MULTI-SORTED LOGIC AND LOGICAL GEOMETRY: SOME PROBLEMS

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Abstract. The paper has a form of a survey on basics of logical geometry and consists of three parts. It is focused on the relationship between many-sorted theory, which leads to logical geometry and one-sorted theory, which is based on important model-theoretic concepts. Our aim is to show that both approaches go in parallel and there are bridges which allow to transfer results, notions and problems back and forth. Thus, an additional freedom in choosing an approach appears. A list of problems which naturally arise in this field is another objective of the paper.

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Contents

1. Introduction

This paper is focused on relationships between many-sorted and one-sorted theories. An insight based on ideas of many-sorted logic leads to logical geometry, while a one-sorted theory is ultimately related to important model-theoretic concepts. Our aim is to show that both approaches go in parallel and there are bridges which allow to transfer results, notions and problems back and forth. Thus, an additional freedom in choosing an approach appears.

The paper can be viewed as a survey of ideas, results and problems collected under the roof of logical geometry. In our opinion, some simple proofs make the paper more vital.

The first part of the paper contains main notions, the second one is devoted to logical geometry, the third part describes types and isotypeness. The problems are distributed in the corresponding parts. The whole material is oriented towards universal algebraic geometry (UAG), i.e., geometry in an arbitrary variety of algebras Θ. We will distinguish between the equational algebraic geometry and the logical geometry. In the equational geometry equations have the form $w \equiv w'$, where w and w' are elements of the free in Θ algebra $W(X)$. In the logical geometry the elements of the multi-sorted first-order logic play the role of equations. We consider logical geometry (LG) as a part of UAG. This theory is strongly influenced by model theory and ideas of A.Tarski and A.I.Malcev.

We fix a variety of algebras Θ. Let $W = W(X)$ be the free in Θ algebra over a set of variables X. The set X is assumed to be finite, if the opposite is not stated explicitly. In the latter case we use the notation X^0. All algebras under consideration are algebras in Θ. Logic is also related to the variety Θ. As usual, the signature of Θ may contain constants.

2. Main notions

In this section we consider a system of notions we are dealing with. Some of them are not formally defined in this paper. For the precise definitions and references use [8], [18], [22], [23], [29], [33].

The general picture of relations between these notions brings forward a lot of new problems, formulated in Sections 3 and 4. These problems are the main objective of the paper.

2.1. **Equations, points, spaces of points and algebra of formulas $\Phi(X)$.** Consider a system T of equations of the form $w = w'$, $w, w' \in W(X)$. Each system T determines an algebraic set of points in the corresponding affine space.

Let $X = \{x_1, \dots, x_n\}$ and let H be an algebra in the variety Θ. We have an affine space H^X of points $\mu : X \to H$. For every μ we have also the n-tuple $(a_1, \dots, a_n) = \bar{a}$ with $a_i = \mu(x_i)$. For the given Θ we have the homomorphism

$$\mu : W(X) \to H$$

and, hence, the affine space is viewed as the set of homomorphisms

$$Hom(W(X), H).$$

The classical kernel $Ker(\mu)$ corresponds to each point $\mu : W(X) \to H$.

Every point μ has also the logical kernel $LKer(\mu)$. Along with the algebra $W(X)$ we will consider the algebra of formulas $\Phi(X)$. Logical kernel $LKer(\mu)$ consists of all formulas $u \in \Phi(X)$ valid on the point μ.

The algebra $\Phi(X)$ will be defined later on, but let us note now that it is an extended Boolean algebra (Boolean algebra, in which quantifiers $\exists x, x \in X$ act as operations, and equalities (Θ-equalities) $w \equiv w'$, $w, w' \in W(X)$ are defined). It is also defined what does it mean that the point μ satisfies a formula $u \in \Phi(X)$. These u are treated as equations. For $T \subset \Phi(X)$, in $Hom(W(X), H)$ we have an elementary set (definable set) consisting of points μ which satisfy every $u \in T$.

Each kernel $LKer(\mu)$ is a Boolean ultrafilter in $\Phi(X)$. Note that

$$Ker(\mu) = LKer(\mu) \cap M_X,$$

where M_X is the set of all $w \equiv w'$, $w, w' \in W(X)$.

2.2. **Extended Boolean algebras.** Let us make some comments regarding the definition of the notion of extended Boolean algebra.

Let B be a Boolean algebra. The *existential quantifier* on B is an unary operation $\exists : B \to B$ subject to conditions

(1) $\exists(0) = 0$,
(2) $a \leq \exists(a)$,
(3) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

The *universal quantifier* $\forall : B \to B$ is defined dually:

(1) $\forall(1) = 1$,
(2) $a \geq \forall(a)$,
(3) $\forall(a \vee \forall b) = \forall a \vee \forall b$.

Here the numerals 0 and 1 are zero and unit of the Boolean algebra B and a, b are arbitrary elements of B.

As usual, the quantifiers $\exists$ and $\forall$ are coordinated by: $(\exists a) = \neg(\forall(\neg a))$, and $(\forall a) = \neg(\exists(\neg a))$.

Now suppose that a variety of algebras Θ is fixed and $W(X)$ is the free in Θ algebra over the set of variables X. These data allow to define the extended Boolean algebra. This is a Boolean algebra where the quantifiers $\exists x$ are defined for every $x \in X$ and

$$\exists x \exists y = \exists y \exists x$$

for every x and y from X. Besides that, for every pair of elements $w, w' \in W(X)$ in an extended Boolean algebra the equality $w \equiv w'$ is defined. These equalities are considered as nullary operations, that is, as constants. Each equality satisfies conditions of an equivalence relation, and for every operation ω from the signature of algebras from Θ we have

$$(w_1 \equiv w'_1) \wedge \ldots \wedge (w_n \equiv w'_n) \to (w_1 \ldots w_n \omega) \equiv (w'_1 \ldots w'_n \omega).$$

Algebra of formulas $\Phi(X)$ is an example of extended Boolean algebra in Θ. Now consider another example.

