

# Equations in simple Lie algebras

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## Abstract

We prove that for a given element  $P(X_1, \dots, X_d)$  of the finitely generated free Lie algebra  $\mathcal{L}_d$ , the induced map  $P: \mathfrak{g}^d \rightarrow \mathfrak{g}$  is dominant for any Chevalley algebra  $\mathfrak{g}$ , provided that  $K$  is of characteristic  $\neq 2$ , and  $P$  is not an identity in  $\mathfrak{sl}(2, K)$ . We prove that for the Engel monomials  $[[[X, Y], Y], \dots, Y]$  and for their linear combinations this map is surjective onto the set of non-central elements of  $\mathfrak{g}$  provided that the ground field  $K$  is big enough.

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## 1. Introduction

For a given element  $P(X_1, \dots, X_d)$  of the finitely generated free Lie algebra  $\mathcal{L}_d = \mathcal{L}(X_1, \dots, X_d)$  defined over a given field  $K$ , and a given Lie algebra  $\mathfrak{g}$  over  $K$ , one can ask the following question:

**Question 1.1.** Is the equation  $P(X_1, \dots, X_d) = A$  solvable **a)** for all  $A \in \mathfrak{g}$ , or, at least, **b)** for a generic  $A \in \mathfrak{g}$ ?

In the present paper we consider this question in a particular case of “classical” semisimple Lie algebras, i.e., quotients of Chevalley algebras modulo the centre, see Section 2.2 for precise definitions. Our motivation is two-fold. The primary inspiration came from widely discussed group-theoretic analogues of Question 1.1:

**Question 1.2.** Let  $w(x_1, \dots, x_d)$  be an element of the finitely generated free group  $\mathcal{F}_d = \mathcal{F}(x_1, \dots, x_d)$  (i.e., a word in  $x_i$  and  $x_i^{-1}$ ), and let a group  $G$  be given. Is the equation  $w(x_1, \dots, x_d) = g$  solvable **a)** for all  $g \in G$ , or, at least, **b)** for a generic  $g \in G$ ?

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If  $G$  is a connected semisimple algebraic  $K$ -group, a theorem of Borel [5], stating that the word map  $G^d \rightarrow G$  is dominant whenever  $w \neq 1$ , gives a positive answer to part b). One can, however, easily produce examples where the word map is not surjective and so the answer to part a) is negative (see [5] and references therein). Some particular words have been extensively studied, and Question 1.2a has been answered in the affirmative. Say, if  $d = 2$  and  $w(x, y) = [x, y]$  (the commutator), the positive answer is known for wide classes of groups ([19], [27], [31], [14], [39], [40], [25]).

The second motivating example is associative algebras, for which analogues of Questions 1.1 and 1.2 have also been intensely investigated (see [24] and references therein). Going over to infinitesimal analogues, one can first mention that the equation  $[X, Y] = A$  is solvable for all  $A$  in any classical split semisimple Lie algebra  $\mathfrak{g}$ , under the assumption that the ground field  $K$  is sufficiently large [9], [20].

Our first results (Section 3) concern the general case where we are given an element  $P(X_1, \dots, X_d)$  of the finitely generated free Lie algebra  $\mathcal{L}_d$  defined over a field  $K$ . Then for any Lie algebra  $\mathfrak{g}$  defined over  $K$  we can consider the induced polynomial map  $P: \mathfrak{g}^d \rightarrow \mathfrak{g}$ . Assuming that  $K$  is an arbitrary field of characteristic  $\neq 2$ , we prove that if  $P$  is not an identity in  $\mathfrak{sl}(2, K)$ , then this map is dominant for any Chevalley algebra  $\mathfrak{g}$ . Going over from dominance to surjectivity (Section 4), we prove that for the Engel monomials  $[[[X, Y], Y], \dots, Y]$  and, more generally, for their linear combinations, the image of the corresponding map contains the set of noncentral elements of  $\mathfrak{g}$  provided that the ground field  $K$  is big enough. We show that for Engel monomials of large degree this image contains no nonzero central elements. We also discuss consequences of these results for polynomial maps of associative matrix algebras as well as some other possible generalizations (Section 5).

## 2. Preliminaries

Our notation is standard. In particular,  $\mathbb{Q}$  and  $\mathbb{F}_p$  denote the field of rational numbers and the finite field of  $p$  elements, respectively.  $\mathbb{A}^n$  denotes the  $n$ -dimensional affine space. The cardinality of a set  $A$  is denoted by  $|A|$ . We use the Zariski topology throughout, and  $\bar{A}$  denotes the closure of  $A$ . The group of automorphisms of any object  $X$  (group, algebra, variety) is denoted by  $\text{Aut}(X)$ . The orbit of an element  $h$  of a set  $H$  with respect to an action of a group  $W$  is denoted by  $Wh$ , and  $WH'$  denotes the union of  $W$ -orbits of all elements of  $H' \subseteq H$ . If  $R$  is a ring on which a group  $G$  acts,  $R^G$  denotes the ring of invariants. The reader is referred to [7] and [22] for all unexplained notions and facts concerning Lie algebras.

### 2.1. Dominant maps

Recall that a  $K$ -morphism  $f: V \rightarrow W$  of algebraic  $K$ -varieties (=reduced  $K$ -schemes of finite type) is called *dominant* if its image  $f(V)$  is Zariski dense in  $W$ . We will mostly deal with the case where  $V$  and  $W$  are irreducible. In

such a case  $f(V)$  contains a non-empty open set  $U$  (see, e.g., [28, Th. IV.3.7]). If  $L/K$  is a field extension, then  $f$  is dominant if and only if the  $L$ -morphism  $f_L: V_L \rightarrow W_L$  obtained by extension of scalars is dominant.

## 2.2. Chevalley and classical Lie algebras

Let  $\mathbf{R}$  be a root system and let  $\Pi$  be a simple root system corresponding to  $\mathbf{R}$ . Further, let  $L(\mathbf{R}, \mathbb{C})$  be a semisimple complex Lie algebra. Then there exists a Chevalley basis  $\{h_\alpha\}_{\alpha \in \Pi} \cup \{e_\beta\}_{\beta \in \mathbf{R}}$  of  $L(\mathbf{R}, \mathbb{C})$  such that

- 1)  $[e_\alpha, e_{-\alpha}] = h_\alpha$  for every  $\alpha \in \Pi$ ;
- 2)  $h_\beta := [e_\beta, e_{-\beta}] \in \sum_{\alpha \in \Pi} \mathbb{Z} h_\alpha$  for every  $\beta \in \mathbf{R}$ ;
- 3)  $[h_\beta, h_\gamma] = 0$  for every  $\beta, \gamma \in \mathbf{R}$ ;
- 4)  $[h_\beta, e_\gamma] = q_{\beta, \gamma} e_\gamma$  for every  $\beta, \gamma \in \mathbf{R}$  (note that  $q_{\beta, \gamma} = 0, \pm 1, \pm 2, \pm 3$ );
- 5)  $[e_\beta, e_\gamma] = 0$  if  $\beta + \gamma \notin \mathbf{R}$ ;
- 6)  $[e_\beta, e_\gamma] = p_{\beta, \gamma} e_{\beta+\gamma}$  if  $\beta + \gamma \in \mathbf{R}$  (note that  $p_{\beta, \gamma} = \pm 1, \pm 2, \pm 3$ ).

