[a', b'] \neq 0$.

We have proved parallel theorems for three particular cases of Ω-groups. It would be very desirable to get the similar result for arbitrary multioperator groups. The main difficulty is that there is no good description of ideals in terms of generators (see [Pl8]).

3. Stable algebras

Taking into account theorems 1, 2, 3 we define algebras H, anticommutative with respect to an algebra G, which serves as an algebra of constants. Definition 3 G-group H with $h : G \to H$ is called anticommutative (in respect to G), if for all elements a and b, from H, $a, b \neq 1$ there exists elements c_1 and c_2 from G, such that the elements $(c_1^h)^{-1} a c_1^h$ and $(c_2^h)^{-1} b c_2^h$ does not commute.

Theorem 4 If G-group H is anticommutative, then H is stable in the variety of all G-groups $\Theta(G)$. Proof Take free in $\Theta(G)$ G-group $W(X) = W = G * W_0(X)$ with the embedding $i_G : G \to W$. Let now A and B be two algebraic varieties over H connected with W. Take $A' = T_1, B' = T_2$. T_1 and T_2 are normal subgroups in W. Check that

$$A \cup B = (T_1 \cap T_2)'$$

It is sufficient to show that if $\mu : W \to H$, does not lie in $A \cup B$, then μ does not lie in $(T_1 \cap T_2)'$. Since $\mu \notin A$, there is $u \in T_1$ such that $u^\mu = a \neq 1$. Since $\mu \notin B$ there is $v \in T_2$ such that $v^\mu = b \neq 1$. Find constants c_1 and $c_2 \in G$ such that $(c_1^{-1})^h a c_1^h$ and $(c_2^{-1})^h b c_2^h$ does not commute. Take element $(c_1^{i_G})^{-1} u c_1^{i_G}$ in T_1 and $(c_2^{i_G})^{-1} v c_2^{i_G}$ in T_2 and let w be their commutator. It belongs to $T_1 \cap T_2$. Take the diagram

$$\begin{tikzcd} G \arrow[rr, "i_G"] \arrow[dr, "h"'] & & W \arrow[dl, "\mu"] \\ & H & \end{tikzcd}$$

and compute w^μ. We have

$$w^\mu = [(c_1^{i_G})^{-1} u c_1^{i_G}, (c_2^{i_G})^{-1} v\, c_2^{i_G}]^\mu =$$

$$= [(c_1^h)^{-1} a\, c_1^h, (c_2^h)^{-1} b\, c_2^h] \neq 1.$$

Thus, $w \notin \mathrm{Ker}\mu$, $T_1 \cap T_2 \not\subset \mathrm{Ker}\mu, \mu \notin (T_1 \cap T_2)'$. ∎

Now take the variety of rings for Θ, consider $\Theta(G), G \in \Theta$ and take the embedding $h: G \to H$. Definition 4 G-ring H is called anticommutative, if for any non-zero elements $a, b \in H$, there exist constants c_1, c'_1, c_2, c'_2 such that $c_1^h a\, c_1'^h$ and $c_2^h b\, c_2'^h$ does not commute Theorem 5 If G-ring H is anticommutative, then it is stable in the variety $\Theta(G)$. Proof Take $W = W(X) = G * W_0(X)$ with $i_G: G \to W$. Let A and B be algebraic varieties over H, defined in W. Take in W the corresponding ideals $T_1 = A', T_2 = B'$. Show that $A \cup B = (T_1 \cap T_2)'$. Take $\mu: W \to H$ and let $\mu \notin A \cup B$. Choose $u \in T_1, v \in T_2$ such that $u^\mu = a \neq 0,\ v^\mu = b \neq 0$. For a and b find constants c_1, c'_1, c_2, c'_2 in G, such that elements $c_1^h\, a\, c_1'^h$ and $c_2^h\, b\, c_2'^h$ does not commute. Let for example

$$c_2^h\, b\, c_2'^h \cdot c_1^h a c_1'^h \neq 0.$$

Take $u' = c_1^{i_G} u c_1'^{i_g} \in T_1$ and $v' = c_2^{i_G} v c_1'^{i_G} \in T_2$. Consider $w = v' \cdot u' \in T_1 \cap T_2$. We have $w^\mu = v'^\mu \cdot u'^\mu = c_2^h b\, c_2'^h \cdot c_1^h a\, c_1'^h \neq 0,\ \ w \notin \mathrm{Ker}\mu, T_1 \cap T_2 \not\subset \mathrm{Ker}\mu,\ \ \mu \notin (T_1 \cap T_2)'$. ∎

Let now Θ be the variety of Lie algebras over a field, $G \in \Theta$. Definition 5 G-algebra H with $h : G \to H$ is called anticommutative, if for any non-zero $a, b \in H$ there exist $c_1, \ldots, c_n$ and $c'_1, \ldots, c'_m$; $c_i, c'_i \in G$ such that the elements $a' = [a; c_1^h, \ldots, c_n^h]$ and $b' = [b, c'^h{}_1, \ldots, c'^h{}_n]$ does not commute, i.e., $[a', b'] \neq 0$. Theorem 6 If G-algebra H is anticommutative, then H is stable in $\Theta(G)$. Proof It goes in a similar way. Take $W = W(X) = G * W_0(X)$ and $i_G: G \to W$. Let A, B be algebraic varieties over H. Take $A' = T_1$ and $B' = T_2$ and check

$$A \cup B = (T_1 \cap T_2)'.$$

Take $\mu: W \to H$ and let $\mu \notin A \cup B$. Choose $u \in T_1$, $v \in T_2$, such that $u^\mu = a \neq 0,\ \ v^\mu = b \neq 0$. For a and b in G there are $c_1, \ldots, c_n; c'_1, \ldots, c'_m$, such that $a' = [a, c_1^h, \ldots, c_n^h]$, $b' = b, c_1'^h, \ldots, c_m'^h]$ does not commute. Take $u' = [u, c_1^{i_G}, \ldots, c_n^{i_G}] \in T_1$ and $v' = [v, c_1'^{i_G}, \ldots, c_m'^{i_G}]$ $\in T_2$ and let $w = [u', v'] \in T_1 \cap T_2$.

We have $w^\mu = [a', b'] \neq 0$, $w \notin \mathrm{Ker}\mu$, $T_1 \cap T_2 \not\subset \mathrm{Ker}\mu$, $\mu \notin (T_1 \cap T_2)'$. ∎

All three theorems can be applied for the case $H = G$. Then we have (absolutely) anticommutative Ω-group G and the following theorems holds: Theorem $4'$ If a group G

is anticommutative, then G is stable in the variety of all G-groups. Theorem 5′ If a ring K is anticommutative, then K is stable in the variety of all K-rings. Theorem 6′ If a Lie algebra L is anticommutative, then L is stable in the variety of all L-algebras,

Simple groups, free groups, free Lie and associative algebras, simple Lie and associative algebras, fields and skew fields: all of them are stable.

Let us notice the following general problem. For an arbitrary variety of algebras Θ find necessary and sufficient conditions on algebra $G \in \Theta$ to be stable in the variety of G-algebras $\Theta(G)$.

In such general form the problem is hard to solve. However, it is solved for groups, Lie algebras and associative algebras. It turns out that sufficient conditions introduced above, are also necessary conditions (A. Berzins) Theorem 7 A group G is stable in the variety $\Theta(G)$, where Θ is the variety of all groups, if and only if G is anticommutative. The same is true for associative and Lie algebras. Proof Sufficiency has been proved above.
Necessity. Let first, Θ be the variety of all groups, and G stable in $\Theta(G)$. The free group in $\Theta(G)$ has the form $F(X) = G * F_0(X)$, F_0 is the free group in Θ. Take $X = \{x, y\}$.

Let A be the variety over G, defined by equation $x = e, e$ is identity element in G, B be the variety, defined by $y = e$. The corresponding affine space is represented as $G \times G$. A consists of points (e, b), B of points (a, e), $a, b \in G$. Take $A \cup B$ and consider $(A \cup B)' = T \triangleleft F(X)$. T consists of "polynomials" $f(x, y)$, such that $f(e, b) = e = f(a, e)$, for every a and $b \in G$. Suppose G is not anticommutative. This means that there are elements $a, b, (a, b \neq e)$, such that for any inner automorphisms σ and τ elements a^{σ} and b^{τ} commute, i.e., a and b absolutely commute.

It is clear that every element $f(x, y) \in F(X)$ can be represented as a product of elements of the form $x^{\sigma}, x^{-1\sigma}, y^{\sigma}, y^{-1\sigma}$, where σ are inner automorphisms, defined by elements of G and by one more multiple $c \in G$ in the end of the presentation. Apply this to $f(x, y) \in T$. Since $f(e, e) = e$, then $c = e$. Take for x and y in $f(x, y)$ absolutely commuting a and b. Then $f(a, b) = \prod a_i \prod b_i$, where $\prod a_i$ belongs to G-normal closure of a and $\prod b_i$ belongs to the same closure of b.

We have $\prod a_i = f(a, e) = e$; $\prod b_i = f(e, b) = e$, then $f(a, b) = e$. Now $(a, b) \in T'$ $= (A \cup B)'', (a, b) \notin A \cup B$. Thus, if G is not anticommutative, then the union $A \cup B$ is not a variety and consequently G is not stable.

Consider now the case of associative algebras. Let Θ be a variety of associative algebras, $K \in \Theta$, and take $\Theta(K)$. Let K be anticommutative. This means that there are $a, b; a, b \neq 0$, such that $acb = bca = 0$ for every $c \in K$.

Again, take the varieties A and B over K defined by the equalities $x = 0$, and $y = 0$ respectively. If $f(x, y) \in (A \cup B)'$, then $f(x, 0) = f(0, y) = 0$.

We have

$$f(x, y) = f_1(x) + f_2(y) + f_3(x, y) + c,$$

where all monoms of f_1 contain x, of f_2 contain y and of f_3 contain x and y.

Since $f(0, 0) = 0$, then $c = 0$. Let now a and b absolutely commute. Then $f_3(a, b) = 0$,

$$f(a, b) = f_1(a) + f_2(b) = f(a, 0) + f(0, b) = 0.$$

Therefore $(a, b) \in (A \cup B)''$ and clearly $(a, b) \notin A \cup B$.

For Lie algebras note that if L is not anticommutative, then there exist non-zero a and b, such that every monom, containing a and b is equal to zero.

In the following proposition variety Θ is arbitrary and G is an arbitrary algebra of Θ. Proposition 1 [Be2] If algebra $H \in \Theta(G)$ has non-trivial cartesian decomposition $H = H_1 \times H_2$, then H is not stable in $\Theta(G)$. Proof We have $h_1 \colon G \to H_1, h_2 \colon G \to H_2$ and $h \colon G \to H, h(g) = (h_1(g), h_2(g))$. Take $X = \{x\}$, and $W(X) = G * W_0(X)$. For every point $\mu \colon W(X) \to H$ we have

$$\begin{tikzcd} G \arrow[r, "i_G"'] \arrow[dr, "h"'] & W(X) \arrow[d, "\mu"] \\ & H \end{tikzcd}$$

For every word $w = w(x, c_1, \dots, c_n)$, $c_i \in G$, c_i is identified with $i_G c_i$, we have $w^\mu = w(x^\mu, c_1^h, \dots, c_n^h)$. Here w is a Θ-word. Take two points $\nu_1 \colon W(x) \to H$ and $\nu_{2'} : W(x) \to H$ by the rule

$$\nu_1(x) = h(g_1) = (h_1(g_1), h_2(g_1)) = (a_1, b_1),$$

$$\nu_2(x) = h(g_2) = (h_1(g_2), h_2(g_2)) = (a_2, b_2),$$

$g_1, g_2 \in G$. The points ν_1 and ν_2 are varieties over H. These varieties A and B respectively, are defined by equalities $x \equiv g_1$ and $x \equiv g_2$. Show that $A \cup B$ is not a variety. Take an arbitrary equation $w = w'$, $w = w(x_1, c_1, \dots, c_n)$, $w' = w'(x, d_1, \dots, d_m)$. Points ν_1 and ν_2 satisfy this equation.

We have

$$w^{\nu_1} = w(x^{\nu_1}, c_1^h, \dots, c_n^h) = w((a_1, b_1), (c_1^{h_1}, c_1^{h_2}), \dots, (c_n^{h_1}, c_n^{h_2})) =$$
$$(w(a_1, c_1^{h_1}, \dots, c_n^{h_1}), w(b_1, c_1^{h_2}, \dots, c_n^{h_2})).$$

Similarly:

$$w'^{\nu_1} = (w'(a_1, d_1^{h_1}, \dots, d_m^{h_1}), w'(b_1, d_1^{h_2}, \dots, d_m^{h_2})).$$

Now $w^{\nu_1} = w'^{\nu_1}$ gives

1. $w(a_1, c_1^{h_1}, \dots, c_n^{h_1}) = w'(a_1, d_1^{h_1}, \dots, d_m^{h_1})$

2. $w(b_1, c_1^{h_2}, \dots, c_n^{h_2}) = w'(b_1, d_1^{h_2}, \dots, d_m^{h_2})$

Similarly $w^{\nu_2} = w'^{\nu_2}$ gives

3. $w(a_2, c_1^{h_1}, \dots, c_n^{h_1}) = w'(a_2, d_1^{h_1}, \dots, d_m^{h_1})$

4. $w(b_2, c_1^{h_2}, \dots, c_n^{h_2}) = w'(b_2, d_1^{h_2}, \dots, d_m^{h_2})$.