2.3. **Important example.** We start from an affine space $Hom(W(X), H)$. Let $Bool(W(X), H)$ be the Boolean algebra of all subsets of $Hom(W(X), H)$. Extend this algebra by adding quantifiers $\exists x$ and equalities. For $A \in Bool(W(X), H)$ we set: $B = \exists x A$ is the set ("cylinder") of points $\mu : W(X) \to H$ such that there is $\nu : W(X) \to H$ in A and $\mu(x') = \nu(x')$ for $x' \in X$, $x' \neq x$. It is, indeed, an existential quantifier for every $x \in X$.

Define an equality $[w \equiv w']_H$ in $Bool(W(X), H)$ for every $w \equiv w'$ in M_X, setting $\mu \in [w \equiv w']_H$ if $(w, w') \in Ker(\mu)$, i.e., $w^\mu = w'^\mu$.

Remark 2.1. The set $[w \equiv w']_H$ can be empty. Thus, we give the following definition. The equality $[w \equiv w']_H$ is called *admissible* for the given Θ, if for every $H \in \Theta$ the set $[w \equiv w']_H$ is not empty. If Θ is the variety of all groups, then each equality is admissible. The same is true for the variety of associative algebras with unity over complex numbers. However, for the field of real numbers this is not the case. Here $x^2+1=0$ is not an admissible equality.

We assume that in each algebra of formulas $\Phi(X)$ lie all Θ-equalities. To arbitrary equality $w \equiv w'$ corresponds either a non-empty equality $[w \equiv w']_H$ in $H \in \Theta$, or the empty set in $H \in \Theta$ which is the zero element of this Boolean algebra.

We have arrived to an extended Boolean algebra, denoted now by $Hal_\Theta^X(H)$. We shall emphasize that this algebra and the algebra of formulas $\Phi(X)$ have the same signature.

2.4. **Homomorphism Val_H^X.** We will proceed from the homomorphism

$$Val_H^X : \Phi(X) \to Hal_\Theta^X(H)$$

with the condition $Val_H^X(w \equiv w') = [w \equiv w']_H$ for equalities, if $[w \equiv w']_H$ is non-empty, or 0 otherwise. This homomorphism will be defined in subsection 2.9. The existence of such a homomorphism is not a trivial fact, since the equalities M_X do not generate (and, of course, do not generate freely) the algebra $\Phi(X)$. If, further, $u \in \Phi(X)$, then $Val_H^X(u)$ is a set of points in the affine space $Hom(W(X), H)$. We say that a point μ satisfies the formula u if μ belongs to $Val_H^X(u)$. Thus, $Val_H^X(u)$ is precisely the set of points satisfying the formula u. Define the *logical kernel* $LKer(\mu)$ of a point μ as the set of all formulas u such that $\mu \in Val_H^X(u)$.

We have

$$Ker(\mu) = LKer(\mu) \cap M_X.$$

Here $Ker(\mu)$ is the set of all formulas of the form $w \equiv w'$, $w, w' \in W(X)$, such that the point μ satisfies these formulas. In parallel, $LKer(\mu)$ is the set of all formulas u, such that the point μ satisfies these formulas.

Then,

$$Ker(Val_H^X) = Th^X(H),$$

$$\bigcap_{\mu : W(X) \to H} LKer(\mu) = Th^X(H).$$

Here $Th^X(H)$ is a set of formulas $u \in \Phi(X)$, such that $Val_H^X(u)$ is the unit in $Bool(W(X), H)$. That is, $Val_H^X(u) = Hom(W(X), H)$ and, thus, $Th^X(H)$ is an X-component of the elementary theory of the algebra H.

In general we have a multi-sorted representation of the elementary theory

$$Th(H) = (Th^X(H), X \in \Gamma),$$

where Γ is a certain system of sets, see Section 2.5.

It follows from the previous considerations that the algebra of formulas $\Phi(X)$ can be embedded in $Hal_\Theta^X(H)$ modulo elementary theory of the algebra H. This fact will be used in the sequel.

2.5. **Multi-sorted logic: first approximation.** Let, further, X^0 be an infinite set of variables and Γ a system of all finite subsets X in X^0.

So, in the logic under consideration we have an infinite system Γ of finite sets instead of one infinite X^0. This leads to multi-sorted logic. This approach is caused by relations with UAG. In the field of universal algebraic geometry one can consider equational geometry and logical geometry. Correspondingly, we have algebraic sets of points and definable sets of points in each affine space.

In the final part of the paper, along with the system of sorts Γ, we also use a system of sorts Γ^* where one initial infinite set X^0 is added to the system Γ.

2.6. **Algebra** $Hal_\Theta(H)$**.** All these algebras and corresponding categories present universal semantics for the logic concerned with a variety Θ. Syntax of this logic is given by the algebra $\tilde{\Phi}$. The homomorphism

$$Val_H : \tilde{\Phi} \to Hal_\Theta(H)$$

gives the correspondence between syntax and semantics. This homomorphism and the homomorphism

$$Val_H^X : \tilde{\Phi}(X) \to Hal_\Theta^X(H)$$

will be defined at the end of the section.

We start with the category $\Theta^*(H)$ of affine spaces. Its objects are spaces $Hom(W(X), H)$, where $X \in \Gamma$.

Morphisms

$$\widetilde{s} : Hom(W(X), H) \to Hom(W(Y), H)$$

of $\Theta^*(H)$ are mappings induced by homomorphisms $s : W(Y) \to W(X)$ according to the rule $\widetilde{s}(\nu) = \nu s$ for every $\nu : W(X) \to H$.

Given a variety of algebras Θ, define the category Θ^0. Its objects are free in Θ algebras $W(X)$ and morphisms s are homomorphisms of algebras. The correspondences $W(X) \to Hom(W(X), H)$ and $s \to \widetilde{s}$ give rise to a contravariant functor

$$\varphi : \Theta^0 \to \Theta^*(H).$$

Morphisms $\tilde{s}$ and s act in the opposite direction. Note that if s is surjective, then $\tilde{s}$ is injective, and if s is injective, then $\tilde{s}$ is surjective.