One can now define the corresponding Lie algebra over any prime field  $F$  using the same basis and relations 1)–6) in the case  $F = \mathbb{Q}$  or the same basis and relations 1)–6) modulo  $p$  in the case  $F = \mathbb{F}_p$ . Then one can define the Lie algebra  $L(\mathbf{R}, K)$  over any field  $K$  using the same basis and relations induced by 1)–6). We will denote such an algebra by  $L(\mathbf{R}, K)$  and call it a *Chevalley* algebra. The Chevalley algebra  $L(\mathbf{R}, K)$  decomposes into the sum  $\sum_i L(\mathbf{R}_i, K)$  where  $\mathbf{R} = \cup_i \mathbf{R}_i$  is the decomposition of the root system  $\mathbf{R}$  into the disjoint union of irreducible root subsystems. The Lie algebras  $L(\mathbf{R}_i, K)$  are not simple if the characteristic of  $K$  is not a “very good prime” [11]. However, if  $\mathbf{R}_i \neq \mathbf{A}_1, \mathbf{B}_r, \mathbf{C}_r, \mathbf{F}_4$  when  $\text{char}(K) = 2$  and  $\mathbf{R}_i \neq \mathbf{G}_2$  when  $\text{char}(K) = 3$ , the algebra  $L(\mathbf{R}_i, K)/Z_i$  is simple (here  $Z_i$  is the centre of  $L(\mathbf{R}_i, K)$ ). Thus, if the Lie algebra  $L(\mathbf{R}_i, K)$  has no components pointed out above, the quotient  $L(\mathbf{R}, K)/Z$  (where  $Z$  is the centre of  $L(\mathbf{R}, K)$ ) is a semisimple Lie algebra.

Let  $L(\mathbf{R}, K)$  be a Chevalley algebra over a field  $K$  which corresponds to an irreducible reduced root system  $\mathbf{R}$ . Denote by  $\mathbf{R}^+$  (resp.  $\mathbf{R}^-$ ) the set of positive (resp. negative) roots and put

$$H = \sum_{\alpha \in \Pi} K h_\alpha, \quad U^\pm = \sum_{\beta \in \mathbf{R}^\pm} K e_\beta, \quad U = U^- + U^+.$$

Then  $L(\mathbf{R}, K) = H + U = H + U^- + U^+$ . The number  $r = \dim H = |\Pi|$  is called the *rank* of  $L(\mathbf{R}, K)$ .

Let now  $i: L(\mathbf{R}, K) \rightarrow \text{End}(V)$  be a linear representation. Then one can construct the corresponding Chevalley group  $G(\mathbf{R}, K) \leq GL(V)$ , which is generated by the so-called root subgroups  $x_\beta(t)$  (see [38], [4]), and a homomorphism  $j: G(\mathbf{R}, K) \rightarrow \text{Aut}(i(L(\mathbf{R}, K)))$ .

Suppose  $K$  is an algebraically closed field and  $i$  is a representation of  $L(\mathbf{R}, K)$  such that the group of weights of  $i$  coincides with the group generated by fundamental weights. Then  $G(\mathbf{R}, K)$  is a simple, simply connected algebraic group,  $i(L(\mathbf{R}, K))$  is the Lie algebra of  $G(\mathbf{R}, K)$ , and the homomorphism  $j$  defines the adjoint action of  $G(\mathbf{R}, K)$  on its Lie algebra  $i(L(\mathbf{R}, K))$  [4, 3.3].

Below we will always consider the Chevalley group  $G(\mathbf{R}, K)$  constructed through a faithful representation  $i$  such that  $G(\mathbf{R}, K)$  is simply connected. We also identify the Lie algebra  $i(L(\mathbf{R}, K))$  with  $L(\mathbf{R}, K)$ . The group  $j(G(\mathbf{R}, K)) \leq \text{Aut}(L(\mathbf{R}, K))$  will be denoted by  $G$ . Note that  $G$  is the group generated by the images  $j(x_\beta(t))$  of the root subgroups which will also be denoted by  $x_\beta(t)$ .

An element  $x \in L(\mathbf{R}, K)$  is called *semisimple* (resp. *nilpotent*) if for a faithful linear representation  $\rho: L(\mathbf{R}, K) \rightarrow \text{End}(V)$  the operator  $\rho(x)$  is semisimple (resp. nilpotent). Every  $x \in L(\mathbf{R}, K)$  has the Jordan decomposition  $x = x_s + x_n$  where  $x_s$  is semisimple,  $x_n$  is nilpotent,  $[x_s, x_n] = 0$ . Let  $K$  be an algebraically closed field. Then:

- a. *Every element of the Lie algebra  $L(\mathbf{R}, K)$  is  $G$ -conjugate to an element  $x = x_s + x_n$  such that  $x_s \in H$ ,  $x_n \in U^+$ ,  $[x_s, x_n] = 0$ .*
- b. [36, II.3.20] *The  $G$ -orbit of an element  $x \in L(\mathbf{R}, K)$  is closed if and only if  $x$  is semisimple.*
- c. *For every root  $\beta \in \mathbf{R}$  there is a linear map  $\beta: H \rightarrow K$  defined by the formula  $[h, e_\beta] = \beta(h)e_\beta$ . Suppose there is a regular element  $h \in H$ , i.e.,  $\beta(h) \neq 0$  for every  $\beta \in \mathbf{R}$ . Then the set of all elements in  $L(\mathbf{R}, K)$  which are  $G$ -conjugate to elements from  $H$  is dense in  $L(\mathbf{R}, K)$ .*
- d. *There is a  $G$ -equivariant dominant morphism  $\pi: L(\mathbf{R}, K) \rightarrow \mathbf{Y}$  where  $\mathbf{Y}$  is an affine variety and the map  $\bar{\pi} = \pi|_H: H \rightarrow \mathbf{Y}$  satisfies the following condition:  $\bar{\pi}^{-1}(\pi(h)) = Wh$  where  $W$  is the Weyl group, which acts naturally on  $H$ .*

*Proof.* Put  $L = L(\mathbf{R}, K)$ , and let  $S = K[L]$  be the algebra of polynomial functions on  $L$  (i.e., the symmetric algebra of the underlying vector space of  $L$ ). Since  $G$  is a simple algebraic group,  $R = S^G$  is finitely generated (see, e.g., [35, Cor. 2.4.10]), say, by  $f_1, \dots, f_k$ . Consider the map  $\pi: L \rightarrow \mathbb{A}^k$  given by the formula  $\pi(x) = (f_1(x), \dots, f_k(x))$ . If  $x = x_s + x_n$  is the Jordan decomposition then  $\pi(x) = \pi(x_s)$ . Indeed, the  $G$ -orbit  $O_{x_s}$  of  $x_s$  is contained in the Zariski closure  $\overline{O_x}$  of the  $G$ -orbit of  $x$  (the fact that the semisimple part of an element  $y$  of a semisimple group  $G$  is contained in the closure of the conjugacy class is well known, see [36, II, 3.7]; for  $G$ -orbits of the Lie algebra of  $G$  the proof of the corresponding part is almost the same, see, e.g., [23, Prop. 2.11]).

Since  $\pi$  is a regular map constant on the orbit  $O_x$ , it is constant on  $\overline{O_x}$ . Hence  $\mathbf{Y} := \overline{\text{Im } \pi} = \overline{\text{Im } \bar{\pi}}$ . Further, functions from  $R$  separate closed orbits in  $L$  (see, e.g., [29, Chapter 1, § 1.2]). Hence  $\bar{\pi}^{-1}(\pi(h)) = H \cap O_h$  where  $O_h$  is the orbit of  $h$ . Since  $H \cap O_h = Wh$  [36, 3.16], we are done.  $\square$

**Remark 2.1.** If  $\text{char}(K)$  is not a torsion prime for  $G(\mathbf{R}, K)$ , then there is an isomorphism  $\pi': L(\mathbf{R}, K)/G \xrightarrow{\sim} H/W$ , and the quotient  $H/W$  is isomorphic to  $\mathbb{A}^r$  [36], [34, 3.12]. Hence in this case  $\mathbf{Q} \cong H/W \cong \mathbb{A}^r$ . In general, the natural morphism  $H/W \rightarrow L(\mathbf{R}, K)/G$ , induced by the inclusion  $H \rightarrow L(\mathbf{R}, K)$ , is dominant; it is an isomorphism if and only if  $\mathbf{R} \neq \mathbf{C}_r$ ,  $r \geq 1$ ,  $\text{char}(K) = 2$  [12].