Take $\mu_1 \colon W(x) \to H$ by the rule $\mu_1(x) = (a_1, b_2)$. Combining (1) and (4) we see $w^{\mu_1} = w'^{\mu_1}$. Analogously, combining (3) and (2) we have $w^{\mu_2} = w'^{\mu_2}$ if $\mu_2(x) = (a_2, b_1)$.

Thus, μ_1 and μ_2 belong to the closure $(A \cup B)''$ and does not lie in $A \cup B$.

Apply this proposition. Let H be stable in $\Theta(G)$. Take $H \times H$, which is not stable and equivalent to H. Thus, two algebras H_1 and H_2 where H_1 is a stable algebra while H_2 is not stable, can be equivalent.

Let us mention the following two problems Problem 1 For which H the lattices $Alv_H(W)$ are always distributive? Is it true that such H is equivalent to some stable algebra? Problem 2 For which H are the lattices $Alv_H(W)$ always modular? How does it look in the variety of all groups? Is it true that the lattices $Alv_H(W)$ are always modular if H is abelian?

Let us add some comments about the definition of anticommutative G-group. We take the set of equations

$$T = \{[x^g, y^{g'}] = 1,\ g, g' \in G\}.$$

It is easy to see that a G-group H is anticommutative if and only if the algebraic variety $A = T'_H$ is trivial. So the notion of anticommutativity, which is a structure notion, can be given in terms of algebraic geometry over the group H.

§4. Semiisomorphisms and geometric similarity of algebras

1. Preliminary remarks Let Θ be an arbitrary variety of algebras, $G \in \Theta$ and $\Theta(G)$ be the corresponding variety of G-algebras.

Two algebras H_1 an H_2 from $\Theta(G)$ are called semiisomorphic, if there is a commutative diagram in Θ:

$$\begin{tikzcd} G \arrow[r, "h_1"] \arrow[d, "\sigma"'] & H_1 \arrow[d, "\mu"] \\ G \arrow[r, "h_2"] & H_2 \end{tikzcd}$$

where σ is automorphism of constant algebra G, μ is Θ-isomorphism of algebras H_1 and H_2. The main topic of this section is the following theorem. Theorem 8 If G-algebras H_1 and H_2 are semiisomorphic, then they are similar and, therefore, the categories $K_{\Theta(G)}(H_1)$ and $K_{\Theta(G)}(H_2)$ are isomorphic. For the classical situation this theorem has been proved by A. Berzins [Be1]. He noticed that semiisomorphism of algebras H_1 and H_2 does not imply equivalency. Proof of the theorem We will prove the theorem in a few steps and first note some facts on free products in Θ.

1 Let A and B be algebras from Θ and $A*B$ their free product. Denote the projections by $i_A: A \to A * B$ and $i_B: B \to A * B$. Given

$$\alpha: A \to H \ \text{ and } \ \beta: B \to H,$$

corresponds

$$\alpha * \beta: A * B \to H.$$

Then, $i_A(\alpha * \beta) = \alpha, \ \ i_B(\alpha * \beta) = \beta$. We have $i_A * i_B: A * B \to A * B$, and

$$\varepsilon = \varepsilon_{A*B} = i_A * i_B.$$

Indeed, $i_A\varepsilon = i_A, i_B\varepsilon = i_B$ and $\varepsilon_{A\times B} = i_A * i_B$, since extension of morphisms is unique. Take $\alpha: A * B \to H$. We have $i_A\alpha: A \to H$, $i_B\alpha: B \to H$, and $\alpha = i_A\alpha * i_B\alpha: A * B \to H$. In particular, once more $\varepsilon = \varepsilon_{A\times B} = i_A\varepsilon * i_B\varepsilon = i_A * i_B$.

2 Apply remarks above to free product $W = G * W_0$ with projections $i_G: G \to W$; $i_{W_0}: W_0 \to W$. Let σ be an automorphism of algebra G. Define

$$\alpha = \sigma \cdot i_G: G \to G * W_0 = W,$$

$$\beta = i_{W_0}: W_0 \to G * W_0 = W.$$

Now let

$$\sigma_W = \sigma i_G * i_{W_0} = \alpha * \beta: G * W_0 \to G * W_0.$$

Check that σ_W is an automorphism of of $G * W_0$. Take $(\sigma^{-1})_W = \sigma^{-1} i_G * i_{W_0}$ and check that $(\sigma_W)^{-1} = (\sigma^{-1})_W$. We have

$$i_G \sigma_W = i_G(\sigma i_G * i_{W_0}) = \sigma i_G,$$

$$i_{W_0} \sigma_W = i_{W_0},$$

$$i_G (\sigma^{-1})_W = \sigma^{-1} i_G,$$

$$i_{W_0}(\sigma^{-1})_W = i_{W_0},$$

$$\begin{aligned}&(\sigma^{-1})_W \sigma_W = i_G(\sigma^{-1})_W \sigma_W * i_{W_0})(\sigma^{-1})_W \sigma_W = \\ =&(\sigma^{-1} i_G)\sigma_W * i_{W_0}\sigma_W = \sigma^{-1}\sigma i_G * i_{W_0} = \\ =&i_G * i_{W_0} = \varepsilon\end{aligned}$$

Analogously: $\sigma_W(\sigma^{-1})_W = \varepsilon$. Thus, σ_W^{-1} exists and it coincides with $(\sigma^{-1})_W$. This means that σ_W is an automorphism of $G * W_0 = W$. We have to check that the pair (σ, σ_W) is a semiautomorphism of G-algebra $G * W_0$. In other words, we have to check the commutativity of the diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & G * W_0 \arrow[d, "\sigma_W"] \\ G \arrow[r, "i_G"] & G * W_0 \end{tikzcd}$$

This was already done. 3 In the category $\Theta(G)^0$ consider automorphism $\varphi: \Theta(G)^0 \to \Theta(G)^0$. For every $W = G * W_0$ set $\varphi(W) = W$. Let $W^1 = G * W_0^1$ and $W^2 = G * W_0^2$ be two objects in $\Theta(G)$ and let the morphism

$$\begin{tikzcd} G \arrow[r, "i_G^1"] \arrow[dr, "i_G^2"'] & G * W_0^1 \arrow[d, "\nu"] \\ & G * W_0^2 \end{tikzcd}$$

be given. Define $\varphi(\nu) = \sigma_{W^1}^{-1}\nu\sigma_{W^2}\colon G * W_0^1 \to G * W_0^2$ and check that this is a morphism in the $\Theta(G)$. We have to check that the diagram

$$\begin{tikzcd} G \arrow[r, "i_G^1"] \arrow[dr, "i_G^2"'] & G * W_0^1 \arrow[d, "\varphi(\nu)"] \\ & G * W_0^2 \end{tikzcd}$$

is commutative. We have

$$i_G^1\sigma_{W^1}^{-1}\nu\sigma_{W^2} = \sigma^{-1}i_G^1\nu\sigma_{W^2} = \sigma^{-1}i_G^2\sigma_{W^2} = \sigma^{-1}\sigma i_G^2 = i_G^2.$$

Thus, φ is defined on objects and morphisms of the category $\Theta(G)^0$, is invertible and is compatible with multiplication of morphisms. So, φ is automorphism of $\Theta(G)^0$.

4 Let T be a congruence in $W = G * W_0$. Define a new congruence $\sigma_W T$ by the rule:

$$w(\sigma_W T)w' \Leftrightarrow w^{\sigma_W}Tw'^{\sigma_W}$$

Taking into account that a congruence of G-algebra H is the same as a congruence of H in Θ we get that $\sigma_W T$ is also a congruence in W. Elements w and w' are elements in G-algebra W.

5 Now, we take the semiisomorphism

$$\begin{tikzcd} G \arrow[r, "h_1"] \arrow[d, "\sigma"'] & H_1 \arrow[d, "\mu"] \\ G \arrow[r, "h_2"] & H_2 \end{tikzcd}$$

and all constructions above apply to the given σ. Check that the congruence T is H_1-closed if and only if $\sigma_W^{-1}T$ is H_2-closed. Consider the diagram

$$\begin{tikzcd} W \arrow[r, "(\sigma,\sigma_W)"] \arrow[d, "\alpha"'] & W \arrow[d, "\beta"] \\ H_1 \arrow[r, "(\sigma,\mu)"] & H_2 \end{tikzcd}$$

Here, in horizontal rows stand semiisomorphisms and in vertical ones stand morphisms in $\Theta(G)$. Arrows α and β can be represented as $(1,\alpha)$ and $(1,\beta)$. Then we can use pairwise multiplication of semimorphisms.

We have

$$(1,\beta)=(\sigma^{-1},\sigma_W^{-1})(1,\alpha)(\sigma,\mu)=(1,\sigma_W^{-1}\alpha\mu)$$

and, thus

$$\beta=\sigma_W^{-1}\alpha\mu: \qquad \alpha=\sigma_W\beta\mu^{-1}.$$

Thus, morphisms σ_W and μ give a one to one correspondence between α and β. Check that $Ker\alpha=\sigma_W(Ker\beta)$.

Take w and $w'\in W$ and let $w(Ker\alpha)w'$, $w^\alpha=w'^\alpha$. Take $w_1=w^{\sigma_W}$, $w'_1=w'^{\sigma_W}$. Then $w_1^{\sigma_W^{-1}\alpha}=w'^{\sigma_W^{-1}\alpha}$; $w_1^{\sigma_W^{-1}\alpha\mu}=w_1'^{\sigma_W^{-1}\alpha\mu}$; $w_1^\beta=w_1'^\beta$, $w_1(Ker\beta)w_1'$, $w^{\sigma_W}(Ker\beta)w'^{\sigma_W}$, $w(\sigma_W Ker\beta)w'$.

Conversely, let $w(\sigma_W Ker\beta)w'$, $w^{\sigma_W}(Ker\beta)w'^{\sigma_W}, w^{\sigma_W\beta}=w'^{\sigma_W\beta}$. Then $w^{\sigma_W\beta\mu^{-1}}=w'^{\sigma_W\beta\mu^{-1}}$ and $w^\alpha=w'^\alpha$, $w(Ker\alpha)w'$. Thus $Ker\alpha=\sigma_W(Ker\beta)$; $Ker\beta=\sigma_W^{-1}(Ker\alpha)$.

Let the congruence T be H_1-closed, i.e. $T=\bigcap_{\alpha\in A}Ker\alpha$. Then $T=\bigcap_{\alpha\in A}\sigma_W(Ker\beta)=\sigma_W(\bigcap_{\alpha\in A}Ker\beta)$. Let $B=\{\beta=\sigma_W^{-1}\alpha\mu,\alpha\in A\}$, $T_1=\bigcap_{\beta\in B}Ker\beta$. Then $T=\sigma_W T_1, T_1=\sigma_W^{-1}T$. So, we get that T_1 is H_2-closed if T is H_1-closed. Since one can proceed from H_2-closed congruence, we get that T is H_1-closed if and only if $T_1=\sigma_W^{-1}T$ is H_2-closed.

6 Connect now T and $\sigma_W^{-1}T$ using automorphism φ (see [Pl2]). For given $T\triangleleft W$ consider the equivalence $\rho=\rho(T)$ in End W. It is defined by the rule: $\nu\rho\nu'$, $\nu,\nu'\in EndW$ if $w^\nu Tw^{\nu'}$ for every $w\in W$. Analogously, for $T_1=\sigma_W^{-1}T$ take $\rho^*=\rho(T_1)$. We want to check that $\rho^*=\varphi(\rho)$. This means that $\nu\rho\nu'\Leftrightarrow\varphi(\nu)\rho^*\varphi(\nu')$.

We take $\nu\rho\nu$, so $\forall w\in W$, $w^\nu Tw^{\nu'}$. This gives $w^{\nu\sigma_W\sigma_W^{-1}}Tw^{\nu'\sigma_W\sigma_W^{-1}}$ and, therefore, $w^{\nu\sigma_W}(\sigma_W^{-1}T)w^{\nu'\sigma_W}$, $w^{\sigma_W(\sigma_W^{-1}\nu\sigma_W)}T_1w^{\sigma_W(\sigma_W^{-1}\nu'\sigma_W)}$. Element $w_1=w^{\sigma_W}$ is an arbitrary element in W and we have $w_1^{\varphi(\nu)}T_1w_1^{\varphi(\nu')}$. This means $\varphi(\nu)\rho^*\varphi(\nu')$. The converse can be checked in a similar way and $\rho^*=\varphi(\rho)$. This means that φ is compatible with the transition $T\to\sigma_W^{-1}T$.

7 We have to check one more compatibility. Let $W^1=G*W_0^1$ and $W^2=G*W_0^2$ be two objects in $\Theta(G)$ and let T be a congruence in W^2. Define the relation $\tau=\tau(T)$ on the set $Hom(W^1,W^2)$ by the rule:

$$s\tau s', s,s'\in Hom(W^1,W^2) \text{ if } w^sTw^{s'} \text{ for every } \quad w\in W^1.$$

Similarly, $\tau^* = \tau(T_1), T_1 = \sigma_{W^2}^{-1}T$. We need to check that $\tau^* = \varphi(\tau)$.