Proposition 2.2. *Functor $\varphi : \Theta^0 \to \Theta^*(H)$ defines a duality of categories if and only if the variety $Var(H)$ generated by H coincides with Θ.*

Proof. The condition of duality implies that if $s_1 \neq s_2$ for the given morphisms $s_1, s_2 : W(Y) \to W(X)$ then $\widetilde{s_1} \neq \widetilde{s_2}$.

Let us assume that $Var(H) = \Theta$ and the categories are not dual, so there are morphisms s_1 and s_2 such that $s_1 \neq s_2$ but $\widetilde{s_1} = \widetilde{s_2}$. Take some $y \in Y$ such that $s_1(y) = w_1$, $s_2(y) = w_2$ and $w_1 \neq w_2$. We will show that the non-trivial identity $w_1 \equiv w_2$ holds in H. Take an arbitrary homomorphism

$\nu : W(X) \to H$. The equality $\tilde{s}_1 = \tilde{s}_2$ implies $\tilde{s}_1(\nu) = \tilde{s}_2(\nu)$ or $\nu s_1 = \nu s_2$. We apply this morphism to the variable y:

$$\nu s_1(y) = \nu s_2(y) \text{ or } \nu w_1 = \nu w_2.$$

Since $\nu : W(X) \to H$ is an arbitrary homomorphism, then $w_1 \equiv w_2$ is an identity of the algebra H. But $Var(H) = \Theta$, which means that there are no non-trivial identities in H. We have a contradiction and the condition $Var(H) = \Theta$ implies duality of the given categories.

Now we show that if $Var(H) \subset \Theta$, then there is no duality. Let $w_1 \equiv w_2$ be some non-trivial identity of the algebra H. Take $Y = \{y_0\}$ and let $s_1(y_0) = w_1$, $s_2(y_0) = w_2$. For any $\nu : W(X) \to H$ we have

$$\nu w_1 = \nu w_2,\ \nu s_1(y_0) = \nu s_2(y_0),\ \widetilde{s_1}(\nu)(y_0) = \widetilde{s_2}(\nu)(y_0).$$

Since the set Y contains only one element y_0, then $\widetilde{s_1}(\nu) = \widetilde{s_2}(\nu)$. As ν is arbitrary, then $\widetilde{s_1} = \widetilde{s_2}$ and there is no duality of the categories. $\square$

Define further the category $Hal_\Theta(H)$. Its objects are algebras $Hal_\Theta^X(H)$. Proceed from $s : W(X) \to W(Y)$ and pass to $\tilde{s} : Hom(W(Y), H) \to Hom(W(X), H)$. Recall that, $Hal_\Theta^X(H) = Bool(W(X), H)$. Take $A \subset Hom(W(X), H)$. Define

$$s_*(A) = \tilde{s}^{-1}(A) = B \subset Hom(W(Y), H).$$

By definition, $\mu \in B$ if and only if $\mu s = \tilde{s}(\mu) \in A$. This determines a morphism

$$s_* = s_*^H : Hal_\Theta^X(H) \to Hal_\Theta^Y(H).$$

Here s_* is well coordinated with the Boolean structure, and relations with quantifiers and equalities are coordinated by identities from Definition 2.3. The category $Hal_\Theta(H)$ can be also treated as a multi-sorted algebra

$$Hal_\Theta(H) = (Hal_\Theta^X(H), X \in \Gamma).$$

2.7. **Variety of Halmos algebras Hal_Θ.** Algebras in Hal_Θ have the form

$$\mathfrak{L} = (\mathfrak{L}_X,\ X \in \Gamma).$$

Here all domains $\mathfrak{L}_X$ are X-extended Boolean algebras. The unary operation

$$s_* : \mathfrak{L}_X \to \mathfrak{L}_Y$$

corresponds to each homomorphism $s : W(X) \to W(Y)$. Besides, we will define a category $\mathfrak{L}$ of all $\mathfrak{L}_X$, $X \in \Gamma$ with morphisms $s_* : \mathfrak{L}_X \to \mathfrak{L}_Y$. The transition $s \to s_*$ determines a covariant functor $\Theta^0 \to \mathfrak{L}$. Informally, operations of s_*-type make logics dynamical.

Every $\mathfrak{L}_X$ is an X-extended Boolean algebra. Denote its signature by

$$L_X = \{\vee, \wedge, \neg, \exists x, M_X\},\ \text{for all } x \in X.$$

Here M_X stands for the set of all symbols of relations of equality of the form $w \equiv w'$.

Denote by $S_{X,Y}$ the set of symbols of operations s_* of type $\tau = (X;Y)$, where $X, Y \in \Gamma$. Define the signature

$$L_\Theta = \{L_X, S_{X,Y}; X, Y \in \Gamma\}.$$

The signature L_Θ is multi-sorted. We take L_Θ as the signature of an arbitrary algebra from the variety of multi-sorted algebras Hal_Θ. The constructed multi-sorted algebras $Hal_\Theta(H)$ possess this signature with the natural realization of all operations from L_Θ.

There is a series of axioms which determine algebras from the variety Hal_Θ. For example, every s_* respects Boolean operations in $\mathfrak{L}_X$ and $\mathfrak{L}_Y$. Correlations of s_* with equalities and quantifiers are described by more complex identities. Below we give the complete list of axioms for Hal_Θ (see also [31], [33]).

Definition 2.3. We call an algebra $\mathfrak{L} = (\mathfrak{L}_X, X \in \Gamma)$ in the signature L_Θ a *Halmos algebra*, if

(1) Every domain $\mathfrak{L}_X$ is an extended Boolean algebra in the signature L_X.

(2) Every mapping $s_* : \mathfrak{L}_X \to \mathfrak{L}_Y$ is a homomorphism of Boolean algebras. Let $s : W(X) \to W(Y)$, $s' : W(Y) \to W(Z)$, and let $u \in \mathfrak{L}_X$. Then $s'_*(s_*(u)) = (s's)_*(u)$.

(3) Conditions controlling the interaction of s_* with quantifiers are as follows:

   (a) $s_{1*}\exists xa = s_{2*}\exists xa$, $a \in \mathfrak{L}(X)$, if $s_1(y) = s_2(y)$ for every $y \neq x$, $x, y \in X$.

