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BURNSIDE-TYPE PROBLEMS RELATED TO SOLVABILITY

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In the paper we pose and discuss new Burnside-type problems, where the role of nilpotency is replaced by that of solvability.

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1. Introduction

The Burnside problems explore consequences of information on cyclic subgroups of a group (namely orders of elements). In this paper we explore consequences of properties of two-generated subgroups. Another context of the Burnside problems is the study of the identity $x^n = 1$, and of related Engel identities and their relations to nilpotency. Here we focus on two variable identities which are related to solvability.

In particular, we shall formulate certain new Burnside-type problems where the role of nilpotency is replaced by the solvability property, thus extending the club of Burnside-type problems. In this context we use the recent results on Thompson-like characterization of the solvable radical of finite groups and finite dimensional Lie algebras [21] and Engel-like descriptions of finite solvable groups [7, 9] in terms of two-variable words. We also discuss some old problems from the perspective of two generated subgroups and so-called one and a half generated subgroups (see definitions below).

We start with a short historical background (see [30, 31, 33, 44, 48–50] and references therein). The general Burnside problem asks: is a torsion group locally finite? Golod constructed infinite finitely generated residually finite torsion groups, thus giving a negative answer to the general Burnside problem. Another Burnside problem deals with the identity $x^n \equiv 1$; namely, is every group of finite exponent locally finite? A negative solution for the Burnside problem was obtained by Novikov–Adian, and later by Olshanskii and Rips. Denote by $B(r, n)$ the free group with r generators in the Burnside variety $x^n \equiv 1$. The restricted Burnside problem states that the group $B(r, n)$ has a unique maximal finite quotient. The final (positive) solution of the restricted Burnside problem was obtained by Zelmanov and is mostly based on studying infinite dimensional Lie algebras and Engel or Engel-like identities (Kostrikin, Kostrikin–Zelmanov). Thus, it took about 90 years to complete the solution of the Burnside problems. These three Burnside problems have various applications and give rise to numerous questions of Burnside type.

Engel's theorem [28, Chap. II, Sec. 3] characterizes nilpotent Lie algebras in the class of finite dimensional Lie algebras by identities in two variables: a finite dimensional Lie algebra L is nilpotent if and only if it satisfies one of the identities $e_n(x, y) := [x, y, y, \ldots, y] \equiv 0$.

In a similar way, Zorn's theorem [27, 51, Satz III.6.3] characterizes nilpotent groups in the class of finite groups:

Theorem 1.1. *A finite group G is nilpotent if and only if it satisfies one of the identities $e_n(x, y) := [x, y, y, \ldots, y] \equiv 1$.*

Here and throughout this paper $[x, y] = xyx^{-1}y^{-1}$, $[x, y, y] = [[x, y], y]$, etc.

Baer's theorem [4] describes the nilpotent radical of a finite group G as the set of all (left)-Engel elements:

Theorem 1.2. *The nilpotent radical of a finite group G coincides with the collection of all Engel elements of G*, i.e. *the elements $y \in G$ such that for all $x \in G$ we have $e_n(x, y) = 1$ for some $n = n(x, y)$.*

Engel's and Zorn's theorems provide a tool for recognition of the nilpotency property for finite dimensional Lie algebras and finite groups in terms of explicit sequences of monotonic (in the sense of profinite topology) words in **two variables**. This explicit two-variable characterization gives rise to a number of well-known applications (see, for instance, [2, 27, 41]).

Baer, Gruenberg and B. Plotkin paid attention to the following Burnside-type problem:

Problem 1.3. *Is a group G satisfying the identity $e_n(x, y) \equiv 1$ for some fixed n locally nilpotent? In other words, is every Engel group G locally nilpotent?*

Although this problem most likely has a negative solution (for sufficiently large n), it remains open up to now.

The purpose of this paper is to formulate some analogous problems and results with emphasis on the solvability property.

The paper is organized as follows. Section 2 deals with Engel-like sequences. In Sec. 3, we pose Burnside-type questions related to sequences introduced in Sec. 2. Section 4 is devoted to two-generated subgroups and the associated Thompson-type problems. In Sec. 5, we consider Burnside-type problems for particular elements of a group. In Sec. 6, we give a general approach to characterization of radicals in finite groups. Finally, Sec. 7 deals with variations on the Baer–Suzuki theorem which characterizes the nilpotent radical of a finite group in terms of two conjugates of an element. We discuss characterizations of the solvable radical in terms of a larger number of conjugates.