### 2.3. Cartan subalgebras and regular elements

Let  $\beta \in \mathbf{R}$ . We have (cf. [12, Lemma 2.3.2])

$$\beta \equiv 0 \Leftrightarrow R = C_r, r \geq 1, \text{char}(K) = 2, \beta \text{ is a long root} \quad (2.1)$$

(here  $C_1 = A_1$ ,  $C_2 = B_2$ ). Thus, if we are not in the case  $R = C_r$ ,  $r \geq 1$ ,  $\text{char}(K) = 2$ , the subalgebra  $H$  is a *Cartan* subalgebra, that is, a nilpotent subalgebra coinciding with its normalizer. In the case  $R = C_r$ ,  $r \geq 1$ ,  $\text{char}(K) = 2$ , the subalgebra  $H$  is a Cartan subalgebra of  $[L(R, K), L(R, K)] \cong L(D_r, K)$  (here  $L(D_1, K) = K, L(D_2, K) = \mathfrak{sl}(2, K) \times \mathfrak{sl}(2, K)$ ).

e. Suppose we are not in the case  $R = C_r$ ,  $r \geq 1$ ,  $\text{char}(K) = 2$ . Then if  $|K| \geq |R^+|$ , the subalgebra  $H$  contains a regular element. Moreover, if  $|K| > m|R|$  for some  $m \in \mathbb{N}$ , then for every subset  $S \subset K$  of size  $m$  there exists  $h \in H$  such that  $\beta(h) \notin S$  for every  $\beta \in R$ .

*Proof.* For infinite fields the statement is trivial. If  $K$  is a finite field, then  $|H| = |K|^r$ , and the hyperplane  $H_{x, \beta}$  of  $H$  defined by the equation  $[h, e_\beta] = x$ ,  $x \in S$ , consists of  $|K|^{r-1}$  points. Thus,

$$|\bigcup_{x \in S, \beta \in R} H_{x, \beta}| \leq |S| \cdot |H_{x, \beta}| = m|R| \cdot |K|^{r-1} < |H|,$$

and therefore we can take  $h \in H \setminus \bigcup_{x \in S, \beta \in R} H_{x, \beta}$ . The first statement can be proved by the same arguments for  $S = \{0\}$  using the fact that  $0 \in H_{0, \beta} = H_{0, -\beta}$  for every  $\beta \in R$ .  $\square$

#### 2.4. Exceptional cases

The Chevalley algebra  $L(R, K)$  modulo the centre is not simple in the following cases (see, e.g., [21]):

$$R = A_1, B_r, C_r, F_4 \text{ if } \text{char}(K) = 2, \quad R = G_2 \text{ if } \text{char}(K) = 3. \quad (2.2)$$

Namely:

- 1) Let  $R = A_1$  and  $\text{char}(K) = 2$ . Then  $L(A_1, K) \cong \mathfrak{sl}(2, K)$  is a nilpotent algebra satisfying the identity  $[[X, Y], Z] \equiv 0$ .
- 2) Let  $R = B_2$  and  $\text{char}(K) = 2$ . Then  $L(B_2, K) \cong \mathfrak{so}(5, K)$  is a solvable algebra satisfying the identity  $[[X, Y], [Z, T]] \equiv 0$ .
- 3) Let  $R = B_r, r > 2$  and  $\text{char}(K) = 2$ . Then  $L(B_r, K)$  contains the nilpotent ideal  $I$  generated by  $\{e_\beta \mid \beta \text{ is a short root}\}$ , and  $L(B_r, K)/I \cong L(D_r, K)/Z'$  where  $Z' \leq Z(L(D_r, K))$ .
- 4) Let  $R = F_4$  and  $\text{char}(K) = 2$ . Then  $L(F_4, K)$  contains the ideal  $I$  generated by  $\{e_\beta \mid \beta \text{ is a short root}\}$ , and  $L(F_4, K)/I \cong L(D_4, K)/Z'$  with  $Z' \leq Z(L(D_4, K))$  where  $Z(L(D_4, K))$  is the centre.
- 5) Let  $R = G_2$  and  $\text{char}(K) = 3$ . Then  $L(G_2, K)$  contains the ideal  $I \cong \mathfrak{sl}(3, K)$  generated by  $\{e_\beta \mid \beta \text{ is a short root}\}$ , and the algebra  $L(G_2, K)/I$  is isomorphic to  $\mathfrak{sl}(3, K)/Z(\mathfrak{sl}(3, K))$ .
- 6) Let  $R = C_r, r > 2$  and  $\text{char}(K) = 2$ . Then  $L(C_r, K)$  contains the ideal  $I \cong L(D_r, K)$  generated by  $\{e_\beta \mid \beta \text{ is a short root}\}$ , and the algebra  $L(C_r, K)/I$  is abelian.

The other Chevalley algebras  $L(\mathbf{R}, K)$  corresponding to irreducible root systems  $\mathbf{R}$  are simple modulo the centre  $Z$  [37]. The simple algebras  $\mathfrak{g} = L(\mathbf{R}, K)/Z$  are classical. The classical semisimple Lie algebras and the corresponding Chevalley algebras form a natural class to consider polynomial maps  $P(X_1, \dots, X_d)$  on their products. However, note that the algebras appearing in “bad cases” 3)–5) are perfect, i.e., satisfy the condition  $[L(\mathbf{R}, K), L(\mathbf{R}, K)] = L(\mathbf{R}, K)$ , and therefore we can also raise the question on dominance of polynomial maps on such algebras.

### 2.5. Prescribed Gauss decomposition

We will use the following generalization of the classical Gauss decomposition.

**f.** *Suppose we are not in the cases appearing in list (2.2). Fix an arbitrary non-central element  $h \in H$ . Then for every non-central element  $l \in L(\mathbf{R}, K)$  there is  $g \in G$  such that  $g(l) \in h + U$  [16, Proposition 1] (actually we mostly need below a particular case  $h = 0$  treated in [9, Lemma II]).*

## 3. Dominance of polynomial maps on Chevalley algebras

In this section  $K$  is an algebraically closed field.

**3.1** We are interested in the following analogue of the Borel dominance theorem for semisimple Lie algebras:

**Question 3.1.** For a given element  $P(X_1, \dots, X_d)$  of the free Lie  $K$ -algebra  $\mathcal{L}_d$  on the finite set  $\{X_1, \dots, X_d\}$  over a given algebraically closed field  $K$ , and a given semisimple Lie algebra  $\mathfrak{g}$  over  $K$ , is the map  $P(X_1, \dots, X_d): \mathfrak{g}^d \rightarrow \mathfrak{g}$  dominant under the condition that  $P(X_1, \dots, X_d)$  is not an identity on  $\mathfrak{g}$ ?

We do not know the answer to this question. However, we can get it under some additional assumption. Our main result is

**Theorem 3.2.** *Let  $L(\mathbf{R}, K)$  be a Chevalley algebra. If  $\text{char}(K) = 2$ , assume that  $\mathbf{R}$  does not contain irreducible components of type  $C_r$ ,  $r \geq 1$  (here  $C_1 = A_1$ ,  $C_2 = B_2$ ). Suppose  $P(X_1, \dots, X_d)$  is not an identity of the Lie algebra  $\mathfrak{sl}(2, K)$ . Then the induced map  $P: L(\mathbf{R}, K)^d \rightarrow L(\mathbf{R}, K)$  is dominant.*

Below we repeatedly use the following construction. Put  $\mathfrak{I} = \overline{P(L(\mathbf{R}, K)^d)}$ . Then  $\mathfrak{I}$  is an irreducible affine variety which is  $G$ -invariant, and therefore  $\pi(\mathfrak{I}) \hookrightarrow \mathbf{Y}$  is an irreducible closed subset of an  $r$ -dimensional affine variety  $\mathbf{Y}$  (see 2.2.d). If  $\overline{\pi(\mathfrak{I})} = \mathbf{Y}$ , then, by 2.2.d, the set  $\mathfrak{I}$  contains all elements which are  $G$ -conjugate to elements of  $H$  (because  $\pi(x) = \pi(x_s)$ ,  $O_{x_s} \subset \overline{O_x}$  and  $\mathfrak{I}$  is  $G$ -invariant). This implies, by 2.2.c, that  $\mathfrak{I} = L(\mathbf{R}, K)$ . Thus,

$$P \text{ is dominant} \Leftrightarrow \overline{\pi(\mathfrak{I})} = \mathbf{Y}. \quad (\text{MAIN}).$$

**Lemma 3.3.** *Let  $M \subset L(\mathbf{R}, K)$  be an irreducible closed subset such that*

- (i)  $\dim \overline{\pi(P(M^d))} = r - 1$ ;

$$(ii) \quad \overline{\pi(P(M^d))} \neq \overline{\pi(\mathcal{J})}.$$

Then  $P$  is dominant.