   (b) $s_*\exists xa = \exists(s(x))(s_*a)$, $a \in \mathfrak{L}(X)$, if $s(x) = y$ and y is a variable which does not belong to the support of $s(x')$, for every $x' \in X$ and $x' \neq x$.
   This condition means that y does not participate in the shortest expression of the element $s(x') \in W(Y)$.

(4) Conditions controlling the interaction of s_* with equalities are as follows:

   (a) $s_*(w \equiv w') = (s(w) \equiv s(w'))$.

   (b) $(s^x_w)_*a \wedge (w \equiv w') \leq (s^x_{w'})_*a$, where $a \in \mathfrak{L}(X)$ and $s^x_w \in End(W(X))$ is defined by: $s^x_w(x) = w$ and $s^x_w(x') = x'$, for $x' \neq x$.

Remark 2.4. We should note that all conditions from the definition of a Halmos algebra can be represented as identities, and this is why the class of Halmos algebras is, indeed, a variety.

Define Hal_Θ to be the variety of all Halmos algebras, that is every algebra from Hal_Θ satisfies Definition 2.3.

Proposition 2.5. *Each algebra $Hal_\Theta(H)$ belongs to the variety Hal_Θ.*

This proposition will be proved in Section 2.10. Moreover,

Theorem 2.6 ([30])**.** *All $Hal_\Theta(H)$, where H runs through Θ, generate the variety Hal_Θ.*

In view of Theorem 2.6 one could define from the very beginning the variety Hal_Θ as the variety, generated by all algebras $Hal_\Theta(H)$.

Recall, that every ideal of an extended Boolean algebra is a Boolean ideal invariant with respect to the universal quantifiers action. An extended Boolean algebra is called *simple* if it does not have non-trivial ideals. In the multi-sorted case an ideal is a system of one-sorted ideals which respects all operations of the form s_*. A multi-sorted Halmos algebra is *simple* if it does not have non-trivial ideals. Algebras $Hal_\Theta(H)$ and their subalgebras are simple Halmos algebras, see [34]. Moreover, these algebras are the only simple algebras in the variety Hal_Θ. Finally, every Halmos algebra is residually simple, see [34]. This fact is essential in the next subsection. Note, that all these facts are true because of the choice of the identities in the variety Hal_Θ.

2.8. **Multi-sorted algebra of formulas.** We shall define the algebra of formulas

$$\widetilde{\Phi} = (\Phi(X),\ X \in \Gamma).$$

We define this algebra as the free over the multi-sorted set of equalities

$$M = (M_X, X \in \Gamma)$$

algebra in Hal_Θ. Assuming this property denote it as

$$Hal^0_\Theta = (Hal^X_\Theta,\ X \in \Gamma).$$

So, $Hal^X_\Theta = \Phi(X)$ and $\widetilde{\Phi} = Hal^0_\Theta$.

In order to define Hal^0_Θ we start from the absolutely free over the same M algebra

$$\mathfrak{L}^0 = (\mathfrak{L}^0(X),\ X \in \Gamma).$$

This free algebra is considered in the signature of the variety Hal_Θ. Algebra $\mathfrak{L}^0$ can be viewed as the algebra of pure formulas of the corresponding logical calculus.

Then, $\widetilde{\Phi}$ is defined as the quotient algebra of $\mathfrak{L}^0$ modulo the verbal congruence of identities of the variety Hal_Θ. The same algebra $\widetilde{\Phi}$ can be obtained from $\mathfrak{L}^0$ using the Lindenbaum-Tarski approach. Namely, basing on identities of Hal_Θ we distinguish a system of axioms and rules of inference in $\mathfrak{L}^0$. For every $X \in \Gamma$ consider the formulas

$$(u \to v) \wedge (v \to u),$$

where $u, v \in \mathfrak{L}^0(X)$. Here $u \to v$ means $\neg u \vee v$. We assume that every

$$(u \to v) \wedge (v \to u),$$

is deducible from the axioms if and only if the pair (u, v) belongs to the X-component of the given verbal congruence.

So, $\widetilde{\Phi}$ can be viewed as an algebra of the compressed formulas modulo this congruence.

2.9. **Homomorphism Val_H.** Proceed from the mapping

$$M_X \to Hal_\Theta^X(H),$$

which takes the equalities $w \equiv w'$ in M_X to the corresponding equalities $[w \equiv w']_H$ in $Hal_\Theta^X(H)$. This gives rise also to the multi-sorted mapping

$$M = (M_X, X \in \Gamma) \to Hal_\Theta(H) = (Hal_\Theta^X(H),\ X \in \Gamma).$$

Since the multi-sorted set M generates freely the algebra $\widetilde{\Phi}$, this mapping is uniquely extended up to the homomorphism

$$Val_H : \widetilde{\Phi} \to Hal_\Theta(H).$$

Note that this homomorphism is the unique homomorphism $\widetilde{\Phi} \to Hal_\Theta(H)$, since equalities are considered as constants.

We have

$$Val_H^X : \Phi(X) \to Hal_\Theta^X(H),$$

i.e., Val_H acts componentwise for each $X \in \Gamma$.

Recall that for every $u \in \Phi(X)$ the corresponding set $Val_H^X(u)$ is a set of points $\mu : W(X) \to H$ satisfying the formula u (see Section 2.4). The logical kernel $LKer(\mu)$ was defined in Section 2.1 in these terms. Now we can say, that if a formula u belongs to $\Phi(X)$ and a point $\mu : W(X) \to H$ is given, then

$$u \in LKer(\mu) \text{ if and only if } \mu \in Val_H^X(u).$$

We shall note that a formula u can be, in general, of the form $u = s_*(v)$, where $v \in \Phi(Y)$, Y is different from X. This means that the logical kernel of the point is very big and it gives a rich characterization of the whole theory.

As we have seen, $LKer(\mu)$ is a Boolean ultrafilter containing the elementary theory $Th^X(H)$. Any ultrafilter with this property will be considered as an X-type of the algebra H.

It is clear that

$$Ker(Val_H) = Th(H).$$

This remark is used, for example, in Definition 3.35.