2. Engel-Like Sequences

In this section, we collect facts about explicit two-variable sequences which are related to nilpotence or solvability of finite groups and finite dimensional Lie algebras.

Let $F_2 = F(x, y)$ be the free two generator group, $W_2 = W(x, y)$ the free two generator Lie algebra.

Recall that an element g of a group G is called Engel element if for every $a \in G$ there exists $n = n(a, g)$ such that $e_n(a, g) = [a, g, g, \dots, g] = 1$. An element g of a Lie algebra L is called Engel if for every $a \in L$ there exists $n = n(a, g)$ such that $e_n(a, g) = [a, g, g, \dots, g] = 0$. If every element of a group G (a Lie algebra L) is Engel, then G (the Lie algebra L) is called unbounded Engel. If a group G (a Lie algebra L) satisfies the identity $e_n(x, y) \equiv 1$ (respectively, $e_n(x, y) \equiv 0$), then the group G (the Lie algebra L) is called Engel.

Definition 2.1 [5, 37]. We say that a sequence $\vec{u} = u_1, u_2, u_3, \dots, u_n, \dots$ of elements from F_2 is **correct** if the following conditions hold:

(i) For every group G and elements $a, g \in G$, we have $u_n(a, 1) = 1$ and $u_n(1, g) = 1$ for all sufficiently large n.
(ii) if a, g are elements of G such that $u_n(a, g) = 1$, then for every $m > n$ we have $u_m(a, g) = 1$.

Thus, if the identity $u_n(x, y) \equiv 1$ is satisfied in G, then for every $m > n$ the identity $u_m(x, y) \equiv 1$ also holds in G.

A similar definition can be given for sequences $\vec{u}$ in Lie algebra $W_2 = W_2(x, y)$.

Definition 2.2. For every correct sequence $\vec{u}$ in $F_2(x, y)$ (respectively, in $W_2(x, y)$), define the class of groups (respectively, Lie algebras) $\Theta = \Theta(\vec{u})$ by the rule: a group (respectively, Lie algebra) G belongs to Θ if and only if there is n such that the identity $u_n(x, y) \equiv 1$ (respectively, $u_n(x, y) \equiv 0$) holds in G.

Definition 2.3. For every group (respectively, Lie algebra) G, denote by $G(\vec{u})$ the subset of G defined by the rule: $g \in G(\vec{u})$ if and only if for every $a \in G$ there exists $n = n(a, g)$ such that $u_n(a, g) = 1$ (respectively, 0). Elements of $G(\vec{u})$ are viewed as Engel elements with respect to the given correct sequence $\vec{u}$. We call these elements **$\vec{u}$-Engel-like** or, for brevity, **$\vec{u}$-Engel elements**.

Example 2.4.

(i) If $\vec{e} = e_1, e_2, e_3, \ldots$, where

$$e_1(x, y) = [x, y] = xyx^{-1}y^{-1}, \ldots, e_n(x, y) = [e_{n-1}(x, y), y], \ldots,$$

then $\Theta(\vec{e})$ is the class of all Engel groups. In the case of finite groups, the class $\Theta(\vec{e})$ coincides with the class of finite nilpotent groups [51]. Clearly, $\vec{e}$-Engel elements of any group G are the usual Engel elements in G. If G is finite, the set $G(\vec{e})$ coincides with the nilpotent radical of G [4].

(ii) The correct sequence $\vec{u}$, where

$$u_1(x, y) = x^{-2}y^{-1}x, \ldots, u_n(x, y) = [\,x\, u_{n-1}(x, y)x^{-1}, y\, u_{n-1}(x, y)\, y^{-1}], \ldots,$$

determines the class $\Theta(\vec{u})$.) In the case of finite groups, $\Theta(\vec{u})$ coincides with the class of finite solvable groups. Indeed we have:

Theorem 2.5 [6, 7]. *Let $u_1(x, y) := x^{-2}y^{-1}x$, $u_{n+1}(x, y) := [xu_n(x, y)x^{-1}, yu_n(x, y)y^{-1}]$. A finite group G is solvable if and only if for some n the identity $u_n(x, y) \equiv 1$ holds in G.*

(iii) The correct sequence $\vec{s}$, where

$$s_1(x, y) = x, \ldots, s_n(x, y) = [s_{n-1}(x, y)^{-y}, s_{n-1}(x, y)], \ldots$$

determines the class $\Theta(\vec{s})$. In the case of finite groups, $\Theta(\vec{s})$ also coincides with the class of finite solvable groups. Indeed:

Theorem 2.6 [9]. *Let $s_1(x, y) = x, \ldots, s_n(x, y) = [s_{n-1}(x, y)^{-y}, s_{n-1}(x, y)]$. A finite group G is solvable if and only if for some n the identity $s_n(x, y) \equiv 1$ holds in G.*

(iv) The correct sequence $\vec{w}$, where

$$w_1(x, y) = [x, y], \ldots, w_n(x, y) = [[w_{n-1}, x], [w_{n-1}, y]], \ldots$$

and [,] stands for the Lie bracket in a Lie algebra, determines the class $\Theta(\vec{w})$ of finite dimensional solvable Lie algebras over an infinite field k, $\mathrm{char}(k) \neq 2, 3, 5$ [24]. Indeed:

Theorem 2.7 [24]. *Let $w_1(x, y) = [x, y]$, $w_{n+1}(x, y) = [[w_n(x, y), x], [w_n(x, y), y]]$. Then a finite dimensional Lie algebra L defined over an infinite field of zero characteristic or positive characteristic greater than 5 is solvable if and only if for some n the identity $w_n(x, y) \equiv 0$ holds in L.*

Theorems 2.5–2.7 provide characterizations of the solvability property for finite groups and finite dimensional Lie algebras in terms of correct two-variable sequences $\vec{l}$ and the corresponding classes $\Theta(\vec{l})$.

Remark 2.8. Evidently, the classical Burnside problems are associated with one-variable sequence $\vec{b} = \vec{b}(x) = 1, x, x^2, \ldots, x^k, \ldots$

Remark 2.9. All sequences above can be put into profinite setting, and translated to the language of implicit operations (see the works of Almeida [1] and others).

Let us now consider the descriptions of nilpotent and solvable radicals of finite groups. For the nilpotent radical, a description is given by Baer's theorem. For finite dimensional Lie algebras, the following counterpart of Baer's theorem holds.

Definition 2.10. We say that an element $y \in L$ is *strictly Engel*, if it is Engel and for any $x \in L$ the element $[x, y]$ is Engel.

Proposition 2.11 [5]. *Let L be a finite dimensional Lie algebra over a field k with char$(k) = 0$. The nilpotent radical N of L coincides with the set of all strictly Engel elements in L.*

Thus, both the nilpotency of a finite group and the description of elements of the nilpotent radical of a group are ultimately related to Engel sequence and Engel elements. The same is true for finite dimensional Lie algebras.

For the solvable radical of a finite dimensional Lie algebra we have the following description in terms of sequences:

Theorem 2.12 [5]. (i) *Let L be a finite dimensional Lie algebra over a field k of characteristic zero. Define $\vec{v}$ by $v_1(x, y) = x$, $v_{n+1}(x, y) = [v_n(x, y), [x, y]]$. Then the solvable radical R of L coincides with the set of all $\vec{v}$-Engel elements of L.*
(ii) *Let L be a finite dimensional Lie algebra over an algebraically closed field k of characteristic zero. Then the solvable radical R of L coincides with the set of all $\vec{w}$-Engel elements of L.*

Here $\vec{w}$ is as in Example 2.4(iv) above.

The following natural problems arise.

Problem 2.13. *Describe the class of sequences $\vec{u}$ such that $\Theta(\vec{u})$ is the class of finite dimensional solvable Lie algebras.*

A more concrete related problem is the following.

Problem 2.14. *Let L be a finite dimensional Lie algebra of classical type defined over an infinite field of zero characteristic or positive characteristic greater than* 5. *Is it true that the radical R of L coincides with the set of all $\vec{w}$-Engel elements* (*as in Example* 4) *of L*?

We can also ask for which Lie algebras L the set of elements $y \in L$, such that $w_n(x, y) \equiv 0$ holds in L for all $x \in L$ and for some $n = n(x, y)$, is a locally solvable ideal.

The situation with a description of the solvable radical of a finite group in terms of correct sequences is even more unclear.

Problem 2.15. *Is there a correct sequence of words $q_n(x,y)$ with the following properties*:

(i) *a finite group G is solvable if and only if for some n the identity $q_n(x,y) \equiv 1$ holds in G*;

(ii) *the radical R of a finite group G coincides with the set of q-Engel elements*, *i.e. the set of $y \in G$ such that $q_n(x,y) = 1$ for all $x \in G$ and some $n = n(x,y)$*?

Although (ii) implies (i), we state these problems separately since the question (ii) seems much harder.