Let $s\tau s'$. This means that $w^s T w^{s'}$ for every $w \in W^1$. We have $w^{s\sigma_{W^2}\sigma_{W^2}^{-1}} T w^{s'\sigma_{W^2}\sigma_{W^2}^{-1}}$ and this gives $w^{s\sigma_{W^2}} T_1 w^{s'\sigma_{W^2}}$. Take an arbitrary element $w^{\sigma_{W_1}} = w_1$ in W^1. We have $w_1^{\sigma_{W^1}^{-1}s\sigma_{W^2}} T_1 w^{\sigma_{W^1}^{-1}s\sigma_{W^2}}$, i.e. $w_1^{\varphi(s)} T_1 w_1^{\varphi(s')}$. Thus, $s\tau s'$ implies that $\varphi(s)T_1\varphi(s')$. The converse is true similarly. This means that automorphism φ, determined by σ, induces isomorphism of functors Cl_{H_1} and $Cl_{H_2}\varphi$ (See [Pl5], [Pl6]).

Thus, G-algebras H_1 and H_2 are similar. In fact, we proved that H_1 and H_2 are weakly equivalent, (see Pl5), and this implies similarity.

We see also that if H_1 and H_2 are semiisomorphic, then the categories $K_{\Theta(G)}(H_1)$ and $K_{\Theta(G)}(H_2)$ are isomorphic, i.e., H_1 and H_2 are categorically equivalent. Note also that automorphism φ here is semiinner (see §6).

§5. Automorphisms of the free algebras category

1. Categories $\Theta(G)^0$ and $K^0_{\Theta(G)}$ Take a variety of algebras Θ, and consider $\Theta(G)$, where $G \in \Theta$ is an algebra of constants. Consider the category of free algebras $\Theta(G)^0$. We will study automorphisms of this category. This problem is of interest by itself, and it is also connected with the problem of similarity of G-algebras.

We will proceed from the additional condition $(*)$:

$$G-\text{algebra } \ G \ \text{ generates the variety } \ \Theta(G). \ (*)$$

This condition is fulfilled in the classical situation $Var - P$ if P is infinite, it is fulfilled in the situation where Θ is the variety of all groups and $G = F$ is a free non-commutative group [Me]. It is fulfilled in the case when Θ is the variety of Lie algebras over an infinite field and G is a free Lie algebra (R.Lipjanskii, unpublished).

Condition $(*)$ is of special interest. If algebra G generates a variety Θ, this, probably, does not imply that G-algebra G generates $\Theta(G)$. But how the situation looks if G is a free algebra which generates Θ? This problem is connected with consideration of identities with constants and can be studied for special varieties Θ.

Assume that $(*)$ is fulfilled. Then, every free algebra $W = W(X)$ with finite X is an object in the category $\mathbb{C}_{\Theta(G)}(G)$ (see [Pl1]) and the category $\Theta(G)^0$ can be considered as a subcategory in the category $\mathbb{C}_{\Theta(G)}(G)$, which is dually isomorphic to the category of algebraic varieties $K_{\Theta(G)}(G)$. This isomorphism assigns to each free algebra $W(X) = G * W_0(X)$ in $\Theta(G)$ the algebraic variety $Hom(W,G)$. Morphisms in this category of affine spaces are the same $s: W^1 \to W^2$, but they act in the opposite direction. To each morphism s corresponds the map of sets of points

$$\tilde{s}: Hom(W^2, G) \to Hom(W^1, G)$$

acting by the rule: if $\nu: W^2 \to G$ then $\tilde{s}(\nu) = \nu s$, where first s acts, then ν. Recall that every point $\nu: W \to G$ satisfies the commutative diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[dr, "id_G"'] & W = G * W \arrow[d, "\nu"] \\ & G \end{tikzcd}$$

Denote the category of affine spaces by $K^0_{\Theta(G)}(G) = K^0_{\Theta(G)}$.

Categories $\Theta(G)^0$ and $K^0_{\Theta(G)}$ are dually isomorphic, which gives connection between automorphisms of the categories.

2. Quasiiner automorphism Let $\nu: W \to G$ be a point. Consider the homomorphism

$$W \xrightarrow{\nu} G \xrightarrow{i_G} W$$

Check that $i_G\nu: W \to W$ is an endomorphism of the algebra W in the variety $\Theta(G)$. We have to check that there is the commutative diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[dr, "i_G"'] & W \arrow[d, "i_G\nu"] \\ & W \end{tikzcd}$$

We have

$$(i_G\nu)i_G = i_G(\nu i_G) = i_G id_G = i_G.$$

Denote $i_G \cdot \nu = \bar{\nu}$. For every $w \in W$ element $\bar{\nu}(w) = i_G(\nu(w))$ is a constant in W and, therefore, $\bar{\nu}$ is called a constant endomorphism. Endomorphism $\bar{\nu}$ defines the map

$$\tilde{\bar{\nu}}: Hom(W,G) \to Hom(W,G).$$

Every endomorphism $s: W \to W$ leaves constants and, therefore, $(s\bar{\nu})(w) = \bar{\nu}(w)$, $s\bar{\nu} = \bar{\nu}$. Note that for every $\nu_0: W \to G$ we have $\tilde{\bar{\nu}}(\nu_0) = \nu$. Indeed, $\tilde{\bar{\nu}}(\nu_0) = \nu_0 \cdot \bar{\nu} = \nu_0(i_G\nu) = (\nu_0 i_G)\nu = id_G \cdot \nu = \nu$.

Thus, the map $\tilde{\bar{\nu}}$ takes an arbitrary ν_0 to one and the same element ν, and, therefore $\tilde{\bar{\nu}}$ is a constant map.

Consider an arbitrary $\sigma: W \to W$, such that $s\sigma = \sigma$ for every $s: W \to W$. Since one can take for s an endomorphism taking w to a constant, σ takes any w to a constant.

Show that $\bar{\sigma}$ takes all $\nu: W \to G$ to the same element. Take ν_0 and ν_1. Then $\tilde{\sigma}(\nu_0) = \nu_0\sigma = \nu_0 s\sigma$. For s takes $\bar{\nu}_1$. Then

$$\tilde{\sigma}(\nu_0) = (\nu_0\bar{\nu}_1)\sigma = \tilde{\bar{\nu}}_1(\nu_0) \cdot \sigma = \nu_1\sigma = \tilde{\sigma}(\nu_1).$$

Denote $\tilde{\sigma}(\nu_0) = \nu$. Then $\tilde{\bar{\nu}}(\nu_0) = \nu$, $\tilde{\sigma}(\nu_0) = \tilde{\bar{\nu}}(\nu_0)$ and $\nu_0\sigma = \nu_0\bar{\nu}$ for every ν_0.

Show that $\sigma = \bar{\nu}$. Take an arbitrary free algebra W of a countable rank. The condition $(*)$ implies that for some I there is an injection $\mu: W \to G^I$. Then there is the similar injection for every $W = W(X)$ with finite X.

For every $\alpha \in I$ take a projection $\pi_\alpha: G^I \to G$ and consider $\nu_\alpha = \pi_\alpha\mu: W \to G$.

Suppose that $\nu_\alpha s_1 = \nu_\alpha s_2$, where s_1 and s_2 are endomorphisms of W.

Check that $s_1 = s_2$. Take an arbitrary $w \in W$. Then:

$$\begin{aligned} &\mu s_1(w)(\alpha) = (\pi_\alpha\mu s_1)(w) = (\nu_\alpha s_1)(w) = (\nu_\alpha s_2)(w) = \\ &= \mu s_2(w)(\alpha), \text{ for every } \alpha \in I \text{ and } \mu s_1(w) = \\ &= \mu s_2(w), \quad \mu(s_1 w) = \mu(s_2 w). \end{aligned}$$

Since μ is injection, then $s_1 w = s_2 w$ for every w, i.e., $s_1 = s_2$. In particular, $\sigma = \bar{\nu}$.

Let now $\varphi: \Theta(G)^0 \to \Theta(G)^0$ be an automorphism of the category $\Theta(G)^0$ and let $\tau : K^0_{\Theta(G)} \to K^0_{\Theta(G)}$ be the corresponding automorphism in the category of affine spaces.

For every object $W = W(X)$ in $\Theta(G)^0$, we have $\varphi(W) = W^1 = W(Y)$. We assume that $|X| = |Y|$. We have

$$\tau(Hom(W,G)) = Hom(\varphi(W), G), \qquad \tilde{s}^\tau = \widetilde{\varphi(s)}.$$

As we see, the constant

$$\tilde{\bar{\nu}}: Hom(W,G) \to Hom(W,G)$$

is characterized by the condition $s\bar{\nu} = \bar{\nu}$ for every s. This condition can be rewritten as $\tilde{\tilde{\nu}}\tilde{s} = \tilde{\tilde{\nu}}$. Apply τ. We have

$$(\tilde{\tilde{\nu}}\tilde{s})^\tau = \tilde{\tilde{\nu}}^\tau \cdot \tilde{s}^\tau = \tilde{\tilde{\nu}}^\tau : Hom(W^1, G) \to Hom(W^1, G)$$

Since $\tilde{s}^\tau$ is an arbitrary, $\tilde{\tilde{\nu}}^\tau$ is a constant, $\tilde{\tilde{\nu}}^\tau = \tilde{\tilde{\nu}}_1$, where $\nu_1 \colon W^1 \to G$ is a point. Denote $\nu_1 = \mu(\nu)$. The map

$$\mu = \mu_W \colon Hom(W, G) \to Hom(W^1, G) = Hom(\varphi(W), G)$$

is a bijection. If $\nu \colon W \to G$ then $\tilde{\tilde{\nu}}^\tau = \widetilde{\overline{\mu_W(\nu)}}$.

Now we want to restore the action of the automorphism τ on an arbitrary $\tilde{s}$ in terms of the function μ_W.

Let now $s \colon W^1 \to W^2$, $W^1 = W(X)$, $W^2 = W(Y)$ be given. It corresponds the map

$$\tilde{s} \colon Hom(W^2, G) \to Hom(W^1, G).$$

Take an arbitrary point $\nu \colon W^2 \to G$ with the map

$$\tilde{\tilde{\nu}} \colon Hom(W^2, G) \to Hom(W^2, G).$$

Then take

$$\tilde{s}\tilde{\tilde{\nu}} \colon Hom(W^2, G) \to Hom(W^1, G)$$

and apply τ. We get

$$(\tilde{s}\tilde{\tilde{\nu}})^\tau = \tilde{s}^\tau \tilde{\tilde{\nu}}^\tau = \tilde{s}^\tau \widetilde{\overline{\mu_{W^2}(\nu)}} \colon Hom(\varphi(W^2), G) \to$$
$$\to Hom(\varphi(W^2), G).$$

Check the equality

$$\tilde{s} \cdot \tilde{\tilde{\nu}} = \widetilde{\overline{\tilde{s}(\nu)}} \cdot \tilde{s}.$$

Take a point $\nu_0 \colon W^2 \to G$. We have

$$\tilde{s} \cdot \tilde{\tilde{\nu}}(\nu_0) = \tilde{s}(\tilde{\tilde{\nu}}(\nu_0)) = \tilde{s}(\nu),$$
$$(\widetilde{\overline{\tilde{s}(\nu)}} \cdot \tilde{s})(\nu_0) = \widetilde{\overline{\tilde{s}(\nu)}}(\tilde{s}(\nu_0)) = \tilde{s}(\nu).$$

This gives the equality. Applying τ

$$\tilde{s}^{\tau}\cdot\tilde{\tilde{\nu}}^{\tau}=\widetilde{\tilde{s}(\nu)}^{\tau}\,\tilde{s}^{\tau}=\overline{\mu_{W^1}(\widetilde{\tilde{s}(\nu)})}\cdot\tilde{s}^{\tau},$$
$$\tilde{s}^{\tau}\cdot\overline{\widetilde{\mu_{W^2}(\nu)}}=\overline{\mu_{W^1}(\widetilde{\tilde{s}(\nu)})}\cdot\tilde{s}^{\tau}.$$

Definition 6 An automorphism τ of the category $K^0_{\Theta(G)}$ is called quasiinner, if for an arbitrary $\tilde{s}\colon Hom(W^2,G)\to Hom(W^1,G)$,

$$\tilde{s}^{\tau}=\mu_{W^1}\tilde{s}\mu^{-1}_{W^2}.$$

Theorem 9 (see [Be2] for $Var-P$) Every automorphism τ of the category $K^0_{\Theta(G)}$ is quasiinner. Proof Take any $\nu_0\colon\varphi(W^2)\to G$ and apply to it the equality above. We have:

$$\tilde{s}^{\tau}\overline{\widetilde{\mu_{W^2}(\nu)}}(\nu_0)=\tilde{s}^{\tau}(\mu_{W^2}(\nu))=\tilde{s}^{\tau}\mu_{W^2}(\nu)$$
$$\overline{\mu_{W^1}(\widetilde{\tilde{s}(\nu)})\tilde{s}^{\tau}})(\nu_0)=\overline{\mu_{W^1}(\widetilde{\tilde{s}(\nu)})}(\tilde{s}^{\tau}(\nu_0))=$$
$$=\mu_{W^1}(\tilde{s}(\nu))=(\mu_{W^1}\cdot\tilde{s})(\nu).$$

Thus, for every ν we have

$$\tilde{s}^{\tau}\cdot\mu_{W^2}(\nu)=(\mu_{W^1}\cdot\tilde{s})(\nu).$$

This gives

$$\tilde{s}^{\tau}\mu_{W^2}=\mu_{W^1}\tilde{s},$$

which proves the theorem.

We note here, that realization of the given φ and τ as a quasiinner automorphism is determined by a function μ, which is defined on objects $W=W(X)$ of the category $\Theta(G)^0$. For every W it gives bijection

$$\mu_W\colon Hom(W,G)\to Hom(\varphi(W),G).$$

It depends on φ and τ.