Recall that the algebra $\widetilde{\Phi}$ is residually simple. This fact implies two important observations:

1. Let u, v be two formulas in $\Phi(X)$. These formulas coincide if and only if for every algebra $H \in \Theta$ the equality

$$Val_H^X(u) = Val_H^X(v)$$

holds.

2. Let a morphism $s : W(X) \to W(Y)$ be given. The morphism $s_* : \Phi(X) \to \Phi(Y)$ corresponds to s. Let us take formulas $u \in \Phi(X)$ and $v \in \Phi(Y)$. The equality

$$s_*(u) = v$$

holds true if and only if for every algebra H in Θ we have

$$s_*(Val_H^X(u)) = Val_H^Y(v).$$

The following commutative diagram relates syntax with semantics

$$\begin{tikzcd}
\Phi(X) \arrow[r, "s_*"] \arrow[d, "Val_H^X"'] & \Phi(Y) \arrow[d, "Val_H^Y"] \\
Hal_\Theta^X(H) \arrow[r, "s_*^H"] & Hal_\Theta^Y(H).
\end{tikzcd}$$

2.10. **Identities of the variety Hal_Θ for algebras $Hal_\Theta(H)$.** We have already defined the algebras $Hal_\Theta(H)$. Now we show that these algebras satisfy the axioms of Definition 2.3 and, thus, belong to the variety Hal_Θ. In fact we should check the correspondences between s_* and quantifiers and between s_* and equalities.

First we consider interaction of s_* with quantifiers. This interaction is determined by following propositions.

Proposition 2.7. *Let s_1 and s_2 be morphisms $W(X) \to W(Y)$ and let $s_1(x') = s_2(x')$ for all $x' \in X$, $x' \neq x$. Then the equality*

$$s_{1*}\exists x(A) = s_{2*}\exists x(A),$$

where $A \subset Hom(W(X), H)$, holds in $Hal_\Theta(H)$.

Proof. Let $\mu \in s_{1*}\exists x(A)$. Then $\mu s_1 \in \exists x(A)$. In the set A there is a point ν such that $\mu s_1(x') = \nu(x')$ for $x' \neq x$, $x' \in X$. We also have the following equalities:

$$\mu s_2(x') = \mu s_1(x') = \nu(x')$$

and, hence, $\mu s_2 \in \exists x(A)$. So, $\mu \in s_{2*}\exists x(A)$. In a similar manner if $\mu \in s_{2*}\exists x(A)$, then $\mu \in s_{1*}\exists x(A)$. Thus, $s_{1*}\exists x(A) = s_{2*}\exists x(A)$. □

Taking A to be a point a we obtain the axiom (3.a) of Definition 2.3.

Proposition 2.8. *Let $s : W(X) \to W(Y)$ be a morphism. Take $x \in X$ and let $s(x) = y$ for some $y \in Y$. We assume also that y is not contained in the support of each $s(x')$, $x' \neq x$. Then the equality*

$$s_*\exists x(A) = \exists s(x)s_*(A),$$

where $A \subset Hom(W(X), H)$, holds in $Hal_\Theta(H)$.

Proof. Let $\mu \in \exists s(x)s_*(A)$. Take $\nu \in s_*A$ such that $\mu(y') = \nu(y')$, $y' \neq y = s(x)$, $y' \in Y$. We also have $\nu s = \gamma \in A$ and

$$\mu(s(x')) = \mu s(x') = \nu(s(x')) = \nu s(x') = \gamma(x')$$

for every $x' \neq x$. So we have $\mu s \in \exists x(A)$ and $\mu \in s_*\big(\exists x(A)\big)$.

Before proving the inverse inclusion we give some remarks. First of all we generalize this situation. Instead of the one variable x we will consider a set of variables I. Define the quantifier $\exists(I)$ by: $\mu \in \exists(I)A$ if there is a point ν in A such that $\mu(y) = \nu(y)$ for $y \notin I$. Then we are interested in the following equality

$$s_*\exists(I)A = \exists(s(I))s_*A.$$

Let us assume that $s(I) = J$ and $I \subset s^{-1}(J)$, and consider the equality $s_*\exists(s^{-1}(J))A = \exists(J)s_*A$. We will prove that it is true under the condition: $s(x) = s(y) \in J$ if and only if $x = y$. Note that the latter condition follows from the assumption of our proposition.

As before we check that if $\mu \in \exists(J)s_*A$ then $\mu \in s_*\exists(s^{-1}(J))A$.

Let now $\mu \in s_*\exists(s^{-1}(J))A$. We will show that $\mu \in \exists(J)s_*A$. We have $\mu s \in \exists(s^{-1}(J))A$ and $\nu \in A$ with $\mu s(y) = \nu(y)$ for all $y \notin s^{-1}(J) = I$.

Now we choose a certain element $\gamma \in s_*A$. We assume that $\gamma(x) = \mu(x)$ for $x \notin J$ and $\gamma(x) = \nu(s^{-1}(x))$ if $x \in s(I) \subset J$.

Take $x = s(x')$, $x' \in X$, $x \in J$. Then $x' = s^{-1}(x)$ and x' is uniquely defined by the element x. So, we have

$$\gamma s(x') = \gamma(s(x')) = \nu(s^{-1}s(x')) = \nu(x'),$$

where x is an arbitrary element from the set I.

Let now $x' \notin I$ and $s(x') = x$ does not belong to J. Then

$$\gamma s(x') = \gamma(s(x')) = \mu(s(x')) = \mu s(x') = \nu(x').$$

Thus, $\gamma s(x') = \nu(x')$ for all x'. Then, $\gamma s = \nu \in A$ and $\gamma \in s_*A$. Thus, $\mu \in \exists(J)s_*A$. As a result we have that

$$s_*\exists(s^{-1}(J))A = \exists(J)s_*A.$$

We have started the proof of this equality with the set I and then turned to the set $s(I) = J$. The condition $s(x) = s(y)$ implies $x = y$ and we have $s^{-1}(J) = I$. Now we can rewrite the equality above as follows:

$$s_*\exists(I)A = \exists(s(I))s_*A.$$

If the set I consists of only one element x then the statement of Proposition 2.8 holds. □

Now we consider the correspondence between morphisms and equalities. Here we have two conditions to check in $Hal_\Theta(H)$:

(1) $s_*(w \equiv w') = \big(s(w) \equiv s(w')\big)$,
(2) $s^x_{w*}(A) \cap Val^X_H(w \equiv w') < s^x_{w'*}A$,

where $A \subset Hom(W(X), H)$.