It turns out that the sequence of Theorem 2.5 as well as the sequence which was studied in Theorem 2.6 are not quite suitable for Problem 2.15. We thus need more sequences satisfying conditions (i) and (ii). To obtain such sequences, a deeper understanding of their geometric nature is required.

Problem 2.15, if settled, would give an explicit characterization of the solvable radical of a finite group.

3. Burnside-Type Problems Related to Engel-Like Sequences

Theorem 2.7 on finite dimensional Lie algebras leads to a similar question in the infinite-dimensional case. Namely, the remarkable Kostrikin–Zelmanov theorem on locally nilpotent Lie algebras [31, 49, 50] and Zelmanov's theorem [48] give rise to the following Burnside-type problems for Lie algebras.

Problem 3.1. *Suppose that L is a Lie algebra over a field k*, *the w_n's are defined by the formulas of Theorem* 2.7, *and there is n such that the identity $w_n(x,y) \equiv 0$ holds in L. Is it true that L is locally solvable*? *If k is of characteristic* 0, *is it true that L is solvable*?

It would be of significant interest to consider similar problems for groups. Recall that G is an Engel group if there is an integer n such that the Engel identity $e_n(x,y) \equiv 1$ holds in G. Suppose a sequence $q_n(x,y)$ is chosen as in Theorem 2.5, i.e. $q_n(x,y) = u_n(x,y)$, or as in Theorem 2.6, i.e. $q_n(x,y) = s_n(x,y)$. We call G a *quasi-Engel* group (with respect to the sequence $q_n(x,y)$ or just quasi-Engel) if there is an integer n such that the identity $q_n(x,y) \equiv 1$ holds in G.

The following problems imitate the analogous problems for Engel groups.

Problem 3.2. *Is every quasi-Engel group locally solvable*?

Problem 3.3. *Is every residually finite*, *quasi-Engel group locally solvable*?

For the class of profinite groups the situation looks more promising in view of [46].

Conjecture 3.4. *Every profinite quasi-Engel group is locally solvable.*

It is quite natural to consider restricted versions of Problems 1.3 and 3.2, as for the Burnside problem. Let E_n be the Engel variety defined by the identity $v_n \equiv 1$. Let $F = F_{k,n}$ be the free group with k generators in the variety E_n. One can prove that the intersection of all co-nilpotent normal subgroups H_α in F is also co-nilpotent. Hence there exists a group $F^0_{n,k}$ in E_n such that every nilpotent group $G \in E_n$ with k generators is a homomorphic image of $F^0_{n,k}$. This implies that all locally nilpotent groups from E_n form a variety. In other words, the *restricted Engel problem has a positive solution.* The situation with the restricted quasi-Engel problem is unclear.

Problem 3.5. *Let $F = F_{k,n}$ be the free group with k generators in the variety of all quasi-Engel groups with fixed n. Is it true that the intersection of all co-solvable normal subgroups in $F = F_{k,n}$ is also co-solvable?*

It would be of great interest to consider the restricted quasi-Engel problem for profinite groups.

Remark 3.6. In case of an affirmative solution of Problem 2.15, the corresponding sequence $q_n(x, y)$ should be chosen for the definition of quasi-Engel groups and for the problems above.

Note that the problem of description of the locally solvable radical as the set of quasi-Engel elements makes sense for wide classes of groups (such as Noetherian groups and their extensions, cf. [4, 38, 39] for the Engel counterparts, [34] for topological groups, etc).

4. Burnside-Type Problems Related to Thompson-Type Properties: Two Generated Subgroups

Generally speaking, one can say that Burnside problems ask to what extent finiteness of cyclic subgroups determine finiteness of arbitrary finitely generated subgroups of a group. Another approach to global properties of groups relies on the investigation of their two-generated subgroups. For example, a finite group G is nilpotent if and only if every two-generated subgroup of G is nilpotent.

Thus one can replace the principal Problem 1.3 by the somewhat weaker.

Problem 4.1. *Is every group with uniformly bounded class of nilpotency for the two-generated subgroups locally nilpotent?*

Remark 4.2. Note that Problem 1.3 and hence, Problem 4.1, have positive solutions for $n = 3$, [26] and $n = 4$, [25]. They also have positive solutions for many classes of groups (see [10, 23, 29, 35, 36, 45, 46], etc).

The result for two-generated subgroups with respect to the solvability property is provided by a remarkable theorem of Thompson [11, 43]:

Theorem 4.3. *A finite group G is solvable if and only if every two-generated subgroup of G is solvable.*

The similar fact holds for finite dimensional Lie algebras [21]. We can look at this fact from the perspective of Burnside-type problems:

Problem 4.4. *Let L be a Lie algebra in which every two elements generate a solvable subalgebra of derived length at most d. Does it follow that L is locally solvable*?