*Proof.* Since  $M$  is irreducible and  $\dim \overline{\pi(P(M^d))} = r - 1$ , we conclude that  $\overline{\pi(P(M^d))}$  is an irreducible hypersurface in  $\mathbb{Y}$ . The assertion of the lemma now follows from (ii) and condition (MAIN).  $\square$

We can now start the proof of Theorem 3.2. First of all, the statement is obviously reduced to the case where  $R$  is irreducible. Indeed, if  $R$  is a disjoint union of  $R_i$ , then  $\mathfrak{g} = L(R, K)$  is a direct sum of  $\mathfrak{g}_i = L(R_i, K)$ , and the image  $\text{Im } P$  of the map  $P = P(X_1, \dots, X_d): \mathfrak{g}^d \rightarrow \mathfrak{g}$  is equal to  $\bigoplus_i \text{Im } P_i$  where  $P_i$  is the restriction of  $P$  to  $\mathfrak{g}_i$ .

Note that  $P(X_1, \dots, X_d) = \sum_i a_i X_i + \sum(\text{monomials from } \mathcal{L}_d \text{ of degree } > 1)$  where  $a_i \in K$ . If  $a_i \neq 0$  for some  $i$ , the statement is trivial. Thus we may and will assume  $a_i = 0$  for every  $i$ . First we prove the assertion of the theorem for the case  $R = A_r$  by induction on  $r$ .

By condition (MAIN) it is enough to prove  $\overline{\pi(\mathcal{J})} = \mathbb{Y}$ , i.e.,  $\dim \overline{\pi(\mathcal{J})} = r$ . Note that in the case  $\text{char}(K) = 2$  the statement of the theorem fails for  $r = 1$ . However, as we will see below, the equality  $\overline{\pi(\mathcal{J})} = \mathbb{Y}$  holds even in this case. Thus we can prove that  $\overline{\pi(\mathcal{J})} = \mathbb{Y}$  by induction on the rank  $r$  starting at  $r = 1$ .

We identify  $L(A_r, K)$  with  $\mathfrak{sl}(r+1, K)$ , the algebra of  $(r+1) \times (r+1)$ -matrices with zero trace. We fix the chain of subalgebras  $L_1 \subset L_2 \subset \dots \subset L_r = \mathfrak{sl}(r+1, K)$  where  $L_{i-1} = \mathfrak{sl}(i, K)$  is the subalgebra embedded in the  $i \times i$  upper left corner of the matrix algebra  $L_i = \mathfrak{sl}(i+1, K)$ . We also fix, for each  $i$ , the subalgebra  $H_i \subset L_i$  of diagonal matrices in  $L_i$ . Further, let  $P_i = P|_{L_i^d}$ .

**Induction base:** we prove that  $\dim \overline{\pi(\text{Im } P_1)} = 1$ . Note that according to our assumption  $P_1 \neq 0$ .

Let first  $\text{char}(K) = 2$ . Then according to our assumption and case 1) of Section 2.4, we have  $P_1 = \sum_{i,j} a_{ij} [X_i, X_j]$  where  $a_{ij} \in K$ . Let  $a_{i_0, j_0} \neq 0$ . Set  $P'_1(X_1, \dots, X_d) = a_{i_0, j_0} [X_{i_0}, X_{j_0}]$ . On putting  $X_i = 0$  for all  $i \neq i_0, j_0$ , we see that  $\text{Im } P_1 \supseteq \text{Im } P'_1 = H_1$ , and therefore  $\dim \overline{\pi(\text{Im } P_1)} = 1$  (see 2.2.d). Note that this case requires special consideration only because we cannot refer to Lemma 3.3, as below.

Let us now assume  $\text{char}(K) \neq 2$ . As the map  $P$  is not identically zero, we may apply Lemma 3.3 with  $M = 0$ . We obtain the dominance of  $P_1: L_1^d \rightarrow L_1$  which implies  $\dim \overline{\pi(\text{Im } P_1)} = 1$ .

**Inductive step:** assume

$$\dim \overline{\pi(\text{Im } P_{r-1})} = r - 1 \quad (3.1)$$

and prove that

$$\dim \overline{\pi(\text{Im } P_r)} = r. \quad (3.2)$$

We have  $H_{r-1} = \{x = \text{diag}(\alpha_1, \dots, \alpha_r, 0) \in M_{r+1}(K) \mid \text{tr } x = 0\}$ . (Here  $M_{r+1}(K)$  is the algebra of  $(r+1) \times (r+1)$ -matrices over  $K$ .) According to 2.2.d, we have

$$\overline{\pi(\text{Im } P_{r-1})} = \overline{\pi(H_{r-1})}. \quad (3.3)$$

Suppose that

$$\mathfrak{I} \cap H_r \neq WH_{r-1}. \quad (3.4)$$

Then

$$\overline{\pi(\mathfrak{I})} \supseteq \overline{\pi(\mathfrak{I} \cap H_r)} \neq \overline{\pi(H_{r-1})}. \quad (3.5)$$

Condition (3.1) is condition 1) from Lemma 3.3 with  $M = L_{r-1}$ . Conditions (3.3) and (3.5) give us condition 2) from the same lemma. Note that the dominance of  $P$  implies  $\dim \pi(\mathfrak{I}) = r$ . Hence we have to prove that condition (3.4) holds.

We may assume that the transcendence degree of  $K$  is sufficiently large because this does not have any influence on dominance of  $P$ . Then we may also assume that there exist a subfield  $F \subset K$  and a division algebra  $D_{r+1} \subset M_{r+1}(K)$  with centre  $F$  such that  $D_{r+1} \otimes_F K = M_{r+1}(K)$  [13], [5]. The algebra  $D_{r+1}$  is dense in  $M_{r+1}(K)$ . Hence the set  $[D_{r+1}, D_{r+1}]$  is dense in  $[M_{r+1}(K), M_{r+1}(K)] = \mathfrak{sl}(r+1, K)$ . On the other hand,  $[D_{r+1}, D_{r+1}] \subset D_{r+1}$ . Thus the set  $S_{r+1} = D_{r+1} \cap \mathfrak{sl}(r+1, K)$  is dense in  $\mathfrak{sl}(r+1, K)$ , and therefore the restriction of  $P$  to  $S_{r+1}^d$  is not the zero map. Then there exist  $s_1, \dots, s_d \in S_{r+1}$  such that  $s = P(s_1, \dots, s_d) \neq 0$ . Since  $s_1, \dots, s_d \in D_{r+1}$ , we have  $s \in D_{r+1}$ . As there are no nonzero nilpotent elements in division algebras, all elements of  $D_{r+1}$  are semisimple, so we may assume  $s \in H_r$ . Since  $s$  has no zero eigenvalues,  $s \notin WH_{r-1}$ , and we get (3.4). Thus (3.2) is proven, and the assertion of the theorem for  $\mathsf{R} = \mathsf{A}_r$  is established.