Assume now that we proceed from a pair of functions (μ,φ), defined on objects of $\Theta(G)^0$, such that

1. Function φ is a substitution on the set of objects $W=W(X)$,

2. Function μ for every W gives a bijection

$$\mu_W: Hom(W,G) \to Hom(\varphi(W),G)$$

For such pair of functions (μ,φ) and for every morphism $s: W^1 \to W^2$ one can define

$$\tilde{s}^\tau = \mu_{W^1}\tilde{s}\mu_{W^2}^{-1}:\ Hom(\varphi(W^2),G) \to Hom(\varphi(W^1),G).$$

However, in this definition $\tilde{s}^\tau$ is not necessarily a morphism in the category $K^0_{\Theta(G)}$, therefore τ is given by the formula above is not an automorphism of the category $K^0_{\Theta(G)}$. Correspondingly, φ is not an automorphism of the category $\Theta(G)^0$.

We show another special construction.

§6. Inner and semiinner automorphisms

For the given $\Theta(G)^0$ and $K^0_{\Theta(G)}$ consider pairs of functions (ψ,φ) defined on objects of the category $\Theta(G)^0$, satisfing

1. Function φ is a substitution on objects, and if

$$\varphi(W(X)) = W(Y), \text{ then } |X| = |Y|$$

2. Function ψ attaches to every W a semiisomorphism $\psi_W = (\sigma,\mu^0_W)$ of W and $\varphi(W)$ determined by a commutative diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W = G * W_0 \arrow[d, "\mu^0_W"] \\ G \arrow[r, "i'_G"] & \varphi(W) = G * W'_0 \end{tikzcd}$$

Here, automorphism σ does not depend on W. Theorem 10 For every morphism $s: W^1 \to W^2$ set $\varphi(s) : \varphi(W^1) \to \varphi(W^2)$ by the rule $\varphi(s) = \mu^0_{W^2} s \mu^{0^{-1}}_{W^1}$. This extension of the function φ to morphisms defines the automorphism of the category $\Theta(G)^0$. Proof Consider semiautomorphisms

$$(\sigma,\mu^0_{W^1}), (\sigma,\mu^0_{W^2}) \text{ and } (1,s).$$

Applying pairwise multiplication, we get

$$(\sigma,\mu^0_{W^2}(1,s)(\sigma^{-1}\mu^{0^{-1}}_W) = (1,\mu^0_{W^2} s \mu^{0^{-1}}_{W^1}).$$

The righthand side in this equality is a semimorphism, and since the first component is 1, this is a morphism in the category $\Theta(G)^0$. So, we have that

$$\varphi(s) = \mu^0_{W^2} s \mu^{0^{-1}}_{W^1} \colon \varphi(W^1) \to \varphi(W^2)$$

is a morphism in $\Theta(G)^0$. Let now $s_1 \colon W^1 \to W^2, s_2 \colon W^2 \to W^3$ and $s_2 s_1 \colon W^1 \to W^3$ be given. Then:

$$\varphi(s_2 s_1) = \mu^0_3 s_2 s_1 \mu^{0^{-1}}_1 = \mu^0_3 s_2 \mu^{0^{-1}}_2 \mu^0_2 s_1 \mu^{0^{-1}}_1 =$$
$$= \varphi(s_2)\varphi(s_1) \colon \varphi(W^1) \to \varphi(W^3).$$

Clearly, φ is invertible and, therefore, φ is automorphism. Definition 7 An automorphism φ determined by a pair (ψ, φ) with conditions 1 and 2 is called semiinner. If in (ψ, φ) the automorphism $\sigma = 1$ then the corresponding φ is called inner.

Variety $\Theta(G)$ is called perfect, if every automorphism of $\Theta(G)$ is inner.

Variety $\Theta(G)$ is called semiperfect, if every automorphism of $\Theta(G)$ is semiinner.

It is easy to see that semiinner, as well as inner automorphisms form a subgroup in the group of all automorphisms of the category $\Theta(G)$. This can be checked by studying pairs of the type (ψ, φ). Indeed, if $\varphi = \varphi_1 \varphi_2$, $\varphi \to (\psi, \varphi)$, $\varphi_1 \to (\psi^1, \varphi_1)$, $\varphi_2 \to (\psi^2, \varphi_2)$, then $(\psi, \varphi) = (\psi^1 \psi^2, \varphi_1 \varphi_2)$ and $(\psi^1 \psi^2)_W = \psi^1_{\varphi_2(W)} \psi^2_W$.

Example Let σ be an automorphism of the constant algebra G. For every $W = G * W_0$ there is a semiautomorphism $\psi_W = (\sigma, \sigma_W)$. Consider a pair (ψ, φ), where the substitution φ is trivial. This pair defines semiiner automorphism φ of the category $\Theta(G)^0$, which was considered in §4.

2. Ties with quasiinner automorphisms

Our nearest goal is to present semiinner automorphism φ as a quasiinner automorphism of the category $K^0_{\Theta(G)}$. Let us take φ and define τ by the rule

$$\tau(Hom(W, G)) = Hom(\varphi(W), G).$$

If, further, $s \colon W^1 \to W^2$ is a morphism in $\Theta(G)^0$, then it corresponds

$$\tilde{s} \colon Hom(W^2, G) \to Hom(W^1, G).$$

Let $\tilde{s}^\tau = \widetilde{\varphi(s)}$. Then $(\tilde{s}_1 \cdot \tilde{s}_2)^\tau = \tilde{s}_1^\tau \cdot \tilde{s}_2^\tau$, and τ is automorphism of the category $K^0_{\Theta(G)}$. Define the bijection

$$\mu_W : Hom(W, G) \to Hom(\varphi(W), G),$$

by the rule

$$\tilde{\tilde{\nu}}^\tau = \widetilde{\overline{\mu_W(\nu)}} \text{ for every } \quad \nu : W \to G.$$

Proceeding from (ψ, φ) for the given W consider the diagram

$$\begin{tikzcd} W \arrow[r, "\nu"] \arrow[d, "\mu^0_W"'] & G \arrow[d, "\sigma"] \\ \varphi(W) \arrow[r, "\nu_1"'] & G \end{tikzcd}$$

Here $\nu_1 = \sigma\nu\mu_W^{0^{-1}}$. Define $\mu_W(\nu) = \nu_1 = \sigma\nu\mu_W^{0^{-1}}$. We have a bijection

$$\mu_W : Hom(W, G) \to Hom(\varphi(W), G).$$

Check that this bijection is well coordinated with the automorphism τ. We have

$$\begin{aligned} &\tilde{\tilde{\nu}}^\tau = \widetilde{\varphi(\bar{\nu})} = \mu^0_W \widetilde{i_G \nu \mu_W^{0^{-1}}} = \widetilde{i'_G \sigma\nu\mu_W^{0^{-1}}}, \\ &\widetilde{\overline{\mu_W(\nu)}} = \widetilde{i'_G \sigma\nu\mu_W^{0^{-1}}}. \end{aligned}$$

We obtained the following main equality

$$\widetilde{\overline{\mu_W(\nu)}} = \tilde{\tilde{\nu}}^\tau.$$

Thus the function μ can be constructed from the given pair (ψ, φ). Simultaneously we have (μ, φ). Now for given μ check that

$$\tilde{s}^\tau = \mu_{W^1} \tilde{s} \mu^{-1}_{W^2}.$$

Take $\nu_1 : \varphi(W^1) \to G$. Then

$$\begin{aligned} &\tilde{s}^\tau(\nu_1) = \widetilde{\varphi(s)}(\nu_1) = \nu_1\varphi(s) = \nu_1 \mu^0_{W^2} s \mu^{0^{-1}}_{W^1}, \\ &(\mu_{W^1}\tilde{s}\mu^{-1}_{W^2})(\nu_1) = \mu_{W^1}(\sigma^{-1}\nu_1\mu^0_{W^2}s) = \\ &= \sigma(\sigma^{-1}\nu_1\mu^0_{W^2}s)\mu^{0^{-1}}_{W^1} = \nu_1\mu^0_{W^2}s\mu^{0^{-1}}_{W_1}. \end{aligned}$$

Therefore, $\tilde{s}^\tau = \mu_{W^1}\tilde{s}\mu_{W^2}^{-1}$. Thus, if automorphism φ is semiinner and is defined by a pair (ψ, φ), then this pair defines (μ, φ) which, in its turn, determines presentation of τ as quasiinner automorphism.

3. Additional remarks

Define a substitutional automorphism φ for the category $\Theta(G)^0$. First consider substitution φ on objects W with the standard condition: $\varphi(W(X)) = W(Y)$ implies $|X| = |Y|$. Let $X = \{x_1, \cdots, x_n\}$, $Y = \{y_1, \cdots, y_n\}$. Define isomorphism $s_W: W(X) \to \varphi(W(X))$ by $s_W(x_i) = y_i, i = 1, \cdots, n$. If, further, $\nu: W^1 \to W^2$ is a morphism, then set

$$\varphi(\nu) = s_{W^2}\nu s_{W^1}^{-1}: \varphi(W^1) \to \varphi(W^2).$$

From the substitution φ we come to the automorphism $\bar{\varphi}$. We call $\bar{\varphi}$ substitutional automorphism. This $\bar{\varphi}$ is an inner automorphism of the category $\Theta(G)^0$.

Proposition 2 Every automorphism $\varphi: \Theta(G)^0 \to \Theta(G)^0$ can be decomposed in the form

$$\varphi = \bar{\varphi}\varphi_1,$$

where $\bar{\varphi}$ is a substitutional automorphism and φ_1 does not change objects. Proof. Let φ be an arbitrary automorphism of the category $\Theta(G)^0$. As usual, assume that if $\varphi(W(X)) = W(Y)$ then $|X| = |Y|$. We will dwell on this condition in the sequel.

Take φ as a substitution on objects and take the corresponding $\bar{\varphi}$. Take φ and $\bar{\varphi}$. Let $\varphi_1 = \bar{\varphi}^{-1}\varphi$. For every W we have: $\varphi_1(W) = \varphi^{-1}\varphi(W) = W$. For $\nu: W^1 \to W^2$ we have $\varphi_1(\nu) = s_{W^2}^{-1}\varphi(\nu)s_{W^1}: W^1 \to W^2$. Hence, φ_1 is an automorphism which does not change objects. We have the canonical decomposition $\varphi = \bar{\varphi}\varphi_1$. Since $\bar{\varphi}$ is an inner automorphism, φ_1 is semiinner if and only if φ is semiinner.

We now consider an automorphism φ which does not change objects.

Let us consider a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{i_G} & W = G * W_0 \\ \sigma\downarrow & & \downarrow s \\ G & \xrightarrow{i_G} & W = G * W_0 \end{array}$$

where σ, s are automorphisms in $\Theta, W_0 = W_0(x_1, \ldots, x_n) = W_0(X)$ is free in Θ. Consider also $i_{W_0}: W_0 \to W$.

To each $x_i \in X$ corresponds $si_{W_0}(x) \in W$. Since all x_i are free generators, we have an endomorphism of G-algebra $s_0 \colon W \to W$, represented by

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "1"'] & W \arrow[d, "s_0"] \\ G \arrow[r, "i_G"] & W \end{tikzcd}$$

Following §4, consider a diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W \arrow[d, "\sigma_W"] \\ G \arrow[r, "i_G"] & W \end{tikzcd}$$

Now we can get semiendomorphism

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W \arrow[d, "s_0\sigma_W"'] \\ G \arrow[r, "i_G"] & W \end{tikzcd}$$

Show, that $s = s_0\sigma_W$. For every $g \in G$ we have

$$s_0\sigma_W i_G(g) = i_G\sigma(g) = si_G(g).$$

For every $x \in X$ we have $s_0 i_{W_0}(x) = si_{W_0}(x)$, and $\sigma_W(i_{W_0}(x)) = i_{W_0}(x)$. This gives

$$s_0\sigma_W(i_W(x)) = s_0 i_{W_0}(x) = si_{W_0}(x).$$

Thus, $s_0\sigma_W$ and s coincide on G and on W_0. Then they coincide on $G * W_0$ and $s = s_0\sigma_W$. This gives canonical decomposition of semiautomorphism (σ, s). We have also $s_0 = s\sigma_W^{-1}$ and, hence, s_o is an automorphism of G-algebra W.

Return to the automorphism $\varphi \colon \Theta(G)^0 \to \Theta(G)^0$ which does not change objects and consider τ and the pair (μ, φ).

Suppose that bijection

$$\mu_W \colon Hom(W, G) \to Hom(W, G)$$

is defined by the diagram

$$\begin{tikzcd} W \arrow[r, "\nu"] \arrow[d, "s"'] & G \arrow[d, "\sigma"] \\ W \arrow[r, "\mu_W(\nu)"] & G \end{tikzcd} \qquad \mu_W(\nu) = \sigma\nu s^{-1}$$

Consider semiautomorphism

$$\begin{tikzcd} G \arrow[r, "i_G"] & W \\ G \arrow[u, "\sigma^{-1}"] \arrow[r, "i_G"] & W \arrow[u, "s^{-1}"'] \end{tikzcd}$$

and take canonical decomposition $s^{-1} = s_0\sigma_W^{-1}$, where s_0 is an automorphism of G-algebra W. We have

$$\mu_W(\nu) = \sigma\nu s^{-1} = \sigma\nu s_0\sigma_W^{-1} = \sigma(\tilde{s}_0(\nu)\sigma_W^{-1} =$$

$$= (\sigma\tilde{s}_0)(\nu)\sigma_W^{-1}.$$

Take $w = w(x_1, \dots, x_n) = w(X) \in W$. For every $x \in X$ we have

$$(\sigma\tilde{s_0})(\nu)\sigma_W^{-1}(i_{W_0}(x)) = \sigma\tilde{s}_0(\nu)(i_{W_0}(x)).$$

This means that $\mu_W(\nu)$ and $\sigma\tilde{s}_0(\nu)$ coincide on every $x \in X$. We get

$$\mu_W(\nu)(w) = w(\mu_W(\nu)(x_1), \dots, \mu_W(\nu)(x_n)) =$$

$$= w(\sigma\nu s_0(x_1), \dots, \sigma\nu s_0(x_n)) = (\sigma\nu w_1(x_1, \dots, x_n), \dots,$$

$$\sigma\nu w_n(x_1, \dots, x_n)).$$

Here $w_i(x_1, \dots, x_n) = s_0(x_i)$. If $\sigma = 1$ we have

$$\mu_W(\nu) = \tilde{s}_0(\nu), \mu_W = \tilde{s}_0.$$

We get additional information about bijection μ_W using canonical decomposition of semiautomorphism.