The Golod example [14, 15] shows that every 2-generated subgroup being solvable does not imply that G is solvable for G a finitely generated residually finite group. So we ask:

Problem 4.5. *Let G be a residually finite group, and suppose $\langle x, y\rangle$ is solvable of derived length at most d for every pair of elements $x, y \in G$. Does it follow that G is solvable of derived length at most $f(d)$ for some function f?*

This reduces to considering finite solvable groups. It is also natural to consider the following variation, where we bound the number of generators of G.

Problem 4.6. *Let G be a residually finite group generated by c elements, and suppose $\langle x, y\rangle$ is solvable of derived length at most d for every pair of elements $x, y \in G$. Does it follow that G is solvable of derived length at most $f(c, d)$ for some function f?*

Again this reduces to considering finite solvable groups. In fact, some of the above questions may even be posed in greater generality, namely for arbitrary groups. For example:

Problem 4.7. *Construct a non-locally solvable group G which has a uniformly bounded class of solvability for the two-generated subgroups.*

Of course, a counterexample for Problem 4.1 would imply the answer to Problem 4.7 since every locally solvable Engel group should be locally nilpotent [36].

5. Burnside-Type Problems Related to Weak Engel-Type Properties: One and a Half Generated Subgroups

We start with a recent Thompson-type characterization of the solvable radical of a finite group, obtained in [21].

Definition 5.1. Let G be a group (Lie algebra). We say that $y \in G$ is a radical element if for any $x \in G$ the subgroup (subalgebra) generated by x and y is solvable.

Obviously, an element $y \in G$ is radical if and only if the $\langle y^{\langle x\rangle}\rangle$ is solvable for all x. Note that $\langle y^{\langle x\rangle}\rangle$ denotes the minimal normal subgroup containing y in the group $\langle x, y\rangle$.

Theorem 5.2 [21]. *Let G be a finite group. The solvable radical R of G coincides with the collection of all radical elements in G.*

A similar characterization holds for finite dimensional Lie algebras over a field k of zero characteristic. However, the fact is no longer true for prime characteristic (as noted by Premet, see also [40, p. 27]).

Theorem 5.2 implies

Corollary 5.3. *Let G be a finite group, let $y \in G$, and let $\langle y^G \rangle$ denote the minimal normal subgroup of G containing y. Then $\langle y^G \rangle$ is solvable if and only if the subgroup $\langle y^{\langle x \rangle} \rangle$ is solvable for all $x \in G$.*

The proof of Theorem 5.2 relies on the so-called "one and a half generation" of almost simple groups, proved by Guralnick and Kantor [20] using probabilistic arguments (see the survey [42] for further background). Now we put the notion of a radical element into a more general context which preserves the flavor of one and a half generation.

In the previous section, we considered the properties of the whole group G assuming certain properties of two-generated subgroups. Now we fix an element $g \in G$, assume that its behavior with respect to any element $x \in G$ is prescribed, and we ask if the normal closure of g in G satisfies the same properties. This local-global behavior is a kind of a Burnside-type problem.

Let $\mathfrak{X}$ be a class of groups. Let G be a group.

Definition 5.4. An element $y \in G$ is called locally $\mathfrak{X}$-radical if $\langle y^{\langle x \rangle} \rangle$ belongs to $\mathfrak{X}$ for every $x \in G$. An element $y \in G$ is called globally $\mathfrak{X}$-radical if $\langle y^G \rangle$ belongs to $\mathfrak{X}$.

So we have local and global properties. Obviously, if a class $\mathfrak{X}$ is closed under subgroups then a globally $\mathfrak{X}$-radical element is automatically locally radical.

The main problem is to study for which classes of groups the converse property holds.

Let $\mathfrak{X}$ be the class of locally nilpotent groups. In this case, we call the element y above locally radical. Let $y \in \langle y^{\langle x \rangle} \rangle$. Then it is easy to see that y is an Engel element.

So our main problem for an arbitrary group G with respect to the class of locally nilpotent groups is as follows.

Does any locally radical element of a group G lie in a locally nilpotent normal subgroup of G? That is, does it lie in the locally nilpotent radical of G?