The general case is a consequence of the following observation [5]: every irreducible root system  $\mathsf{R}$  has a subsystem  $\mathsf{R}'$  which has the same rank as  $\mathsf{R}$  and decomposes into a disjoint union of irreducible subsystems  $\mathsf{R}' = \bigcup_i \mathsf{R}'_i$  where each  $\mathsf{R}'_i$  is a system of type  $\mathsf{A}_{r_i}$ . Hence  $L' = \bigoplus_i L(\mathsf{A}_{r_i}, K) \subset L(\mathsf{R}, K)$ ,  $\sum_i r_i = r$ . Thus  $\dim \overline{\pi(P(L'))} = r \Rightarrow \overline{\pi(\mathfrak{I})} = \mathsf{Y}$ , and we get the statement from condition (MAIN). Theorem 3.2 is proved.  $\square$

**Corollary 3.4.** *Let  $\mathfrak{g}$  be a classical semisimple Lie algebra. Suppose that a polynomial  $P(X_1, \dots, X_d)$  is not an identity of the Lie algebra  $\mathfrak{sl}(2, K)$ . Then the induced map  $P: \mathfrak{g}^d \rightarrow \mathfrak{g}$  is dominant.*

*Proof.* Let  $\mathsf{R}$  be the root system corresponding to  $\mathfrak{g}$ . If the Chevalley algebra  $L(\mathsf{R}, K)$  is semisimple, we have  $\mathfrak{g} = L(\mathsf{R}, K)$ , and there is nothing to prove. If  $\mathfrak{g} = L(\mathsf{R}, K)/Z$ , where  $Z$  is the centre, the assertion is an immediate consequence of the following obvious observation: if the Lie polynomial  $P(X_1, \dots, X_d)$  does not contain terms of degree 1, then the map  $P: L(\mathsf{R}, K)^d \rightarrow L(\mathsf{R}, K)$  is trivial on  $Z$ .  $\square$

3.2 Theorem 3.2 reduces the problem of dominance to the class of maps  $P$  which are identically zero on  $\mathfrak{sl}(2, K)$ . The following theorem gives another possibility to reduce the problem of dominance.

**Theorem 3.5.** *Let  $L(\mathsf{R}, K)$  be a Chevalley algebra corresponding to an irreducible root system  $\mathsf{R}$ , and suppose that  $\mathsf{R} \neq \mathsf{C}_r$  if  $\text{char}(K) = 2$ . Suppose that the map  $P: L(\mathsf{R}, K)^d \rightarrow L(\mathsf{R}, K)$  is dominant for  $\mathsf{R} = \mathsf{A}_2$  and  $\mathsf{B}_2$ . Then  $P$  is dominant for every  $L(\mathsf{R}, K)$ ,  $r > 1$ .*

*Proof.* We prove the theorem by induction on  $r$ . Let first  $r = 2$ . The cases  $R = A_2, B_2$  are included in the hypothesis, and the case  $R = G_2$  is established by the same argument as at the end of the proof of Theorem 3.2 because  $G_2$  contains  $A_2$ .

Let now  $r > 2$ , and make the induction hypothesis:  
*the map  $P$  is dominant for every  $L(R, K)$  where  $1 < \text{rank } R < r$ .*

We proceed case by case.

**1.**  $R = A_r$ . The induction step is the same as in the proof of Theorem 3.2.

**2.**  $R = C_r$  ( $r \geq 3$ ) or  $D_r$  ( $r \geq 4$ ). Let  $\Pi = \{\alpha_1, \dots, \alpha_r\}$  be the simple root system numerated as in Bourbaki [7]. Let  $\Pi_1 = \{\alpha_1, \dots, \alpha_{r-1}\}$ ,  $\Pi_2 = \{\alpha_2, \dots, \alpha_r\}$ . Then  $R_1 = \langle \Pi_1 \rangle = A_{r-1}$ ,  $R_2 = \langle \Pi_2 \rangle = C_{r-1}$  or  $D_{r-1}$ , respectively. Let  $H_i = H \cap L(R_i, K)$ . There exists  $h \in H_1$  such that  $h \notin WH_2$ . Indeed, let  $\epsilon_i: H \rightarrow K$  be the weights given by the formula  $\epsilon_i(h_{\alpha_k}) = \frac{2(\epsilon_i, \alpha_k)}{(\alpha_k, \alpha_k)}$ . Then  $\epsilon_1(H_2) = 0$ , and therefore for every  $h' \in WH_2$  we have  $\epsilon_i(h') = 0$  for some  $i$ . On the other hand, since  $R_1 = A_{r-1}$ , we can find  $h \in H_1$  such that  $\epsilon_i(h) \neq 0$  for every  $i$ , and therefore  $h \notin WH_2$ . Note that  $h \in \overline{P(L(R_1, K)^d)}$  because  $P$  is dominant on  $L(R_1, K)$  (see the proof of Theorem 3.2). Then  $h \in \mathfrak{J}$ . On the other hand,  $\pi(h) \notin \overline{\pi(P(L(R_2, K)^d))}$  because  $h \notin WH_2$ . Hence we can apply Lemma 3.3 with  $M = L(R_2, K)$ .

**3.**  $R = B_r, F_4$ . Here we have  $D_r \subset R$ . (The respective embeddings are as follows:  $\mathfrak{so}(2r) \subset \mathfrak{so}(2r+1)$  is a natural inclusion, and  $D_4$  embeds into  $F_4$  as the subsystem consisting of the long roots.)

Then  $H \subset \overline{P(L(D_r, K))}$ , and therefore  $P$  is dominant on  $L(R, K)$ .

**4.**  $R = E_r$ . Consider the extended Dynkin diagram. We obtain a needed subsystem of type  $A$  by removing one of its vertices. In each case the diagram is a trident. We remove the 3-valent vertex in the case  $r = 6$ , and the tooth of length 1 (the lower vertex  $\alpha_2$  in the Bourbaki notation) in the cases  $r = 7, 8$ . We obtain subsystems of types  $A_2 \times A_2 \times A_2$ ,  $A_7$ , and  $A_8$ , respectively. Then we use the same argument as above.  $\square$

**3.3** To use the theorems proven above for practical purposes, the following simple remarks may be useful.

**Remark 3.6.** If  $P(X_1, \dots, X_d) \in \mathcal{L}_d$  is a polynomial containing a monomial of degree  $< 5$ , then the map  $P: L(R, K)^d \rightarrow L(R, K)$  ( $\text{char}(K) \neq 2$ ) is dominant.

The reason is that such a polynomial cannot be an identity in  $\mathfrak{sl}(2, K)$ . Indeed, if it were an identity, so would be its homogeneous component of the lowest degree (because any homogeneous component of any polynomial identity of any algebra of any signature over any infinite field is an identity, see [33, 6.4.14]). On the other hand, any identity of the Lie algebra  $\mathfrak{sl}(2, K)$  ( $\text{char}(K) \neq 2$ ), is an identity of  $\mathfrak{gl}(2, K)$  (because every matrix is a sum of a trace zero matrix and a scalar matrix, and such an identity lifts to an identity of the associative matrix algebra  $M_2(K)$ ). The latter one does not contain identities of degree less than 4 (which is the smallest degree of the so-called standard identity satisfied in  $M_2$ ), hence the same is true for  $\mathfrak{gl}(2)$  (see, e.g., [33, Remark 6.1.18] or [1,

Exercise 2.8.1]). Moreover, a little subtler argument allows one to show that  $\mathfrak{sl}(2, K)$  does not contain identities of degree 4 (see, e.g., [1, Section 5.6.2]).

**Remark 3.7.** Note that Razmyslov [30] found a finite basis for identities in this algebra (assuming  $K$  to be of characteristic zero). Moreover, it turned out that all such identities are consequence of the single identity [15]:

$$P = [[[Y, Z], [T, X]], X] + [[[Y, X], [Z, X]], T],$$

and this result remains true for any infinite field  $K$ ,  $\text{char}(K) \neq 2$  [42].