§7. Category of polynomial maps

1. Category Pol-G In the first section we considered $Hom(W(X), G)$ as an affine space. Now affine space is the usual cartesian product $G^{(n)}$. This is the set of points $a = (a_1, \dots, a_n), a_i \in G$. We have bijection $\alpha_X\colon Hom(W(X), G) \to G^{(n)}$. Here $X = \{x_1, \dots, x_n\}$ and $\alpha_X(\nu) = (\nu(x_1), \dots, \nu(x_n))$, $\nu \in Hom(W, G)$. Fix a variety Θ, take $G \in \Theta$ and define the category Pol-G. Its objects are cartesian products $G^{(n)}$. Morphisms depend on Θ.

Define a morphism $G^{(m)} \to G^{(n)}$. Consider a homomorphism $s\colon W(X) \to W(Y)$, where $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$. We have $s(x_1) = w_1(y_1, \dots, y_m), \cdots, s(x_n) = w_n(y_1, \cdots, y_m)$.

Homomorphism s is determined by Θ-"polynomials" $w_1, \cdots, w_n$. Define $s^\alpha: G^{(m)} \to G^{(n)}$ by the rule

$$s^\alpha(a_1, \ldots, a_m) = (w_1(a_1, \ldots, a_m), \ldots, w_n(a_1, \ldots, a_m)).$$

This is a morphism in Pol-G. Define multiplication of morphisms. Let $s_1^\alpha: G^{(m)} \to G^{(n)}$ and $s_2^\alpha: G^{(n)} \to G^{(p)}$ be given. Take $Z = \{z_1, \ldots, z_p\}$, $W(Z)$ and $s_2: W(Z) \to W(X)$. Proceeding from $s_1 = (w_{11}, \ldots, w_{1n}), s_2 = (w_{21}, \ldots, w_{2p})$ define $s_2^\alpha \cdot s_1^\alpha$ according to the rule $s_2^\alpha s_1^\alpha = (s_1 s_2)^\alpha$. We have

$$(s_1s_2)(z_1) = w_{21}(w_{11}(y_1, \ldots, y_m), \ldots, w_{1n}(y_1, \ldots, y_m)),$$

....

$$(s_1s_2)(z_p) = w_{2p}(w_{11}(y_1, \ldots, y_m), \ldots, w_{1n}(y_1, \ldots, y_m))$$

Define $s_2^\alpha \cdot s_1^\alpha: G^{(m)} \to G^{(p)}$ by the rule:

$$s_2^\alpha \cdot s_1^\alpha(a_1, \ldots, a_m) = (w_{21}(w_{11}(a_1, \ldots, a_m), \ldots, w_{1n}(a_1, \ldots, a_m)),$$
$$\ldots, w_{2p}(w_{11}(a_1, \ldots, a_m), \ldots, w_{1n}(a_1, \ldots, a_m))).$$

Return to the category $K_\Theta^0(G)$ and construct the functor $\alpha: K_\Theta^0(G) \to (Pol - G)$.

If $Hom(W(X), G)$, $X = \{x_1, \ldots, x_n\}$, is an object in $K_\Theta^0(G)$, then it corresponds $G^{(n)}$. To every morphism $s: W(X) \to W(Y)$ in Θ^0 corresponds $\tilde{s}: Hom(W(Y), G) \to Hom(W(X), G)$ in the category $K_\Theta^0(G)$, and in $Pol - G$ we have $s^\alpha: G^{(m)} \to G^{(n)}$. Define $\alpha(\tilde{s}) = s^\alpha$. Here α is a covariant functor.

From the commutative diagram

$$\begin{tikzcd}
Hom(W(Y),G) \arrow[r, "\tilde{s}"] \arrow[d, "\alpha_Y"'] & Hom(W(X),G) \arrow[d, "\alpha_X"] \\
G^{(m)} \arrow[r, "s^\alpha"] & G^{(n)}
\end{tikzcd}$$

we have $s^\alpha = \alpha_X \tilde{s} \alpha_Y^{-1}$, $\tilde{s} = \alpha_X^{-1} s^\alpha \alpha_Y$.

2. Applications to $\Theta(G)$ Apply this construction to the variety $\Theta(G)$ and G-algebra G. Let φ be an automorphism in $\Theta(G)^0$, $\varphi(W(X)) = W(Y)$. Suppose $|X| = |Y|$ and $Y = \varphi(X)$. Automorphism φ defines automorphism τ of the category $K_{\Theta(G)}^0$. Suppose φ acts identically on objects. Then one can define the morphism τ^α of the category $Pol - G$.

Morphisms τ and τ^α also act identically on objects. If, further, $s^\alpha\colon G^{(m)} \to G^{(n)}$ is given, then $\tau^\alpha(s^\alpha) = \alpha_X \tau(\tilde{s})\alpha_Y^{-1}$, where $\tilde{s} = \alpha_X^{-1} s^\alpha \alpha_Y$. Here τ^α is an automorphism of the category $Pol - G$. To automorphism τ corresponds a pair of functions (μ, φ). For every $X = \{x_1, \dots, x_n\}$ we have bijection

$$\mu_X = \mu_{W(X)} = \mu_W\colon Hom(W, G) \to Hom(W, G).$$

In order to consider $Pol - G$ define a bijection

$$\mu_n\colon G^{(n)} \to G^{(n)}.$$

Take $a = (a_1, \dots, a_n) \in G^{(n)}$. Let

$$\mu_n(a) = \alpha_X(\mu_X(\alpha_X^{-1}(a)) = \alpha_X \mu_X \alpha_X^{-1}(a).$$

Then,

$$\mu_n = \alpha_X \mu_X \alpha_X^{-1},\ \mu_X = \alpha_X^{-1} \mu_n \alpha_X.$$

Let us make a remark about constant maps.

Take $\nu\colon W(X) \to G$ and consider a point

$$\alpha_X(\nu) = c = (\nu(x_1), \dots, \nu(x_n)) = (c_1, \dots, c_n) \in G^{(n)}.$$

We have $\bar{\nu} = i_G \nu\colon W(X) \to W(X)$. For every $x_i \in X$ we get

$$\bar{\nu}(x_i) = i_G \nu(x_i) = i_G(c_i) = c_i.$$

Thus, endomorphism $\bar{\nu}$ is represented in the form $\bar{\nu} = (c_1, \dots, c_n)$, where all $c_i = w_i$ are considered as constant polynomials, which are elements of $W(X)$. For every point $a = (a_1, \dots, a_n)$ we obtain

$$\bar{\nu}^\alpha(a) = (c_1(a_1, \dots, a_n), \dots, c_n(a_1, \dots, a_n)) = (c_1, \dots, c_n) = c = \alpha_X(\nu),$$

i.e. $\bar{\nu}^\alpha(a)$ does not depend on a. Denote $\bar{\nu}^\alpha = \bar{c}(a) = c$.

Now make some remarks on semiautomorphisms in $Pol - G$, which are connected with semiautomorphisms in $\Theta(G)^0$.

Consider diagram

$$\begin{array}{ccc} G & \xrightarrow{i_G} & W \\ \sigma \big\downarrow & & \big\downarrow s=\mu^0_W \\ G & \xrightarrow{i_G} & W \end{array}$$

where $W = W(X) = G * W_0(X)$, s and σ are automorphisms in Θ.

As it was done earlier, we can pass to

$$\begin{array}{ccc} W(X) & \xrightarrow{\nu} & G \\ s \big\downarrow & & \big\downarrow \sigma \\ W(X) & \xrightarrow{\mu_W(\nu)} & G \end{array}$$

and define $\mu_W\colon Hom(W,G) \to Hom(W,G)$ by the rule $\mu_W(\nu) = \sigma\nu s^{-1}$. We can write $s^{-1} = s_0\sigma_W^{-1}$, where $s_0\colon W \to W$ is an automorphism of G-algebras. Then we have
$\mu_W(\nu) = \sigma\nu s_0\sigma_W^{-1}$ and

$$\alpha_X(\mu_W(\nu)) = \alpha_X\mu_W\alpha_X^{-1}\alpha_X(\nu) = \mu_n(a),$$

where $a = \alpha_X(\nu)$. Note that application of α_X to the righthandside of the formula requires some commentary. By definition $\alpha_X(\nu) = (\nu(x_1), \ldots, \nu(x_n))$, where $\nu\colon W \to G$ is a homomorphism. The map $\sigma\nu s_0\sigma_W^{-1}\colon W \to G$ can be not a homomorphism. Therefore, we have to define α_X by the same rule for any map $W \to G$. We get

$$\sigma\nu s_0\sigma_W^{-1}(x_i) = \sigma_\nu(s_0(x_i)) = \sigma\nu(w_i(x_1, \ldots, x_n)) =$$
$$= \sigma(w_i(\nu(x_1), \ldots, \nu(x_n)) = \sigma w_i(a_1, \ldots, a_n)), \qquad a_i = \nu(x_i).$$

This gives

$$\alpha_X(\sigma\nu s_0\sigma_W^{-1}) = (\sigma w_1(a_1, \ldots, a_n), \ldots, \sigma w_n(a_1, \ldots, a_n)).$$

Here $(w_1, \ldots, w_n)$ is a presentation for the automorphism s_0.

Now we define

$$\mu_n(a) = (\sigma w_1(a), \ldots, \sigma w_n(a)) = \sigma(w_1(a), \ldots, w_n(a)).$$

This $\mu_n\colon G^{(n)} \to G^{(n)}$ is considered as a semipolynomial map.

Return to $K^0_{\Theta(G)}$ and to its automorphism τ, defined by an automorphism φ of $\Theta(G)^0$. For homomorphism $s\colon W(X) \to W(Y)$, $|X| = n, |Y| = m$, we have $\tilde{s}\colon Hom(W(Y), G) \to Hom(W(X), G)$ and

$$\tilde{s}^\tau = \mu_X\tilde{s}\mu_Y^{-1} = \tilde{s}_1\colon Hom(W(Y), G) \to Hom(W(X), G).$$

Write $\mu_X\tilde{s} = \tilde{s}_1\mu_Y, \tilde{s}_1 = \widetilde{\varphi(s)}$. Let us pass to the category $Pol - G$. We use

$$\tilde{s} = \alpha_X^{-1}s^\alpha\alpha_Y, \tilde{s}_1 = \alpha_X^{-1}s_1^\alpha\alpha_Y, \mu_X = \alpha_X^{-1}\mu_n\alpha_X, \mu_Y = \alpha_Y^{-1}\mu_m\alpha_Y.$$

Then

$$\mu_X\tilde{s} = \alpha_X^{-1}\mu_n\alpha_X\alpha_X^{-1}s^\alpha\alpha_Y =$$

$$= \alpha_X^{-1}\mu_n s^\alpha\alpha_Y;$$

$$\tilde{s}_1\mu_Y = \alpha_X^{-1}s_1^\alpha\alpha_Y\alpha_Y^{-1}\mu_m\alpha_Y =$$

$$= \alpha_X^{-1}s_1^\alpha\mu_m\alpha_Y.$$

Thus,

$$\mu_n s^\alpha = s_1^\alpha\mu_m.$$

3. W and $\varphi(W)$

We consider our condition on connections between W and $\varphi(W)$ for an arbitrary Θ. Consider $s: W(X) \to W(X)$,$X = \{x_1, \ldots, x_n\}$ and present it as $s = (s_1, \ldots, s_n)$, where all s_i, $i = 1, \ldots, n$ are morphisms from $W(x) \to W(X)$. Here, s_i are defined by the condition

$$s_i(x) = s(x_i) = w_i(x_1, \ldots, x_n) = w_i.$$

The presentation $s = (s_1, \ldots, s_n)$ depends on the basis X. We have written earlier $s = (w_1, \ldots, w_n)$. Consider an automorphism $\varphi: \Theta^0 \to \Theta^0$. What can be said about

$$\varphi(s) = (\varphi(s_1), \ldots, \varphi(s_n)).$$

We will see that application of φ preserves the corresponding presentation, but this is a presentation in some special base, connected with φ.