In this form, the answer is negative due to Golod's example. However, there are many classes of groups with the property that any Engel element lies in the locally nilpotent radical. For example, these are solvable groups, noetherian groups, linear groups, PI-groups, etc. For these groups, a locally radical element is also globally radical. Other local-global problems appear if the classes of nilpotency for locally radical elements are uniformly bounded.

Now let $\mathfrak{X}$ be the class of locally solvable groups. In this case, y is a locally radical element (of a solvable type) if $\langle y^{\langle x \rangle} \rangle$ is locally solvable, and y is a globally radical element if $\langle y^G \rangle$ is locally solvable.

We have here the problem:

Problem 5.5. (i) *Determine for which groups every locally radical element is globally radical.*

(ii) *Determine for which groups every radical element is globally radical.*

(iii) *Determine for which groups every locally radical element is a radical element.*

Remark 5.6. Note that there exists a group $G = \langle x, y \rangle$ such that $\langle x^{\langle G \rangle} \rangle$ is locally solvable but G is not solvable [32]. Moreover, his construction gives examples of groups $G = \langle x, y \rangle$ such that $\langle x^{\langle G \rangle} \rangle$ is locally finite and even a p-group, but G is not finite.

In infinite groups, there need not be a locally solvable radical, i.e. a maximal locally solvable normal subgroup (see [8]). So item (ii) deserves to be distinguished separately:

Question 5.7. Which groups G satisfy the condition that the subgroup $\langle y^G \rangle$ is locally solvable for any radical element $y \in G$?

Below are several questions of this type for the case of residually finite groups (cf. [21]).

We have a necessary condition for $\langle y^G \rangle$ to be solvable. If $\langle y^G \rangle$ is solvable of derived length d, then $\langle x, y \rangle$ is solvable of derived length at most $d + 1$ for every $x \in G$.

Question 5.8. Let G be a residually finite group and suppose $y \in G$ satisfies $\langle x, y \rangle$ is solvable of derived length at most d (for some positive integer d independent of x). Does it follow that $\langle y^G \rangle$ is solvable of derived length at most $f(d)$ for some function f?

This question has a negative answer. The following example is due to an unknown referee of the paper [21].

Let B_{np} be the Burnside group of exponent p with np generators. Denote by Γ_n its maximal finite homomorphic image. Clearly, there is an automorphism of Γ_n of order p which acts as a permutation which is a product of n p-cycles on the np generators. Denote this automorphism by y_n. Take G_n to be the semidirect product of Γ_n with $\langle y_n \rangle$. Then the exponent of G_n is p^2, and in particular, G_n is a p-group.

Take an arbitrary $x \in \Gamma_n$ and consider the subgroup $H = \langle x, y_n \rangle$ in G_n. Then H is generated by $\langle y_n \rangle$ and H_0, where H_0 is generated by p elements $\{x, x^{y_n}, x^{y_n^2}, \ldots, x^{y_n^{p-1}}\}$.

Since the number of generators of H_0 coincides with its exponent p, the order of H_0 is bounded by some function of p, independently of x. Correspondingly, the order of H is bounded by a function of p. The group H is a finite p-group, and thus, nilpotent. Hence, the derived length of $H = \langle x, y_n \rangle$ is bounded by some function $f(p)$ for any x.

Now consider the solvable groups Γ_n and take the direct product Γ of all Γ_n. Let G be a semidirect product of Γ and the automorphism y, which acts on each Γ_n as y_n.

The group G is locally finite p-group, and locally solvable. Then $\langle x, y\rangle$ is solvable for any x, and the derived length of $\langle x, y\rangle$ is bounded by $f(p)$. However, the group G is not solvable and $\langle y^G\rangle$ is not solvable as well.

Thus, we state the following questions:

Problem 5.9. *Let G be a residually finite group and suppose $y \in G$ satisfies $\langle x, y\rangle$ is solvable of derived length at most d (for some positive integer d independent of $x \in G$). Does it follow that $\langle y^G\rangle$ is locally solvable?*

Problem 5.10. *Let G be a residually finite group generated by c elements. Fix a positive integer d and an element $y \in G$. Suppose that for all $x \in G$, $\langle x, y\rangle$ is solvable of derived length at most d. Does it follow that $\langle y^G\rangle$ is solvable of derived length at most $f(c, d)$ for some function f?*

Note that this reduces to solving the problem for finite groups. An affirmative answer would give a characterization of the set of elements in a residually finite group whose normal closure is solvable.

Some of the questions above may be posed even in greater generality, namely for arbitrary groups.