Below we illustrate how one can apply Theorem 3.5 using one of the identities appearing in Razmyslov's basis (the reader willing to deduce this identity from Filippov's one mentioned above is referred to Section 2 of [15]).

**Example 3.8.** The polynomial  $[[[[Z, Y], Y], X], Y] - [[[Z, Y], X], Y], Y]$  appears in [30] as one of the elements of a finite basis of identities in  $\mathfrak{sl}(2, K)$  ( $\text{char}(K) = 0$ ). Clearly, the polynomial

$$P(X, Y, Z) = [[[Z, Y], Y], X], Y], [[[Z, Y], X], Y], Y]]$$

is also identically zero in  $\mathfrak{sl}(2, K)$ . It turns out that the map  $P: L(\mathbf{R}, K)^3 \rightarrow L(\mathbf{R}, K)$  is dominant. We check this using computations by MAGMA. In view of Theorem 3.5, we have to check dominance only for  $\mathbf{R} = \mathbf{A}_2, \mathbf{B}_2$ .

Consider the map  $\pi: L(\mathbf{R}, K) \rightarrow \mathbf{Y}$  defined in Section 2.2.d. Since  $\text{char}(K) = 0$ , we have  $\mathbf{Y} \cong H/W \cong \mathbb{A}^r$ , and  $\pi = (f_1, f_2, \dots, f_r)$  where  $f_1, f_2, \dots, f_r$  are  $G$ -invariant homogeneous polynomials on  $L(\mathbf{R}, K)$  which generate the invariant algebra  $K[L(\mathbf{R}, K)]^G \cong K[H]^W$ . Moreover,  $\deg f_1 \deg f_2 \cdots \deg f_r = |W|$  (see Section 2.2.d). In our cases,  $r = 2$  and we have  $\deg f_1 = 2$ ,  $\deg f_2 = 3$  for  $\mathbf{R} = \mathbf{A}_2$  and  $\deg f_1 = 2$ ,  $\deg f_2 = 4$  for  $\mathbf{R} = \mathbf{B}_2$ .

Let  $0 \neq D_1 = P(A, B, C), D_2 = P(A', B', C') \in \mathfrak{I} = \overline{\text{Im } P(L(\mathbf{R}, K))^3}$ . Since  $P$  is a homogeneous map with respect to  $X, Y, Z$ , the lines  $l_j := KD_j$ ,  $j = 1, 2$ , also lie in  $\mathfrak{I}$ , and the curves  $\pi(l_j)$  in the affine space  $\mathbb{A}^2$  with coordinates  $(x_1, x_2)$  are defined by equations of the form  $x_1^{m_1}/x_2^{m_2} = c_j$ , where  $m_1 = \deg f_2$ ,  $m_2 = \deg f_1$ ,  $c_j = \text{const}$ .

Put  $\theta := f_1^{m_1}/f_2^{m_2}$ . We get  $\theta(D_1) \neq \theta(D_2) \Rightarrow \pi(l_1) \neq \pi(l_2) \Rightarrow \mathfrak{I} = \mathbb{A}^2$ .

Using MAGMA, in each of the cases  $\mathbf{R} = \mathbf{A}_2$  and  $\mathbf{R} = \mathbf{B}_2$ , we find a pair of triples  $(A, B, C), (A', B', C')$  such that  $\theta(P(A, B, C)) \neq \theta(P(A', B', C'))$ . Thus,  $P$  is a dominant map for every  $L(\mathbf{R}, K)$ ,  $r > 1$ .

#### 4. From dominance to surjectivity

For some polynomials  $P \in \mathcal{L}_d$  we can say more than in the preceding section. Namely, we present here several cases where the map  $P: \mathfrak{g}^d \rightarrow \mathfrak{g}$  is surjective. We start with the following simple observation (parallel to Remark 3 in [5, §1]).

**Proposition 4.1.** *Let  $P_1(X_1, \dots, X_{d_1})$ ,  $P_2(Y_1, \dots, Y_{d_2})$  be Lie polynomials. Let  $\mathfrak{g}$  be a Lie algebra. Suppose that each of the maps  $P_i: \mathfrak{g}^{d_i} \rightarrow \mathfrak{g}$  is dominant. Let  $d = d_1 + d_2$ ,*

$$P(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = P_1(X_1, \dots, X_{d_1}) + P_2(Y_1, \dots, Y_{d_2}).$$

*Then the map  $P: \mathfrak{g}^d \rightarrow \mathfrak{g}$  is surjective.*

*Proof.* As the underlying variety of  $\mathfrak{g}$  is irreducible, the image of each of the dominant morphisms  $P_i$  ( $i = 1, 2$ ) contains a non-empty open subset  $U_i$ . It remains to notice that  $U_1 + U_2 = \mathfrak{g}$  (see, e.g., [6, Chapter I, § 1, 1.3]).  $\square$

Let us now prove surjectivity for some special maps, which are linear in one variable.

**Definition 4.2.** We call  $E_m(X, Y) = \underbrace{[[\dots [X, Y], Y], \dots, Y]}_{m \text{ times}} \in \mathcal{L}_2$  an Engel polynomial of degree  $(m + 1)$ . We call  $\sum_{i=1}^m a_i E_i(X, Y) \in \mathcal{L}_2$ , where  $a_i \in K$ , a generalized Engel polynomial.

**Theorem 4.3.** *Let  $P(X, Y) \in \mathcal{L}_2$  be a generalized Engel polynomial of degree  $(m + 1)$ , and let  $P: L(\mathbb{R}, K)^2 \rightarrow L(\mathbb{R}, K)$  be the corresponding map of Chevalley algebras. If  $\mathbb{R}$  does not contain irreducible components of types listed in (2.2) and  $|K| > m|R|$ , then the image of  $P$  contains  $(L(\mathbb{R}, K) \setminus Z(L(\mathbb{R}, K))) \cup \{0\}$ . Moreover, if  $P$  is an Engel polynomial, then the same is true under the assumption  $|K| > |R^+|$ .*

*Proof.* Since  $|K| > m|R|$ , for any chosen  $S \subset K$  of size  $m$  there is  $h \in H$  such that  $\beta(h) \notin S$  for all  $\beta \in \mathbb{R}$  (see 2.3.e). Further, for every  $h \in H$  the map  $P_h: L(\mathbb{R}, K) \rightarrow L(\mathbb{R}, K)$ , given by  $X \mapsto P(X, h)$ , is a semisimple linear operator on  $L(\mathbb{R}, K)$  which is diagonalizable in the Chevalley basis. Each  $h_\alpha$  is its eigenvector with zero eigenvalue. Further, there is a degree  $m$  polynomial  $f \in K[t]$  such that  $P(e_\beta, h) = f(\beta(h))e_\beta$  for every  $\beta \in \mathbb{R}$ . (Explicitly, one can take  $f = \sum_{i=1}^m (-1)^i a_i t^i$ .) Define  $S$  as the set of roots of  $f$  in  $K$ . Then  $f(\beta(h)) \neq 0$  for every  $\beta \in \mathbb{R}$ , and therefore  $\text{Im}(P_h) = U$ . Now the statement follows from 2.5.f. If  $P$  is an Engel polynomial of degree  $(m + 1)$ , then one can take  $f = x^m$ , and therefore  $S = \{0\}$ , that is,  $h$  is a regular element. Once again, we can use 2.5.f.  $\square$

**Corollary 4.4.** *Let  $P = P(X, Y) \in \mathcal{L}_2$  be a generalized Engel polynomial of degree  $(m + 1)$ , and let  $\mathfrak{g}$  be a simple classical Lie algebra corresponding to the root system  $\mathbb{R}$ . If  $|K| > m|R|$ , then the map  $P: \mathfrak{g}^2 \rightarrow \mathfrak{g}$  is surjective. Moreover, if  $P$  is an Engel polynomial, the same is true under the assumption  $|K| > |R^+|$ .*

**Remark 4.5.** Corollary 4.4 generalizes Theorem 7 of [41] where Question 1.1a was answered in the affirmative for the words in three variables  $P(X, Y, Z)$  of the form  $[X, Y, \dots, Y, Z]$  and  $\mathfrak{g} = \mathfrak{sl}(n)$ .