Consider a system of injections $(\varepsilon_1, \ldots, \varepsilon_n)$,

$$\varepsilon_i: W(x) \to W(X).$$

Definition 9 We say that $(\varepsilon_1, \ldots, \varepsilon_n)$ freely defines an algebra W, if for any morphisms $f_1, \ldots, f_n$, $f_i: W(x) \to W(X)$, there exist unique $s: W(X) \to W(X)$, such that $f_i = s\varepsilon_i, i = 1, 2, \ldots, n$. Proposition 3 A collection $(\varepsilon_1, \ldots, \varepsilon_n)$ freely defines an algebra W if and only if the elements $\varepsilon_1(x), \ldots, \varepsilon_n(x)$ freely generate W. Proof Let elements $\varepsilon_1(x), \ldots, \varepsilon_n(x)$,

$\varepsilon_i(x) = x'_i$ freely generate W, $X' = \{x'_1, \ldots, x'_n\}$, and let $f_1, \ldots, f_n: W(x) \to W(X)$ be given. Define $\mu: X' \to W$ by the rule $\mu(x'_i) = f_i(x)$. The map μ uniquely defines endomorphism $s: W \to W$, such that $s(x'_i) = \mu(x'_i) = f_i(x)$. Besides $s(x'_i) = s\varepsilon_i(x)$. Thus, $s\varepsilon_i = f_i$, i.e., the set $(\varepsilon_1, \ldots, \varepsilon_n)$ freely generates W.

Let now a set $(\varepsilon_1, \ldots, \varepsilon_n)$ freely define W. Take $x'_i = \varepsilon_i(x), i = 1, 2, \ldots, n$. Show that the set $X' = \{x'_1, \ldots, x'_n\}$ freely generates W. Take an arbitrary $\mu: X' \to W$.

Define $f_1, \ldots, f_n$ by the rule $f_i(x) = \mu\varepsilon_i(x)$. Find s such that $f_i = s\varepsilon_i, i = 1, \ldots, n$. Then

$$f_i(x) = s\varepsilon_i(x) = s(x'_i) = \mu\varepsilon_i(x) = \mu(x'_i),$$
$$s(x'_i) = \mu(x'_i).$$

Hence, the endomorphism s is uniquely defined by the map μ.

We see, also, that if a set $(\varepsilon_1, \ldots, \varepsilon_n)$ freely generates W, then the set $f_1, \ldots, f_n$ defines presentation for the corresponding s in the basis $\varepsilon_1(x), \ldots, \varepsilon_n(x)$. Indeed, for every $x'_i = \varepsilon_i(x)$ we have $s(x'_i) = s\varepsilon_i(x) = f_i(x)$.

Consider, further, automorphism φ of the category Θ^0 with the condition $\varphi(W(x)) = W(y)$. Proposition 4 Let the set of morphisms $(\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_i: W(x) \to W(X)$ freely define $W = W(X), X = \{x_1, \ldots, x_n\}$. Then the set $(\varphi(\varepsilon_1), \ldots, \varphi(\varepsilon_n))$, $\varphi(\varepsilon_i): \varphi(W(x)) = W(y) \to \varphi(W(X)) = W(Y)$ freely defines $W(Y)$. Proof Although the proof is clear, we give the formal computations. Take $f'_1, \ldots, f'_n$, $f'_i: W(y) \to W(Y)$. We have to check that there exists unique $s': W(Y) \to W(Y)$, such that $f'_i = s'\varphi(\varepsilon_i)$, $i = 1, \ldots, n$. Take $f_i: W(x) \to W(X)$, $f_i = \varphi^{-1}(f'_i)$. There is unique s, such that $f_i = s\varepsilon_i$. Then $\varphi(f_i) = f'_i = \varphi(s)\varphi(\varepsilon_i)$. For s' take $\varphi(s), f'_i = s'\varphi(\varepsilon_i)$, $i = 1, \ldots, n$. Since s is unique, s' is unique as well. Corollary Algebra $W(Y)$ has a system of free generators, consisting of n elements. Proof Take $\varepsilon_i: W(x) \to W(X)$, such that $\varepsilon_i(x) = x_i, i = 1, 2, \ldots, n$. Then the set $(\varepsilon_1, \ldots, \varepsilon_n)$ freely defines $W = W(X)$. The set $(\varphi(\varepsilon_1), \ldots, \varphi(\varepsilon_n))$ freely defines $\varphi(W) = W(Y)$. Then the elements $\varphi(\varepsilon_1)(y), \ldots, \varphi(\varepsilon_n)(y)$, freely generates $W(Y)$. Definition 10 A variety Θ is called a regular variety, if for any free algebra $W = W(X), |X| = n$, every other system of free generators of W, also consists of n elements.

Now we can state that if $W(Y) = \varphi(W(X))$, then $|Y| = |X|$ if Θ is regular. Fix

$(\varepsilon_1, \dots, \varepsilon_n), \varepsilon_i(x) = x_i$, take $(\varphi(\varepsilon_1), \dots, \varphi(\varepsilon_n))$ and $y'_1 = \varphi(\varepsilon_1)(y), \dots, y'_n = \varphi(\varepsilon_n)(y), Y' = \{y'_1, \dots, y'_n\}$. Proposition 5 If an endomorphism $s: W \to W$ in the basis $X = \{x_1, \dots, x_n\}$ has presentation $s = (s_1, \dots, s_n)$, then in the base Y' we have $\varphi(s) = (\varphi(s_1), \dots, \varphi(s_n))$. Proof

$$\varphi(s)(y'_i) = \varphi(s)\varphi(\varepsilon_i)(y) = \varphi(s\varepsilon_i)(y) = \varphi(s_i)(y).$$

§8. Perfect and semiperfect varieties

1. Some reductions In this section we consider conditions on $\Theta(G)$ to be perfect or semiperfect. Take an automorphism $\varphi\colon \Theta(G)^0 \to \Theta(G)^0$. It corresponds φ_1 which does not change objects, and such that φ is inner (semiinner) if and only if φ_1 is inner (semiinner). Thus, the first reduction is to consider φ which does not change objects. Now, let the decomposition $\varphi = \varphi_1\varphi_2$ be given. Consider τ defined by φ. For every $W = W(X)$ we have

$$\begin{aligned}
&\tau(Hom(W,G)) = Hom(\varphi(W),G) = \\
&= Hom(\varphi_1(\varphi_2(W)),G) = \tau_1(Hom(\varphi_2(W),G) = \\
&= \tau_1(\tau_2(Hom(W,G)) = \tau_1 \cdot \tau_2(Hom(W,G)).
\end{aligned}$$

Here, τ_1, τ_2 corresponds to φ_1, φ_2 respectively. On objects we have $\tau = \tau_1\tau_2$. Check on morphisms. Let $s\colon W^1 \to W^2$. It corresponds $\tilde{s}\colon Hom(W^2,G) \to Hom(W^1,G)$. We have $\tilde{s}^\tau = \widetilde{\varphi(s)} = \varphi_1\widetilde{\varphi_2}(s) = \varphi_1(\widetilde{\varphi_2(s)})$. $\tilde{s}^{\tau_1\tau_2} = (\tilde{s}^{\tau_2})^{\tau_1} = \widetilde{\varphi(s)}^{\tau_1} = \varphi_1(\widetilde{\varphi_2(s)})$. Thus, $\tau_1\tau_2(\tilde{s}) = \tau(\tilde{s})$ and $\tau = \tau_1\tau_2$ takes place also on morphisms.

Let us pass to the pair of functions (μ, φ) defined by φ and τ, where φ does not change objects. For every $W = W(X)$ an automorphism τ and the function μ are connected by the rule

$$\tilde{\tilde{\nu}}^\tau = \overline{\widetilde{\mu_W(\nu)}}, \quad \text{for} \ \ \nu\colon W \to G.$$

Let now $\tau = \tau_1 \cdot \tau_2$. Then

$$\begin{aligned}
&\tilde{\tilde{\nu}}^{(\tau_1\tau_2)} = \tilde{\tilde{\nu}}^{\tau_2})^{\tau_1} = \overline{\widetilde{\mu^2_W(\nu)}}^{\tau_1} = \mu^1_{\varphi_2(W)}\mu^2_W(\nu) \\
&= \overline{\widetilde{\mu^1_{\varphi_2(W)}\mu^2_W)(\nu)}}.
\end{aligned}$$

Here μ^1 corresponds to τ_1 and μ^2 corresponds to τ^2. Thus, if (μ^1, φ_1) and (μ^2, φ_2) are given, then $(\mu, \varphi) = (\mu^1\mu^2, \varphi_1\varphi_2)$. Here, $(\mu^1\mu^2)_W = \mu^1_{\varphi^2(W)}\mu^2_W$. This rule for multiplication of pairs (μ, φ) was the second reduction. Consider the third reduction.

Let an automorphism $\varphi: \Theta(G)^0 \to \Theta(G)^0$ be given, and let φ does not change objects. For every $W = W(X)$ automorphism φ induces an automorphism of the semigroup $EndW$. Take $W = W(x)$, the free algebra with one generator. Denote by φ_0 the automorphism of $EndW(x)$, induced by φ.

Let $\bar{\varphi_0}: \Theta(G)^0 \to \Theta(G)^0$ be the automorphism of $\Theta(G)^0$, constructed by φ_0, which coincides with φ_0 on $EndW(x)$ and which also does not change objects.

Suppose φ_0 is semiinner and make $\bar{\varphi}_0$ also semiinner. Consider a semiautomorphism

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W(x) = G * W_0(x) \arrow[d, "s"] \\ G \arrow[r, "i_G"] & W(x) = G * W_0(x) \end{tikzcd}$$

such that for every endomorphism $\eta: W(x) \to W(x)$, $\eta^{\varphi_0} = s\eta s^{-1}$ holds. Consider the diagram

$$\begin{tikzcd} W(x) \arrow[r, "\nu"] \arrow[d, "s"'] & G \arrow[d, "\sigma"] \\ W(x) \arrow[r, "\mu_x(\nu)"] & G \end{tikzcd}$$

$$\mu_x(\nu) = \sigma\nu s^{-1}, \ \ s^{-1} = s_0\sigma_W^{-1},$$

where s_0 is automorphism of $W(x)$ as G-algebra. From μ_x we come to $\mu_1: G \to G$ in the category $Pol-G$. A point $\nu(x) = a$ corresponds to homomorphism ν. Then $\mu_1(a) = \sigma w(a)$, where $w(x) = s_0(x)$. Such representation of μ_1 reconstructs φ_0 as semiinner automorphism.

Construct $\bar{\mu}_n: G^{(n)} \to G^{(n)}$. For a point $a = (a_1, \ldots, a_n)$ we set

$$\bar{\mu}_n(a) = \sigma(w(a_1), \ldots, w(a_n)).$$

Having $\bar{\mu}_n$ we construct $\bar{\mu}_W$ for $W = W(X), |X| = n$, $X = \{x_1, \ldots, x_n\}$. Define an automorphism

$$\bar{s_0}: W(X) \to W(X)$$

by the rule $\bar{s}_0(x_i) = w(x_i)$. This is an automorphism and $\bar{\mu}_n(a) = \sigma s_0^{\alpha}(a)$. Now $\mu_W(\nu) = \sigma\nu\bar{s_0}\sigma_W^{-1}$ and we have

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W(X) \arrow[d, "s"] \\ G \arrow[r, "i_G"] & W(X) \end{tikzcd}$$

where $s^{-1} = \bar{s}_0\sigma_W^{-1}$. Take a function ψ by the rule $\psi_W = (\sigma, s)$. It determines semiinner automorphism $\bar{\varphi}_0$ which coincides with φ_0 on $W(x)$.

Return to the initial φ. Let $\varphi = \varphi_1\bar{\varphi}_0, \varphi_1 = \varphi \cdot \bar{\varphi}_0^{-1}$. Decomposition of φ gives rise to decomposition of τ, $\tau = \tau_1\tau_2$, where τ_2 corresponds to the automorphism $\bar{\varphi}_0$. If now μ is a function for τ, then $\mu = \mu^1\mu^2$, $\mu_W = \mu_W^1 \cdot \mu_W^2$. Let now $W_0 = W(x)$. Then $\mu_{W_0} = \mu_{W_0}^1 \cdot \mu_{W_0}^2$. But $\mu_{W_0} = \mu_{W_0}^2$ since φ and $\bar{\varphi}_0$, τ_1 and τ_2 coincide on W_0. Therefore $\mu_{W_0}^1 = 1$.

We will show that φ_1 with this property is an inner automorphism. Let us find out what the condition $\mu_W = 1$ means. Take an arbitrary $\nu: W_0 \to G$ and let $\mu_{W_0} = 1$ for φ. Then

$$\bar{\nu}^\tau = \widetilde{\overline{\mu_{W_0}(\nu)}} = \tilde{\bar{\nu}}.$$

Thus, $\tilde{\bar{\nu}}$ is fixed under τ. Then endomorphism $\bar{\nu}: W_0 \to W_0$ is invariant under automorphism φ. By definition $\bar{\nu} = i_G\nu, \bar{\nu}(x) = i_G(\nu(x)) = i_G(a)$, where $a = \nu(x) \in G$. In other words, $\bar{\nu}$ takes x to constant and φ leaves every such constant fixed.

So, the condition $\mu_{W_0} = 1$ means that automorphism φ does not change automorphisms of $W_0 = W(x)$ which take variable x to a constant.

Every such φ which also does not change objects will be called a *special automorphism* of the category $\Theta(G)^0$.

2. The main theorem Theorem 11 Let the semigroup $EndW_0$ is perfect in $\Theta(G)$. Then the variety $\Theta(G)$ is perfect too. If the semigroup $EndW_0$ is semiperfect then the variety $\Theta(G)$ is semiperfect. Proof Take an arbitrary automorphism $\varphi: \Theta(G)^0 \to \Theta(G)^0$, and show that φ is either inner (in the first case), or semiinner. We can assume that φ does not change objects. Then $\varphi = \varphi_1 \cdot \bar{\varphi}_0$. If φ_0 is inner, or semiinner, then $\bar{\varphi}_0$ is the same type.