6. Finite Groups: The General Situation

Let S be a set of finite simple groups. Denote the class of finite groups G such that all composition factors of G belong to S by $\mathfrak{X} = \mathfrak{X}(S)$. It is easy to see that such $\mathfrak{X}$ is closed under normal subgroups, homomorphic images and extensions. On the other hand, if a class $\mathfrak{X}$ is closed under these three operators and S is the set of all simple groups in $\mathfrak{X}$, then $\mathfrak{X} = \mathfrak{X}(S)$.

It is clear that such class $\mathfrak{X}$ is a radical class. This means that in every finite group G there is a unique maximal normal subgroup $\mathfrak{X}(G)$ which belongs to $\mathfrak{X}$. We want to characterize elements which constitute $\mathfrak{X}(G)$.

Following the previous section, an element $y \in G$ is locally $\mathfrak{X}$-radical if $\langle y^{\langle x\rangle}\rangle$ belongs to $\mathfrak{X}$ for every $x \in G$. An element $y \in G$ is called globally radical if $\langle y^G\rangle$ belongs to $\mathfrak{X}$.

We will use the basic properties of the generalized Fitting subgroup, $F^*(G)$ of a finite group G. See [3].

Theorem 6.1. *Let $\mathfrak{X}$ be a class of finite groups closed under homomorphic images, normal subgroups and extensions (equivalently, let $\mathfrak{X}$ be a class of finite groups with composition factors in some set S of simple groups).*

(i) *If G is a finite group then every locally $\mathfrak{X}$-radical element belongs to $\mathfrak{X}(G)$.*

(ii) *If in addition, $\mathfrak{X}$ is closed under subgroups, then $\mathfrak{X}(G)$ coincides with the set of all $\mathfrak{X}$-locally radical elements.*

Remark 6.2. Theorem 5.2 is a particular case of Theorem 6.1 if $\mathfrak{X}$ is the class of solvable groups. In this case S consists of all cyclic groups of prime order. We can take also classes of p-groups, p is prime, of π-groups, and other interesting classes. However, Theorem 6.1 does not cover the class of nilpotent groups where we have to use Baer's theorem for $\mathfrak{X}(G) = F(G)$, the Fitting subgroup.

Proof If $\mathfrak{X}$ is closed under subgroups, then $\langle g^G \rangle$ in $\mathfrak{X}$ implies that $\langle g^{\langle x \rangle} \rangle$ is in $\mathfrak{X}$, and so we see that the second statement follows from the first. We shall prove the first implication, i.e. we have to prove that if G is a finite group and $g \in G$ with $\langle g^{\langle x \rangle} \rangle$ in $\mathfrak{X}$ for all $x \in G$, then $\langle g^G \rangle$ is in $\mathfrak{X}$.

So assume that G is a minimal counterexample to the first statement. This means that there exists a locally $\mathfrak{X}$-radical element g in G with $\langle g^G \rangle$ not in $\mathfrak{X}$. Consider the properties of this group G.

Reduction 1: Take an arbitrary locally $\mathfrak{X}$-radical element g in G such that $\langle g^G \rangle$ is not in $\mathfrak{X}$. Show that $G = \langle g^G \rangle$. If not, set $H = \langle g^G \rangle$ and suppose that $H < G$. Take an arbitrary element $h \in H$. Then $h = \prod g_i$ where all g_i are conjugate to g. Hence, all g_i are locally $\mathfrak{X}$-radical elements. Since $H < G$, then all elements g_i lie in the radical $\mathfrak{X}(H)$. Then, clearly, h lies in $\mathfrak{X}(H)$. Therefore, $\langle g^G \rangle = H = \mathfrak{X}(H)$ and H is in $\mathfrak{X}$, a contradiction, and we may assume that $G = \langle g^G \rangle$.

Reduction 2: G has a unique minimal normal subgroup. If not, G has two normal subgroups N_1 and N_2 with trivial intersection. Consider $N_1N_2/N_2 \simeq N_1$. Since G/N_2 lies in $\mathfrak{X}$, then $N_1 \in \mathfrak{X}$. Since $G/N_1 \in \mathfrak{X}$, we have $G \in \mathfrak{X}$, a contradiction. In particular, we may assume that every two normal subgroups in G has a non-trivial intersection.

Reduction 3: $\mathfrak{X}(G) = 1$.

If not, pass to $G/\mathfrak{X}(G)$ and so by induction, $g\mathfrak{X}(G)$ is in $\mathfrak{X}(G/\mathfrak{X}(G)) = 1$, whence $g \in \mathfrak{X}(G)$.

Reduction 4: $F(G) = Z(G)$.