Our next result shows that one cannot hope to extend surjectivity to central elements.

**Proposition 4.6.** *Let  $P_m(X, Y) \in \mathcal{L}_2$  be an Engel polynomial of degree  $m$ , and let  $P: L(\mathbb{R}, K)^2 \rightarrow L(\mathbb{R}, K)$  be the corresponding map of Chevalley algebras. Then for  $m$  big enough the image of  $P$  contains no nonzero elements of  $Z(L(\mathbb{R}, K))$ .*

*Proof.* The scheme of the proof is as follows. We distinguish two cases. If  $X$  and  $Y$  centralize the same element of a Cartan subalgebra of  $L(\mathbb{R}, K)$ , we prove that  $P_m(X, Y) = 0$  for  $m$  big enough. Otherwise, we use a partial order on  $\mathbb{R}$  induced by height to show that  $P(X, Y)$  cannot lie in  $H$ . Here is a detailed argument.

We may assume  $K$  algebraically closed. Then, by “bringing to the Jordan form”, we may assume  $Y = h + y$  where  $h \in H$ ,  $y \in U^+$ ,  $[h, y] = 0$ . Further, let  $X = h' + x$ ,  $h' \in H$ ,  $x \in U$ . For brevity, for every  $n$  denote  $z_n = P_n(X, Y)$ .

*Case I.*  $[h, x] = 0$ . Let us prove that  $z_n = P_n(X, y)$  and  $[z_n, h] = 0$ . We use induction on  $n$ . For  $n = 1$  we have  $z_1 = [X, h + y] = [X, h] + [X, y]$ . Since  $[x, h] = [h', h] = 0$ , we have  $z_1 = [X, y]$ . Further,  $[z_1, h] = [[X, y], h] = [[h', y], h] + [[x, y], h]$ . Since  $[h', h] = [x, h] = [y, h] = 0$ , each summand equals zero by the Jacobi identity, so  $[z_1, h] = 0$ . Assume  $z_{n-1} = P_{n-1}(X, y)$  and  $[z_{n-1}, h] = 0$ . We have  $z_n = [z_{n-1}, Y] = [z_{n-1}, h + y] = [z_{n-1}, y] = [P_{n-1}(X, y), y] = P_n(X, y)$  and  $[z_n, h] = [[z_{n-1}, y], h] = 0$  by the Jacobi identity (because  $[z_{n-1}, h] = [y, h] = 0$ ). Thus we have  $P_n(X, Y) = [[X, y], y, \dots, y]$  which is zero for  $n$  big enough because  $y$  is nilpotent.

*Case II.*  $[h, x] \neq 0$ . First suppose that  $y = 0$ , i.e.,  $Y = h$  is semisimple. As  $x \neq 0$ , we can write  $x = \sum_{\beta \in \mathbb{R}} f_\beta e_\beta$ , where  $f_\beta \in K$ . Since  $[h, x] \neq 0$ , there exists  $\beta$  such that  $[h, e_\beta] \neq 0$ . We now observe that if  $f_\beta \neq 0$  then for every  $m$  the term of  $P_m(X, Y)$  containing  $e_\beta$  enters with nonzero coefficient, so  $P_m(X, Y)$  belongs to  $U$  and thus does not belong to the centre. So assume  $y \neq 0$  and write  $y = \sum_{\beta \in \mathbb{R}^+} p_\beta e_\beta$ , where  $p_\beta \in K$ . Put

$$\mathbb{R}_h = \{\beta \in \mathbb{R} \mid \beta(h) \neq 0\}, \quad \hat{\mathbb{R}}_h = \{\beta \in \mathbb{R} \mid \beta(h) = 0\},$$

$$\mathbb{R}_x = \{\beta \in \mathbb{R} \mid f_\beta \neq 0\}, \quad \mathbb{R}_y = \{\beta \in \mathbb{R} \mid p_\beta \neq 0\}.$$

All these sets are non-empty, and  $\mathbb{R}_{h,x} = \mathbb{R}_h \cap \mathbb{R}_x \neq \emptyset$ . We have  $\mathbb{R}_y \subseteq \mathbb{R}^+$ ,  $\mathbb{R}_y \subseteq \hat{\mathbb{R}}_h$ . Let  $\prec$  be the (partial) order on  $\mathbb{R}$  induced by height. Recall that by definition  $\alpha \prec \beta$  if and only if  $\beta - \alpha$  is a sum of positive roots. We fix some minimal  $\gamma$  in  $\mathbb{R}_{h,x}$ . Further, write  $P_n(X, Y) = z_n = \sum_{\beta \in \mathbb{R}} d_{n,\beta} e_\beta + h_n$  where  $h_n \in H$ ,  $d_{n,\beta} \in K$ .

*Claim: a)*  $d_{n,\gamma} \neq 0$ ;

*b)* if  $d_{n,\delta} \neq 0$  and  $\delta \neq \gamma$ , then either  $\delta \in \hat{\mathbb{R}}_h$  or  $\delta \not\prec \gamma$ .

Evidently, a) is enough to establish the assertion of the proposition.

Let us prove the claim by induction on  $n$ . Let first  $n = 1$ . We have

$$[h', y] = \sum_{\beta \in \hat{\mathbb{R}}_h} a_\beta e_\beta, \quad [x, y] = \sum_{\beta \in \mathbb{R}} b_\beta e_\beta + h_1, \quad [x, h] = \sum_{\beta \in \mathbb{R}_h} c_\beta e_\beta, \quad (4.1)$$

where  $h_1 \in H$ , and we have  $d_{1,\beta} = a_\beta + b_\beta$  or  $d_{1,\beta} = b_\beta + c_\beta$ .

a) We have  $a_\gamma = 0, c_\gamma \neq 0$  because  $\gamma \in R_{h,x} \subseteq R_h$ . Let us prove that  $b_\gamma = 0$ . Assume to the contrary that  $b_\gamma \neq 0$ . Then from the middle equality in (4.1) it follows that there are roots  $\alpha \in R_x$  and  $\beta \in R_y$  such that  $[e_\alpha, e_\beta] = e_\gamma$  (and so  $\gamma = \alpha + \beta$ ). Since  $[h, e_\beta] = 0$  and  $[h, e_\gamma] \neq 0$ , we have  $[h, e_\alpha] \neq 0$ . Hence  $\alpha \in R_h$  and therefore  $\alpha \in R_{h,x} = R_h \cap R_x$ . Since  $\gamma = \alpha + \beta$ , we have the inequality  $\alpha \prec \gamma$  because  $\beta$  is a positive root. This is a contradiction with the choice of  $\gamma$  (recall that  $\gamma$  is a minimal root in  $R_{h,x}$  with respect to the partial order  $\prec$ ). Thus  $b_\gamma = 0$  and  $d_{1,\gamma} = c_\gamma \neq 0$ .

b) Suppose  $d_{1,\delta} \neq 0$  and  $\delta \notin \hat{R}_h$ . Then  $\delta \in R_h$  and  $d_{1,\delta} = b_\delta + c_\delta$ . If  $c_\delta \neq 0$ , then  $\delta \in R_x$ . Hence  $\delta \in R_{h,x}$  and  $\delta \not\prec \gamma$  because of the choice of  $\gamma$ . If  $c_\delta = 0$ , then  $d_{1,\delta} = b_\delta \neq 0$ . Then  $e_\delta = [e_\alpha, e_\beta]$  for some  $\alpha \in R_x, \beta \in R_y$ . Since  $\delta \in R_h$  (and so  $[h, e_\delta] \neq 0$ ) and  $\beta \in R_y \subseteq \hat{R}_h$  (and so  $[h, e_\beta] = 0$ ), we have  $[h, e_\alpha] \neq 0 \Rightarrow \alpha \in R_h \Rightarrow \alpha \in R_{h,x}$ . Suppose that  $\delta = \alpha + \beta \prec \gamma$ . Then  $\alpha \prec \gamma$  which is again a contradiction with the choice of  $\gamma$ . Hence  $\delta \not\prec \gamma$ .