Thus, the theorem will be proved, if φ_1 is inner. So, we are going to prove the following fact. Let φ be a special automorphism of the category $\Theta(G)^0$. Then φ is inner automorphism.

Let us pass to the categories $K_{\Theta(G)}^0$ and $Pol - G$. Automorphism τ of $K_{\Theta(G)}^0$ corresponds to the automorphism φ. For every X, $|X| = n$, we have a bijection

$$\mu_W = \mu_{W(X)}: Hom(W, G) \to Hom(W, G)$$

and, correspondingly, we have

$$\mu_n : G^{(n)} \to G^{(n)}.$$

We want to compute μ_W and μ_n. Show that for some automorphism

$$\zeta_W : W \to W, \text{ we have } \mu_W = \tilde{\zeta}_W, \ \mu_n = \zeta_W^\alpha.$$

Take an arbitrary automorphism $s: W \to W$ and let $X = \{x_1, \dots, x_n\}$. Then $s = (s_1, \dots, s_n)$, where $s_i: W(x) \to W(X)$ are morphisms in $\Theta(G)$, $i = 1, \dots, n$, and $s(x_i) = s_i(x) = w_i(x_1, \dots, x_n)$.

Take also $\varepsilon_i: W(x) \to W(X)$, defined by $\varepsilon_i(x) = x_i$. The set $(\varepsilon_1, \dots, \varepsilon_n)$ freely defines W. Take a new basis $Y = \{y_1, \dots, y_n\}$, where $y_i = \varphi(\varepsilon_i)(x)$. In this new basis $\varphi(s) = (\varphi(s_1), \dots, \varphi(s_n))$.

Return to the basis X. Consider an automorphism $\sigma_W: W \to W$, defined by $\sigma_W(x_i) = y_i$. In the base Y we have $\varphi(s)(y_i) = \varphi(s_i)(x) = w'_i(y_1, \dots, y_n)$. In the base X we have

$$\varphi(s)(\sigma x_i) = (\varphi(s) \cdot \sigma)(x_i) = w'_i(\sigma x_1, \dots, \sigma x_n) =$$

$$= \sigma w'_i(x_1, \dots, x_n) = \varphi(s_i)(x).$$

Thus, morphisms $\varphi(s_1), \dots, \varphi(s_n)$ present in the base X the automorphism $\varphi(s) \cdot \sigma$.

Using $s_i: W(x) \to W(X)$, we have

$$\tilde{s}_i^\tau = \mu_x \tilde{s}_i \mu_X^{-1} = \widetilde{\varphi(s_i)}, \qquad \tilde{s}_i = \widetilde{\varphi(s_i)} \mu_X,$$

since $\mu_x = 1$.

In the category $Pol - G$ we have

$$s_i^\alpha = \varphi(s_i)^\alpha \mu_n, \ \ i = 1, 2, \dots, n.$$

We have a system of equations, which defines μ_n.

If $s = (s_1, \dots, s_n)$, then for every $a \in G^{(n)}$ we have

$$s^\alpha(a) = (s_1^\alpha(a), \dots, s_n^\alpha(a)).$$

In our case

$$s^\alpha(a) = (s_1^\alpha(a), \dots, s_n^\alpha(a)) = (\varphi(s_1)^\alpha \mu_n(a), \dots, \varphi(s_n)^\alpha \mu_n(a)) =$$

$$(\varphi(s_1)^\alpha(\mu_n(a)), \dots, \varphi(s_n)^\alpha(\mu_n(a)).$$

As we know, the sequence $(\varphi(s_1), \ldots, \varphi(s_n))$ in the base X presents automorphism $\varphi(s)\sigma$. Therefore,

$$(\varphi(s)\sigma)^{\alpha}(\mu_n(a)) = (\varphi(s_1)^{\alpha}(\mu_n(a)), \ldots, \varphi(s_n)^{\alpha}(\mu_n(a)).$$

Thus, for every $a \in G^{(n)}$, we have $s^{\alpha}(a) = (\varphi(s)\sigma)^{\alpha}(\mu_n(a))$. Hence, $((\varphi(s)\sigma)^{\alpha^{-1}} s^{\alpha}(a) = \mu_n(a)$;

$$((\varphi(s))\sigma)^{-1})^{\alpha} s^{\alpha}(a) = (s\sigma^{-1}\varphi(s)^{-1})^{\alpha}(a) = \mu_n(a),$$

for every a. Then

$$\mu_n = (s\sigma^{-1}\varphi(s)^{-1})^{\alpha}$$

Take $\zeta_W = s\sigma^{-1}\varphi(s)^{-1}$. Now $\mu_n = \zeta_W^{\alpha}$; $\mu_W = \tilde{\zeta}_W$. Here, ζ_W depends on s, but μ_n does not depend on s. Hence, ζ_W, indeed does not depend on s. Take now an arbitrary morphism $s \colon W^1 \to W^2$ in $\Theta(G)^0$. Pass from s to $\tilde{s}$. We have

$$\tilde{s}^{\tau} = \widetilde{\varphi(s)} = \mu_{W^1}\tilde{s}\mu_{W^2}^{-1} = \tilde{\zeta}_{W^1}\tilde{s}\tilde{\zeta}_{W^2}^{-1} = \widetilde{\zeta_{W^2}^{-1}s\zeta_{W^1}}.$$

Hence,

$$\varphi(s) = \zeta_{W^2}^{-1}s\zeta_{W^1}.$$

The theorem is proved.

§9. The classic variety. Problems and applications

1. Varieties $Var - P$ and $Grp - F$

Variety $Var - P$ is a classical variety over the field P. Variety $Grp - F$ is, in fact, $\Theta(G)$, where Θ is the variety of groups and $G = F(a_1, \ldots, a_m)$ is a free group with free generators $a_1, \ldots, a_n$.

In the papers [Be1], [Be2] it was proved that:

1. If $P[x]$ is algebra of polynomials with one variable x, then the semigroup $EndP[x]$ is semiperfect, i.e. every its automorphism is semiinner.

For $\Theta(F) = Grp - F$ every free group is the free product $F * F(X)$.

2. The semigroup $End(F * \{x\})$ is semiperfect.

So we have Theorem 12 The variety $Var - P$ is semiperfect. If the field P does not have automorphisms, then $Var-P$ is perfect. Theorem 13 Variety $Grp-F$ is semiperfect.

Let the variety $Var-P$ be perfect and the field P have no automorphisms. In this case the similarity of the two extensions L_1 and L_2 of the field P became an equivalence [Pl5]. Thus, if the field P does not have non-trivial automorphisms, then the corresponding categories $K_P(L_1)$ and $K_P(L_2)$ are (correctly) isomorphic if and only if the extensions L_1 and L_2 are geometrically equivalent.

2. Problems Consider other varieties $\Theta(G)$. We have here morphisms and semimorphisms. The last are represented by diagrams

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ \sigma\downarrow & & \downarrow\mu \\ G & \xrightarrow{h'} & H' \end{array}$$

where σ is an endomorphism of the algebra of constants. In particular, for every G-algebra H we have the usual semigroup $EndH$ in $\Theta(G)$, and $SEnd(H)$ in $\Theta(G)$ with semimorphisms. Similarly, there are $AutH$ and $SAutH$. *The semigroup $EndH$ is called* **perfect** *if every automorphism of this semigroup is inner, i.e., induced by some invertible element from $EndH$. We say that $EndH$ is* **semiperfect** *if every automorphism of this semigroup is semiinner, i.e., induced by some invertible element from $SEndH$.*

Let now $H = W(X)$. Problem 3 When the semigroup $EndH$ is perfect (semiperfect)?

This problem is similar to the well-known investigations in matrix semigroups, groups and algebras (see, for example [OMe]).

Let Θ be a variety of Lie algebras over a field P, $charP = 0$, and let $L = L(a_1, \dots, a_n)$ be the free algebra in Θ, $\Theta(L) = Lie - L$. As it was noticed, R. Lipjansky has shown that the condition $(*)$ is fulfilled in this situation. Take in $Lie - L$ a free algebra $W(x)$, x is a variable. Problem 4 Is it true that the semigroup $EndW(x)$ is semiperfect?

If yes, then $Lie - L$ is semiperfect. R. Lipjansky also has noticed that the algebra L in $Lie - L$ is anticommutative, and, therefore is stable.

We have here the general problem to consider algebraic geometry in $Lie - L$. In comparison, algebraic geometry in $Grp - F$ now attracts a lot of efforts, see [BMR], [ChM], [Ri], [RS], [Ma], [Ra], etc. Problem 5 As we have seen, if two G-algebras H and H' are semiisomorphic, then they are similar. Thus, relation of semiisomorphism and relation of equivalence generate a relation, which imply similarity. In which cases similarity is generated by these two relations? In other words, the question is as follows. Let H and H' be similar. When one can state that there is a sequence

$$H = H_1, H_2, \ldots, H_n = H'$$

such that H_i and H_{i+1} are semiisomorphic or equivalent.

As we know, similarity of algebras H and H' is defined with the diagram

$$\begin{tikzcd} \Theta^0 \arrow[r, "Cl_H"] \arrow[d, "\varphi"'] & Set \\ \Theta^0 \arrow[ur, "Cl_{H'}"'] & \end{tikzcd}$$

One can ask about the following decomposition of the diagram

$$\begin{tikzcd} \Theta^0 \arrow[d, "\varphi_1"'] \arrow[dr, "Cl_{H_1}"] & \\ {} \arrow[r, "Cl_{H_2}"] \arrow[d, "\varphi_{n-1}"'] & Set \\ \Theta^0 \arrow[ur, "Cl_{H_n}"'] & \end{tikzcd}$$

$\varphi = \varphi_{n-1} \cdots \varphi_1$.

Here, the question is whether it is possible to find an appropriate H_i in case the decomposition of φ is known? (See Theorem 14.)

Problem 6 To study what this decomposition gives for similarity of algebras.

Problem 7 What common features have similar algebras H_1 and H_2.

If H_1 and H_2 are equivalent, then they have the same quasiidentities [PPT]. Do we have some statement of the same nature for similar algebras H_1 and H_2?

Problem 7 can be considered separately in the varieties of the type $\Theta(G)$. What one can say about G-algebras H_1 and H_2, if they are similar and equivalent as algebras in Θ?

Definition G-algebra H is called *algebraically closed* if for every $W = W(X) = G * W_0(X)$ and every proper congruence $T \triangleleft W$ there is G-homomorphism $\mu: W \to H$ with $T \subseteq Ker\mu$.

Problem 8 What can be said on equivalence or similarity of two G-algebras H_1 and H_2 if G is algebraically closed.

From Hilbert Nullstellensatz it follows that if P is algebraically closed, then L_1 and L_2 are equivalent.

It would the of special interest to find out how Hilbert Nullstellensatz looks like for variety $\Theta(G)$ with algebraically closed G.

3. Similarity of algebras in semiperfect varieties $\Theta(G)$

First, let us discuss some details of the notion of similarity. The definition was given in [PL5], [Pl6]. We repeat it here with some modifications in the notations.

Consider two functions β and γ which determine the maps β_W and γ_W defined on the objects of the category Θ^0. The map β_W assigns to each congruence T in W the equivalence $\rho = \beta_W(T)$ on the semigroup $EndW$, defined by the rule: $\nu\rho\nu'$ if and only if $w^\nu T w^{\nu'}$ for every $w \in W$.

The map γ_W assigns to each equivalence ρ on $EndW$ some relation T on W. By the definition, $w_1 T w_2$ if $w_1 = w^\nu$, $w_2 = w^{\nu'}$, for some $w \in W$, $\nu, \nu' \in EndW$, and $\nu\rho\nu'$.

We have: $\gamma_W(\beta_W(T) = T$, if T is a congruence in W.

Let the algebras H,H' from Θ be given. We have the functors

$$Cl_H \colon Var(H)^0 \to Set$$

and

$$Cl_{H'} \colon Var(H')^0 \to Set.$$

The algebras H and H' are similar, if there exists an isomorphism of categories $\varphi \colon Var(H)^0 \to Var(H'))^0$, which induces a commutative diagram

$$\begin{tikzcd} Var(H)^0 \arrow[rr, "\varphi"] \arrow[dr, "Cl_H"'] & & Var(H')^0 \arrow[dl, "Cl_{H'}"] \\ & Set & \end{tikzcd}$$

Here, the commutativity of the diagram means that the functors Cl_H and $Cl_{H'}\varphi$ are isomorphic. We have an isomorphism $\alpha = \alpha(\varphi) \colon Cl_H \to Cl_{H'}\varphi$, which depends on φ.

Now we want to study the relation between α and φ.