We show that the first condition holds. Let $\mu : W(X) \to H$ be a point in $s_*(w \equiv w')$. We have $\mu s \in Val^X_H(w \equiv w')$, $\mu s(w) = \mu s(w')$, $(sw)^\mu = (sw')^\mu$, $\mu \in Val^X_H(s(w) \equiv s(w'))$.

Similarly we can check that if $\mu \in \big(s(w) \equiv s(w')\big)$ then $\mu \in s_*(w \equiv w')$.

Now we show that the second condition is true. Let

$$\mu \in s^x_{w*}(A) \cap Val^X_H(w \equiv w').$$

Then $\mu s^x_w \in A$ and $w^\mu = (w')^\mu$. From the last condition follows that $\mu s^x_w(x) = \mu s^x_{w'}(x)$ and $\mu s^x_w(y) = \mu s^x_{w'}(y)$ for $y \neq x$. This gives $\mu s^x_w = \mu s^x_{w'}$. Since $\mu s^x_w \in A$ then $\mu s^x_{w'} \in A$ and $\mu \in s^x_{w'*}(A)$.

Thus, the correspondence between morphisms and equalities is verified.

So, each algebra $Hal_\Theta(H)$ satisfies the identities of the variety Hal_Θ.

We finished a survey of the notions of multi-sorted logic needed for UAG and in the next section we will relate these notions with the ideas of one-sorted logic used in Model Theory. Note also that we cannot define algebras of formulas $\Phi(X)$ individually. They are defined only in the multi-sorted case of algebras $\tilde{\Phi} = (\Phi(X), X \in \Gamma)$.

In fact, the definition of the algebra of formulas $\widetilde{\Phi}$ and the system of algebras $\Phi(X)$ is the main result of the first part of the paper. They are essentially used throughout the paper.

3. Logical geometry

3.1. **Introduction.** The setting of logical geometry looks as follows. As before, we fix a variety of algebras Θ. Let $X = \{x_1, \ldots, x_n\}$ be a finite set of variables, $W(X)$ the free in Θ algebra over X, H an algebra in Θ. The set

$$Hom(W(X), H)$$

of all homomorphisms $\mu : W(X) \to H$ is viewed as the affine space of the sort X over H.

Take the algebra of formulas $\Phi(X)$ which was defined in Section 2.8. Consider various subsets T of $\Phi(X)$. We will establish a Galois correspondence between such T and sets of points A in the space $Hom(W(X), H)$. This Galois correspondence gives rise to logical geometry in the given Θ.

The notion of the logical kernel plays a major role in this correspondence. Recall (see Section 2.4), that for every point $\mu : W(X) \to H$ there exists its logical kernel $LKer(\mu)$ which is a Boolean ultrafilter in $\Phi(X)$, containing the elementary theory $Th^X(H)$.

Having in mind the context of the theory of models (see the next section), we view $LKer(\mu)$ as an LG-type (that is, logically-geometric type) of the point μ. Denote $LKer(\mu) = LG^X_H(\mu)$.

Note that the variety Θ is arbitrary and, correspondingly, the system of notions and statements of problems is of a universal character. However, even in the classical situation $\Theta = Com - P$ of the commutative and associative algebras with unit over the field P, many new problems and results appear.

3.2. **Galois correspondence in the Logical Geometry.** Let us start with a particular case when the set of formulas T in $\Phi(X)$ is a set of equations of the form $w = w'$, $w, w' \in W(X)$, $X \in \Gamma$.

We set
$$A = T'_H = \{\mu : W(X) \to H \mid T \subset Ker(\mu)\}.$$
Here A is an algebraic set in $Hom(W(X), H)$, determined by the set T.

Let, further, A be a subset in $Hom(W(X), H)$. We set
$$T = A'_H = \bigcap_{\mu \in A} Ker(\mu).$$
Congruences T of such kind are called H-closed in $W(X)$. We have also Galois-closures T''_H and A''_H.

Let us pass to the general case of logical geometry. Let now T be a set of arbitrary formulas in $\Phi(X)$. We set
$$A = T^L_H = \{\mu : W(X) \to H \mid T \subset LKer(\mu)\}.$$
We have also
$$A = \bigcap_{u \in T} Val^X_H(u).$$
Here A is called a *definable set* in $Hom(W(X), H)$, determined by the set T (cf., Section 3.10). We use the term "definable" for A of such kind, meaning that A is defined by some set of formulas T.

For the set of points A in $Hom(W(X), H)$ we set
$$T = A^L_H = \bigcap_{\mu \in A} LKer(\mu).$$

We have also
$$T = A^L_H = \{u \in \Phi(X) \mid A \subset Val^X_H(u)\}.$$

Here T is a Boolean filter in $\Phi(X)$ determined by the set of points A. Filters of such kind are Galois-closed and we can define the Galois-closures of arbitrary sets T in $\Phi(X)$ and A in $Hom(W(X), H)$ as T^{LL} and A^{LL}.

Proposition 3.1. [34] *Intersection of H-closed filters is also an H-closed filter.*

3.3. **AG-equivalent and LG-equivalent algebras. LG-isotypic algebras.** Let us formulate two key definitions and the corresponding results (see, for example, [29], [32]).

Definition 3.2. Algebras H_1 and H_2 are AG-equivalent, if for every X and every system of equations T holds $T''_{H_1} = T''_{H_2}$.

Definition 3.3. Algebras H_1 and H_2 are LG-equivalent, if for every X and every set of formulas T in $\Phi(X)$ holds $T^{LL}_{H_1} = T^{LL}_{H_2}$.