Let us now assume **a)**  $d_{n-1,\gamma} \neq 0$ ; **b)** if  $d_{n-1,\delta} \neq 0$  and  $\delta \neq \gamma$ , then either  $\delta \in \hat{R}_h$  or  $\delta \not\prec \gamma$ , and prove the same assertions for  $n$ . Consider

$$z_n = [z_{n-1}, h + y] = \underbrace{\sum_{\beta \in R} d_{n-1,\beta} [e_\beta, h]}_I + \underbrace{\sum_{\beta \in R} d_{n-1,\beta} [e_\beta, y]}_{II} + \underbrace{[h_{n-1}, y]}_{III}.$$

The induction hypotheses imply that

$$z_n = \underbrace{\sum_{\delta \in \hat{R}_h} q_\delta e_\delta + s_\gamma e_\gamma}_{\spadesuit} + \underbrace{\sum_{\delta \in R_h, \delta \not\prec \gamma} s_\delta e_\delta + h_n}_{\heartsuit},$$

where  $h_n \in H$  and  $s_\gamma \neq 0$ . Indeed, sum I has only terms of types  $\heartsuit$  and the term  $s_\gamma e_\gamma \neq 0$ . Further, sum II has terms of types  $\spadesuit$  and  $\heartsuit$  and elements of  $H$ . Sum III has only terms of type  $\spadesuit$  because  $R_y \subseteq \hat{R}_h$ . Thus conditions a) and b) hold for  $z_n$ , and  $z_n = P(X, Y) \notin Z(L(R, K))$ .  $\square$

**Remark 4.7.** Suppose we are in one of the exceptional cases listed in (2.2). Let us exclude abelian and solvable cases 1), 2) of Section 2.4. Also in case 6) in Theorem 4.3 we may consider the Lie algebra  $[L(R, K), L(R, K)]$  instead of  $L(R, K)$ . In cases 3), 4), 5) the algebra  $L(R, K)$  contains an ideal  $I$  (generated by short roots) such that the quotient  $\bar{L} = L(R, K)/I$  is not on list (2.2), and therefore the assertion of Theorem 4.3 on surjectivity of  $P$  holds for  $\bar{L}$ .

**Remark 4.8.** In the group case the phenomenon of Proposition 4.6 can be observed already for  $m = 1$ : some quasisimple groups contain central elements that are not commutators, see [39], [14] for infinite groups and [3] for finite groups.

**Example 4.9.** In the following example we show that non-Engel maps are not necessarily surjective. Let

$$P = P(X, Y) = [[[X, Y], X], [X, Y], Y]: \mathfrak{sl}(2, K) \times \mathfrak{sl}(2, K) \rightarrow \mathfrak{sl}(2, K)$$

where  $\text{char}(K) \neq 2$  and  $K$  is an algebraically closed field. Let  $\{e, f, h\}$  be the standard basis of  $\mathfrak{sl}(2)$ :  $[e, f] = h$ ,  $[h, e] = 2e$ ,  $[h, f] = -2f$ . First note that  $P(X + mY, Y) = P(X, Y + mX) = P(X, Y)$ . Therefore we may assume that  $X = ae + bf$ ,  $Y = cf + dh$ . A straightforward calculation then gives  $P(X, Y) = 4a^2csh + 8a^2dse + 8abdsf$  where  $s = 4bd^2 - ac^2$ . This implies that in  $\text{Im}(P)$  there are no elements of the form  $me$  or  $mf$  with  $m \neq 0$ .

## 5. Concluding remarks and possible generalizations

**Remark 5.1.** The method used in the proof of Theorem 3.2 (which goes back to [13] and [5]) is applicable to the problem of dominance of polynomial maps on associative matrix algebras (which is attributed to Kaplansky, see [24] and references therein). More precisely, let  $P(X_1, \dots, X_d) \in K\langle X_1, \dots, X_d \rangle$  be an associative, noncommutative polynomial (i.e., an element of the finitely generated free associative algebra), and let  $P: M_n(K)^d \rightarrow M_n(K)$  denote the corresponding map. Then the same inductive argument as in the proof of Theorem 3.2 shows that if  $P(X_1, \dots, X_d)$  is not identically zero on  $M_1(K)^d$  then the map  $P$  is dominant for all  $n$ . In the situation where  $P(X_1, \dots, X_d)$  is identically zero on  $K^d$ , one can consider the induction base  $n = 2$  and prove that if the restriction of  $P$  to  $M_2(K)^d$  is dominant then so is  $P$ . The assumption made above holds, for instance, for any semi-homogeneous, non-central polynomial having at least one  $2 \times 2$ -matrix with nonzero trace among its values [24, Theorem 1]. If, under the same assumptions on  $P$ ,  $\text{Im}(P)$  lies in  $\mathfrak{sl}(n, K)$ , then  $\overline{\text{Im}(P)} = \mathfrak{sl}(n, K)$ .

**Remark 5.2.** It would be interesting to consider maps  $P$  with some fixed  $X_i = A_i$ . Then one could find an approach to the dominance calculating the differential map of  $P$ .

**Remark 5.3.** It would be interesting to consider a more general set-up when we have a polynomial map  $P: L^d \rightarrow L^s$ . In [17] some dominance results were obtained for the multiple commutator map  $P: L \times L^d \rightarrow L^d$  given by the formula  $P(X, X_1, \dots, X_d) = ([X, X_1], \dots, [X, X_d])$ .

**Remark 5.4.** In a similar spirit, one can consider generalized word maps  $w: G^d \rightarrow G^s$  on simple groups. Apart from [17], see also a discussion of a particular case  $w = (w_1, w_2): G^2 \rightarrow G^2$  in [8, Problem 1].

**Remark 5.5.** One could try to extend some of results of this paper to the case where the ground field is replaced with some sufficiently good ring. One has to be careful in view of [32]: there are rings  $R$  such that not every element of  $\mathfrak{sl}(n, R)$  is a commutator.

**Remark 5.6.** It would be interesting to understand the situation with *infinite-dimensional* simple Lie algebras (as well as with finite-dimensional algebras of Cartan type in positive characteristics). The first question is whether every element of such an algebra can be represented as a Lie product of two other elements. Note that the question on the existence of a simple *group* not every

element of which is a commutator remained open for a long time. First examples of such groups appeared in geometric context [2], where the groups under consideration were infinitely generated; later on there were constructed finitely generated groups with the same property [26]. These are counter-examples in very strong sense: the so-called commutator width, defined as supremum of the minimal number of commutators needed for a representation of a given element as a product of commutators, may be arbitrarily large or even infinite [26, Theorems 4 and 5].

**Remark 5.7.** Apparently, one cannot hope to extend dominance and surjectivity results to polynomial maps on algebras which are far from simple. Indeed, in the group case the simplicity assumption is essential. As pointed out to us by D. Calegari, if  $G$  is non-elementary word-hyperbolic group and  $w$  is a nontrivial word, then one cannot hope to have a generic element of  $G$  in the image of the word map induced by  $w$  (this can be deduced from [10]).

**Remark 5.8.** One can ask questions similar to Questions 1.1 and 1.2 for other classes of algebras (beyond groups, Lie algebras and associative algebras). The interested reader may refer to [18] for the case of values of commutators and associators on alternative and Jordan algebras.

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