By the definition:

$$\alpha(\varphi)_W(T) = \gamma_{\varphi(W)}(\varphi(\beta_W(T)), \quad T \in Cl_H(W).$$

Here, $\varphi(\beta_W(T))$ is the relation in $End\varphi(W)$ defined by the rule

$$\mu\varphi(\beta_W(T))\mu' \Leftrightarrow \mu = \varphi(\nu), \quad \mu' = \varphi(\nu'), \qquad \nu, \nu' \in EndW, \ \nu(\beta_W(T))\nu'.$$

Simultaneously,

$$\varphi(\beta_W(T)) = \beta_{\varphi(W)}(\alpha(\varphi)_W(T)).$$

For every W we have a bijection $\alpha(\varphi)_W : Cl_H(W) \to Cl_{H'}(\varphi(W))$, and for every $\nu : W_1 \to W_2$ there is a commutative diagram

$$\begin{tikzcd} Cl_H((W_2) \arrow[r, "\alpha(\varphi)_{W_2}"] \arrow[d, "Cl_H(\nu)"'] & Cl_{H'}\varphi(W_2) \arrow[d, "(Cl_{H'}\varphi)(\nu)"] \\ Cl_H((W_1) \arrow[r, "\alpha(\varphi)_{W_1}"] & Cl_{H'}\varphi(W_1) \end{tikzcd}$$

We assume that the isomorphism α should satisfy the additional condition. Namely, consider a function τ, defined for every pair of objects W and W' in $Var(H)^0$. It takes a congruence T in W_2 to the relation $\rho = \tau_{W_1,W_2}(T)$ on $Hom(W_1, W_2)$ by the rule: $s\rho s'$, where s and $s' \in Hom(W_1, W_2)$ if and only if $w^s T w^{s'}$ for every $w \in W_1$.

The isomorphism α should be coordinated with the function τ in the following sense

$$\tau_{\varphi(W_1),\varphi(W_2)}(\alpha(\varphi)_{W_2}(T)) = \varphi(\tau_{W_1,W_2}(T)).$$

Note here, that the bijection $\alpha(\varphi) : Cl_H(W) \to Cl_{H'}(\varphi(W))$ preserves the natural ordering for congruences.

Assume further, that for φ the decomposition $\varphi = \varphi_2\varphi_1$ is given. Here,

$$\varphi_1 : Var(H)^0 \to Var(H_1)^0, \qquad \varphi_2 : Var(H_1)^0 \to Var(H')^0,$$

for some H_1, and φ_1, φ_2 are isomorphisms.

We want to find out how α is coordinated with this decomposition. Along with Cl_H and $Cl_{H'}$ we have also the functor Cl_{H_1}. We will calculate $\alpha(\varphi_2\varphi_1)$.

Take $T \in Cl_H(W)$. Then

$$\varphi_W(\beta_W(T)) = (\varphi_2\varphi_1)_W(\beta_W(T)) = \varphi_{2\varphi_1(W)}\varphi_{1W}(\beta_W(T).$$

Further,

$$\alpha(\varphi)_W(T) = \gamma_{\varphi(W)}(\varphi_W(\beta_W(T)) = \gamma_{\varphi(W)}(\varphi_{2\varphi_1(W)}\varphi_{1W}(\beta_W(T)).$$

Since $\alpha_1 = \alpha(\varphi_1)$ and $\alpha_2 = \alpha(\varphi_2)$, we have $\varphi_{1W}(\beta_W(T)) = \beta_{\varphi_1(W)}(\alpha(\varphi_1)_W(T))$.

Denote $\alpha(\varphi_1)_W(T) = T^*$. This is a congruence in $Cl_{H_1}(\varphi_1(W))$. Then,

$$\alpha(\varphi)_W(T) = \gamma_{\varphi(W)}(\varphi_{2\varphi_1(W)}\beta_{\varphi_1(W)}(T^*)) = \gamma_{\varphi_2(\varphi_1(W))}(\varphi_{2\varphi_1(W)}\beta_{\varphi_1(W)}(T^*)) =$$

$$\alpha(\varphi_2)_{\varphi_1(W)}(T^*) = \alpha(\varphi_2)_{\varphi_1(W)}\alpha(\varphi_1)_W(T).$$

Thus,

$$\alpha(\varphi_2\varphi_1)_W = \alpha(\varphi_2)_{\varphi_1(W)}\alpha(\varphi_1)_W.$$

We note now the proposition, which, in fact, is contained in [Pl5, Proposition 8]. Proposition 6. Let $Var(H) = Var(H') = \Theta$, and H and H' are similar with respect to an inner automorphism $\varphi: \Theta^0 \to \Theta^0$. Then H and H' are equivalent.

Let us pass now to the our main goal

Theorem 14 Let H and H' be algebras from the variety $\Theta(G)$, $Var(H) = Var(H') = \Theta(G)$, and let they are similar with respect to semiinner automorphism $\varphi: \Theta(G)^0 \to \Theta(G)^0$. Then there exists an algebra $H_1 \in \Theta(G)$, such that H and H_1 are semiisomorphic, H_1 and H' are equivalent.

Proof Let φ is determined by the function $\psi = (\sigma, s)$ as a semiinner automorphism. Using H and $\sigma \in Aut(G)$, construct an algebra H_1. We have $h: G \to H$. Take $h_1 = h\sigma$, $h_1: G \to H$. Denote the new G-algebra by H_1. The algebra H_1 coincides with H as an algebra in Θ, but in $\Theta(G)$ these H and H_1 are semiisomorphic. We have

$$\begin{array}{ccc} G & \xrightarrow{h_1} & H_1 \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\mu=1} \\ G & \xrightarrow{h} & H \end{array}$$

By Theorem 8 G-algebras H and H_1 are similar, and the similarity is given by a semiinner automorphism $\varphi_1: \Theta(G)^0 \to \Theta(G)^0$. The corresponding $\psi_1 = (\sigma, s_1)$ is defined by $\psi_{1W} = (\sigma, \sigma_W)$, $s_{1W} = \sigma_W$, φ_1 does not change objects.

Decomposing $\varphi = \varphi_2\varphi_1$ we get $\varphi_2 = \varphi\varphi_1^{-1}$. We show that φ_2 is inner automorphism which determines the equivalence of algebras H_1 and H'.

For φ we have $\psi = (\sigma, s))$ and for W

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W = G * W_0 \arrow[d, "s_W"] \\ G \arrow[r, "i'_G"] & \varphi(W) = G * W'_0 \end{tikzcd}$$

We have also

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W \arrow[d, "\sigma_W"] \\ G \arrow[r, "i_G"] & W \end{tikzcd}$$

This gives decomposition

$$(\sigma, s_W) = (1, s_W^0)(\sigma, \sigma_W), \quad s_W = s_W^0 \sigma_W,$$

where s_W^0 is an isomorphism of G-algebras W and $\varphi(W)$.

We have also

$$\psi = \psi^2\psi^1, \quad \psi_W = (\sigma, s_W) = \psi^2_{\varphi_1(W)}\psi^1_W = \psi^2_W\psi^1_W = \psi^2_W(\sigma, \sigma_W).$$

Hence, $\psi^2_W = (1, s^0_W)$. Since φ and φ_1 are semiinner, the automorphism φ_2 is also semiinner. It is defined by the function $\psi^2 = (1, s^0)$. Hence φ_2 is inner automorphism.

Now we have to check that ψ^2 gives the similarity of the algebras H_1 and H'. This means that if we define $\alpha_2 = \alpha(\varphi_2)$ by the rule

$$\alpha_{2W}(T) = \gamma_{\varphi_2(W)}(\varphi_{2W}(\beta_W(T)),$$

then we get a bijection $\alpha_{2W}\colon Cl_{H_1}(W) \to Cl_{H'}(\varphi_2(W)$, and α_2 defines an isomorphism of functors Cl_{H_1} and $Cl_{H'}\varphi_2$, which is coordinated with the corresponding function τ.

For $\varphi = \varphi_2\varphi_1$ we have

$$\alpha_(\varphi)_W = \alpha(\varphi_1\varphi_2)_W = \alpha_(\varphi_2)_{\varphi_1(W)}\alpha(\varphi_1)_W.$$

Here, $\alpha(\varphi)_W$ is the bijection $Cl_H(W) \to Cl_{H'}(\varphi(W))$, and $\alpha(\varphi_1)_W$ is the bijection $Cl_H(W) \to Cl_{H_1}(W)$. Therefore, $\alpha(\varphi_2)_{\varphi_1(W)} = \alpha_{2W}$ is the bijection $Cl_{H_1}(W) \to Cl_{H'}(\varphi_2(W))$.

We need to check that α_2 is an isomorphism of functors. Take a morphism of G-algebras $\nu: W_1 \to W_2$. Since α is isomorphism of functors Cl_H and $Cl_{H'}\varphi$, for any $T \in Cl_H(W_2)$ we have

$$\alpha(\varphi)_{W_1}(\nu T) = \varphi(\nu)\alpha(\varphi)_{W_2}(T).$$

Since,

$$\alpha(\varphi)_{W_1} = \alpha(\varphi_2)_{W_1}\alpha(\varphi_1)_{W_1},$$

$$\alpha(\varphi)_{W_2} = \alpha(\varphi_2)_{W_2}\alpha(\varphi_1)_{W_2},$$

then

$$\alpha(\varphi_2)_{W_1}\alpha(\varphi_1)_{W_1}(\nu T) = \varphi_2(\varphi_1(\nu))\alpha(\varphi_2)_{W_2}\alpha(\varphi_1)_{W_2}(T).$$

We have also $\alpha(\varphi_1)_{W_1}(\nu T) = \varphi_1(\nu)\alpha(\varphi_1)_{W_2}(T)$. This gives

$$\alpha(\varphi_2)_{W_1}\varphi_1(\nu)\alpha(\varphi_1)_{W_2}(T) = \varphi_2(\varphi_1(\nu)\alpha(\varphi_2)_{W_2}\alpha(\varphi_1)_{W_2}(T).$$

Denote $\varphi_1(\nu) = \nu_1: W_1 \to W_2$, and $\alpha(\varphi_1)_{W_2}(T) = T^* \in Cl_{H_1}(W_2)$. Then,

$$\alpha(\varphi_2)_{W_1}(\nu_1 T^*) = \varphi_2(\nu_1)\alpha(\varphi_2)_{W_2}T^*.$$

This means that α_2 defines isomorphism of functors Cl_{H_1} and $Cl_{H'}\varphi_2$.

It remains to check the coordination with τ.

Take W_1 and W_2 from $\Theta(G)$. Consider $Hom(W_1, W_2)$, $T \triangleleft W_2$, $\tau_{W_1,W_2} = \rho$. We have

$$\tau_{(\varphi(W_1),\varphi(W_2)}(\alpha(\varphi)_{W_2}(T)) = \varphi(\tau_{W_1,W_2}(T)).$$

Since $\varphi = \varphi_2\varphi_1$,

$$\tau_{(\varphi_2(\varphi_1(W_1),(\varphi_2(\varphi_1(W_2)))}(\alpha(\varphi_2)_{W_1}\alpha(\varphi_1)_{W_1}(T)) =$$

$$\varphi_2(\varphi_1(\tau_{W_1,W_2}(T)) = \varphi_2(\tau_{\varphi_1(W_1),\varphi_1(W_2)}(\alpha(\varphi_1)_{W_2}(T)).$$

We have $\varphi_1(W_1) = W_1$, $\varphi_1(W_2) = W_2$, and let $\alpha(\varphi_1)_{W_2}(T) = T^*$. Then,

$$\varphi_2(\tau_{(W_1),(W_2)}(T^*) = \tau_{\varphi_2(W_1),\varphi_2(W_2)}(\alpha(\varphi_2)_{W_1}(T^*)).$$

This gives compatibility α_2 with τ. Thus, φ_2 gives similarity of algebras H_1 and H'. Since φ_2 is inner, by the proposition 6 algebras H_1 and H' are equivalent. Corollary If the variety

$\Theta(G)$ is semiperfect, then its algebras H and H' such that $Var(H) = Var(H') = \Theta(G)$ are similar if and only if there exists H_1 such that H_1 is semiisomorphic to H and equivalent to H'.

The existence of such H_1 is regarded as an equivalence of H and H' up to some semiisomorphism.

This corollary solves Problem 5 for semiperfect variety $\Theta(G)$. In particular, this can be applied to $Var - P$ and $Grp - F$.

In the classical situation we have, in particular,

Theorem 15 Let P be an infinite field, and L_1, L_2 two its extensions. Categories of algebraic varieties $K_P(L_1)$ and $K_P(L_2)$ are (correctly) isomorphic if and only if L_2 and L_1 are equivalent up to some semiisomorphism.

Let , further, H and H' are similar in $\Theta(G)$. We are interested in correspondence between the identities of H and H' . Suppose that $\Theta(G)$ is semiperfect. Then there is an algebra H_1 in $\Theta(G)$, which is semiisomorphic to H and equivalent to H'. Algebras H_1 and H' have the same identities. Therefore it is sufficient to take H_1 with the semiisomorphism

$$\begin{tikzcd} G \arrow[r, "h_1"] \arrow[d, "\sigma"'] & H_1 \arrow[d, "\mu=1"] \\ G \arrow[r, "h"] & H \end{tikzcd}$$

Take $W = G * W_0$ and consider the diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\sigma"'] & W = G * W_0 \arrow[d, "\sigma_W"] \\ G \arrow[r, "i_G"] & W = G * W_0 \end{tikzcd}$$

Let T be the congruence of identities of the algebra H in W. It is defined by the algebraic variety $Hom(W, H)$. We have $T^* = \sigma_W^{-1} T$ (see §4). This is a congruence of identities of the algebra H_1. Indeed, T is the minimal H-closed congruence in W, hence, T^* is the minimal H_1 closed congruence in W. It coincides with the congruence of identities of the algebra H_1.

We have $T = \sigma_W T^*$ and $w_1 T w_2 \Leftrightarrow w_1^{\sigma_W} T w_2^{\sigma_W}$. Thus, $w_1 \equiv w_2$ is an identity of the algebra H if and only if $w_1^{\sigma_W} \equiv w_2^{\sigma_W}$ is an identity of the algebra H_1. In particular, H and H' have the same identities without constants.

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