Suppose that the Fitting subgroup $F(G) \neq 1$. Suppose that $F(G) \neq Z(G)$. Since each Sylow subgroup of $F(G)$ is normal in G, it follows that $F(G)$ is a p-group for some prime p. Since $G = \langle g^G \rangle$, it suffices to show that g commutes with $F(G)$. If not, then taking $y \in F(G)$ with $[g, y] \neq 1$ shows that $1 \neq g^{-1}g^y$ and so $\langle g^{\langle y \rangle} \rangle$ has a composition factor of order p. Thus $\mathfrak{X}(G) \geq F(G) \neq 1$, a contradiction.

We now complete the proof. Since G is not abelian, and since $G/Z(G)$ acts faithfully on $F^*(G)$, there is a component Q of G (otherwise, $F^*(G) = F(G) = Z(G)$ and G is abelian).

If $Z(G) \neq 1$, then $Z(G) \cap Q \neq 1$ (otherwise the normal closure of Q would be a minimal normal subgroup).

We may assume that g does not commute with Q (for if g commuted with every component of G, then so would $G = \langle g^G \rangle$, which is a contradiction). Let $H = \langle Q, g \rangle$. Set N to be the (central) product of the distinct conjugates of Q under $\langle g \rangle$. Then N

is clearly perfect and H/N is cyclic (generated by g). Also, $Z(N) = \Phi(N) \leq \Phi(H)$ and $N/Z(N)$ is a minimal normal subgroup of $H/Z(N)$. So applying [21, Lemma 3.4] to $H/Z(N)$, we see that $H = \langle g, h\rangle$ for some $h \in N$.

We claim that $H = J := \langle g^{\langle h\rangle}\rangle$. Clearly, J is normal in H (since it is normalized by $g \in J$ and by h (by definition)). Clearly, H/J is abelian (since $g \in J$ and so $[h, g] \in J$). Thus, J contains $[H, H] = N$. Now $H = \langle Q, g\rangle = \langle N, g\rangle$, so once $N < J$, since g is in J, $H = J$.

Since $H = \langle g^{\langle h\rangle}\rangle$, all composition factors of H are $\mathfrak{X}$-groups. If $Z(G) \neq 1$, then $Z(G)$ is an $\mathfrak{X}$-group and if $Z(G) = 1$, then Q is an $\mathfrak{X}$-group, Q lies in $\mathfrak{X}(G)$. In either case, $\mathfrak{X}(G) \neq 1$, a contradiction.

7. Variation on the Baer-Suzuki Result

Recall that the Baer–Suzuki theorem [19] characterizes the nilpotent radical of a finite (or linear) group by the property that g is in the nilpotent radical of G if and only if any two conjugates of g generate a nilpotent group. The proof of this is relatively elementary.

One can ask if there is a similar result for the characterization of the solvable radical. Note that, if $g^2 = 1$, then any two conjugates of g generate a dihedral group and in particular, a metabelian group. Hence, a similar assertion for two conjugates does not characterize the solvable radical.

In [16, 17], the following theorem is established:

Theorem 7.1. *Let G be a finite* (*or linear*) *group. Then g is in the solvable radical of G if and only if any eight conjugates of g generate a solvable group. Thus, a group G is solvable if and only if in any conjugacy class every eight elements generate a solvable subgroup.*

Remark 7.2. In [12], a similar result for 10 conjugates is proved. The proof in [12] does not use the Classification of finite simple groups. However, better estimates certainly depend on the Classification.

Moreover, Flavell, Guest and Guralnick [13] and, independently, Gordeev *et al* [18] announced a sharp analog of the Baer–Suzuki theorem:

Theorem 7.3. *Let G be a finite* (*or linear*) *group. Then g is in the solvable radical of G if and only if any four conjugates of g generate a solvable group.*

The example of g a transposition in S_n $(n > 4)$ shows that four is best possible. However, this is rare. Indeed, one has the following result [13].

Theorem 7.4. *Let G be a finite group. Let g be an element of prime order $p > 3$. If any two conjugates of g generate a solvable group, then g is in the solvable radical of G.*

There are also other criteria when two conjugates are enough (cf. [17, Theorem 1.15]).

Note the following variation of Baer–Suzuki result (see [22, 47]).

If g is a p-element, then g is in the nilpotent radical of the finite group G if and only if $[g, x]$ is a p-element for all $x \in G$.

These results use the 3/2 generation theorem of Guralnick and Kantor [20] and the Classification of finite simple groups.

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