Let now
$$\Big(\bigwedge_{(w,w') \in T} (w \equiv w')\Big) \to (w_0 \equiv w'_0)$$
be a quasi-identity. We will also write
$$T \to w_0 \equiv w'_0.$$

This quasi-identity can be infinitary if the set T is infinite. Note that $w_0 \equiv w'_0 \in T''_H$ if and only if the quasi-identity $T \to w_0 \equiv w'_0$ holds true in the algebra H.

Algebras H_1 and H_2 in Θ are AG-equivalent, if and only if each quasi-identity $T \to w_0 \equiv w'_0$ which holds true in H_1 is a quasi-identity of the algebra H_2.

In particular, if H_1 and H_2 are AG-equivalent then they generate the same quasi-variety. The inverse statement is not true (see [18]). Recall that quasi-varieties are generated by systems of finitary quasi-identities.

Consider the following formula:

$$(\bigwedge_{u \in T} u) \to v, \;\; v \in \Phi(X)$$

or

$$T \to v.$$

The set T can be infinite and then we speak about infinitary formulas.

Proposition 3.4. *A formula v belongs to T^{LL}_H if and only if the formula $T \to v$ holds true in the algebra H.*

Proof. Take $A = T^L_H$. We have $v \in T^{LL}_H$ if and only if $A \subset Val^X_H(v)$. A point μ belongs to A if and only if μ satisfies every $u \in T$. The formula $T \to v$ holds true in H if and only if for every point μ satisfying all formulas $u \in T$ this point satisfies the formula v, i.e. $\mu \in Val^X_H(v)$. Thus, $A \subset Val^X_H(v)$ whenever $T \to v$ holds in H. □

From this proposition follows:

Proposition 3.5. *Algebras H_1 and H_2 are LG-equivalent if for every $X \in \Gamma$ and $T \subset \Phi(X)$ the formula $T \to v$ holds true in the algebra H_1 if and only if it is true in the algebra H_2.*

Denote by $ImTh(H)$ the implicative theory of the algebra H. Recall that the implicative theory is the set of all formulas of the form $T \to u$, for different $X \in \Gamma$, which hold true in the algebra H. So, algebras H_1 and H_2 are LG-equivalent if their implicative theories coincide, i.e.,

$$ImTh(H_1) = ImTh(H_2).$$

Now we give one more approach to the notion of LG-equivalence. Let T be a set of formulas from $\Phi(X)$ and let $T^\vee$ be the set of all disjunctions of the formulas $u \in T$ and $\widetilde{T}^\vee$ be the set of all disjunctions of the formulas $\neg u$ for $u \in T$. Here we have the following properties

$$\neg(\bigwedge_{u \in T} u) = \widetilde{T}^\vee; \;\; \neg(\bigwedge_{u \in T} \neg u) = T^\vee.$$

We want to consider the disjunctive theory of the algebra H. The disjunctive theory of the algebra H is the set of all possible formulas $T^\vee$, for all $T \subset \Phi(X)$ and different $X \in \Gamma$, which hold true in the algebra H.

Note that a formula $T \to v$ holds true in the algebra H if and only if the formula $\widetilde{T}^{\vee} \vee v$ is true in H. Thus if the disjunctive theories of two algebras H_1 and H_2 coincide then these algebras are LG-equivalent. Moreover, there is the following

Proposition 3.6. *Algebras H_1 and H_2 are LG-equivalent if and only if their disjunctive theories coincide.*

Proof. Let algebras H_1 and H_2 be LG-equivalent. We take a set of formulas $T \subset \Phi(X)$ and consider the formula $\widetilde{T}^{\vee} \vee v$, where v is the formula $(x \equiv y) \wedge (x \not\equiv y)$. There is no point $\mu : W(X) \to H$ satisfying the formula v. So μ satisfies the formula $\widetilde{T}^{\vee} \vee v$ if and only if μ satisfies the formula $\widetilde{T}^{\vee}$. It means that there is $u \in T$ such that the point μ does not satisfy the formula u and so this point satisfies the formula $T \to v$.

Now let $H = H_1$ and let $T^{\vee}$ be a formula which hold true in the algebra H_1. An arbitrary point $\mu_1 : W(X) \to H_1$ satisfies $T^{\vee}$ and $\widetilde{T}^{\vee} \to v$. Since the algebras H_1 and H_2 are LG-equivalent, then every point $\mu_2 : W(X) \to H_2$ satisfies the formula $\widetilde{T}^{\vee} \to v$ and, hence, it satisfies the formula $T^{\vee} \vee v$. So, each point $\mu_2 : W(X) \to H_2$ satisfies the formula $T^{\vee}$. Thus, the disjunctive theories of H_1 and H_2 coincide. ☐

Note that

Proposition 3.7. *If algebras H_1 and H_2 are LG-equivalent then they are elementarily equivalent.*

Proof. Let us consider the formula $u \to v$, where u is the formula $x \equiv x$. This formula holds true in the algebra H if and only if the formula v is true in H, i.e., $v \in Th(H)$. If algebras H_1 and H_2 are LG-equivalent then the formula $u \to v$ holds in H_1 if and only if it is true in H_2. Thus $v \in Th(H_1)$ if and only if $v \in Th(H_2)$, that is, $Th(H_1) = Th(H_2)$. ☐

Definition 3.8. Two algebras H_1 and H_2 are called LG-isotypic (cf. Section 4.4) if for every point $\mu : W(X) \to H_1$ there exists a point $\nu : W(X) \to H_2$ such that $LKer(\mu) = LKer(\nu)$ and, conversely, for every point $\nu : W(X) \to H_2$ there exists a point $\mu : W(X) \to H_1$ such that $LKer(\nu) = LKer(\mu)$.

The main theorem is the following [48]

Theorem 3.9. *Algebras H_1 and H_2 are LG-equivalent if and only if they are LG-isotypic.*

Proof. Let H_1 and H_2 be LG-equivalent algebras. By definition for any finite set X and any H_1-closed filter T from $\Phi(X)$ we have:

$$T = T^{LL}_{H_1} = T^{LL}_{H_2}.$$

So, T is H_1-closed if and only if it is H_2-closed.

Let $T = LKer(\mu)$ be the logical kernel of a point $\mu : W(X) \to H_1$. Then $T^{L}_{H_1} = A$, where $A = \{\mu\}$ and $T^{LL}_{H_1} = A^{L}_{H_1} = LKer(\mu) = T$. So, T is an

H_1-closed filter. Hence, T is an H_2-closed filter. Since $T = LKer(\mu)$, the filter T is maximal. Since H_1 and H_2 are LG-equivalent, there exists a set B in $Hom(W(X), H_2)$ such that $B^L_{H_2} = T$. Then $T = \bigcap_{\nu \in B} LKer(\nu)$. Since the filter T is maximal, $LKer(\nu) = T = LKer(\mu)$ for all points $\nu \in B$.

Note that we used the fact that $T^L_{H_2}$ is not empty. Indeed, if we assume that $T^L_{H_2} = \{\emptyset\}$ then $T^{LL}_{H_2} = \{\emptyset\}^L_{H_2} = \Phi(X) = T^{LL}_{H_1} = T$, but T is a proper filter.

In the similar way one can prove that if $T = LKer(\mu)$ is the logical kernel of a point $\nu : W(X) \to H_2$, then there exists a point $\mu : W(X) \to H_1$ such that $LKer(\nu) = LKer(\mu)$. Hence, H_1 and H_2 are isotypic.

Let, further, H_1 and H_2 be isotypic algebras. This means that if $T = LKer(\mu)$ is the logical kernel of a point $\mu : W(X) \to H_1$, then $T = LKer(\nu)$ is the logical kernel for some $\nu : W(X) \to H_2$ as well, and vice versa. Recall, that every logical kernel is a closed filter, so T is H_1- and H_2-closed filter.

Let, now, T be an arbitrary H_1-closed filter in $\Phi(X)$. We will show that T is H_2-closed.

Let $T^L_{H_1} = A$, then $T = T^{LL}_{H_1} = A^L_{H_1} = \bigcap_{\mu \in A} LKer(\mu)$. Since H_1 and H_2 are isotypic, there exist points $\nu : W(X) \to H_2$ such that

$$\bigcap_{\nu \in Hom(W(X),H)} LKer(\nu) = \bigcap_{\mu \in A} LKer(\mu).$$

According to Proposition 3.1, the intersection of H-closed filters is also an H-closed filter, hence T is an H_2-closed filter.

Similarly, we can prove that each H_2-closed filter is H_1-closed. Hence, H_1 and H_2 are LG-equivalent. ☐

From this theorem follows

Corollary 3.10. *If the algebras H_1 and H_2 are isotypic, then they are elementarily equivalent.*

Proof. Take a formula $x = x \to u$, where $u \in \Phi(X)$. This formula holds in H_1 if and only if u holds in H_1. Since H_1 and H_2 are isotypic, then (Proposition 3.5) $x = x \to u$ holds in H_1 if and only if it holds in H_2. So if u belongs to the elementary theory of H_1, then it belongs to the elementary theory of H_2 and vice versa.

☐

3.4. **Categories of algebraic and definable sets over a given algebra H.** Recall that we introduced (Section 2.6) the category of affine spaces $\Theta^*(H)$. It is natural to assume that $Var(H) = \Theta$. If this condition does not hold, the situation when for two different morphisms $s_1 : W(Y) \to W(X)$ and $s_2 : W(Y) \to W(X)$ the corresponding morphisms $\widetilde{s}_1$ and $\widetilde{s}_2$ in $\Theta^*(H)$ coincide, is possible. This breaks duality between Θ^0 and Θ^* (Proposition 2.2) and, as we will see, leads to a lot of other disadvantages. The condition $Var(H) = \Theta$ plays also a crucial role in the problem of sameness of geometries over different algebras.

Define now a category of algebraic sets $AG_\Theta(H)$ and a category of definable sets $LG_\Theta(H)$.

Define first a category $Set_\Theta(H)$. Its objects are pairs (X, A) with A being a subset in $Hom(W(X), H)$ and $X \in \Gamma$.

Given $s : W(Y) \to W(X)$, a morphism s_* takes (X, A) to (Y, B), where B contains the points $\nu : W(Y) \to H$ such that $\nu = \mu s$ for $\mu \in A$.

Now, $AG_\Theta(H)$ is a full subcategory in $Set_\Theta(H)$, whose objects are pairs (X, A), where A is an algebraic set.

If for A we take definable sets, then we have the category $LG_\Theta(H)$ which is a full subcategory in $Set_\Theta(H)$.

Two key results are as follows (see, for example, [29], [32]).

Theorem 3.11. *If H_1 and H_2 are AG-equivalent, then categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ are isomorphic.*

Theorem 3.12. *If H_1 and H_2 are LG-equivalent, then categories $LG_\Theta(H_1)$ and $LG_\Theta(H_2)$ are isomorphic.*

Remark 3.13. In view of Theorem 3.8, the geometric notion of LG-equivalent algebras coincides with a model theoretic notion of isotypic algebras. Thus, if algebras H_1 and H_2 are isotypic, then the categories of definable sets $LG_\Theta(H_1)$ and $LG_\Theta(H_2)$ are isomorphic for every Θ.

Theorems 3.11 and 3.12 provide sufficient conditions for isomorphisms of categories of algebraic and definable sets, respectively. Other necessary and sufficient conditions will be treated in the sequel.

Beforehand, we shall slightly modify the categories $AG_\Theta(H)$ and $LG_\Theta(H)$. First of all, modify the definition of the category $AG_\Theta(H)$. Objects $AG^X_\Theta(H)$ of $AG_\Theta(H)$ are not pairs (X, A), where A is an algebraic set, but systems of all algebraic sets in the space $Hom(W(X), H)$, where X is fixed. Analogously, an object $LG^X_\Theta(H)$ is the system of all definable sets in the space $Hom(W(X), H)$.

Note that all definable sets under the given X constitute a lattice, while all algebraic sets are just a poset. So, one can say that objects $AG^X_\Theta(H)$ of $AG_\Theta(H)$ are posets of algebraic sets in $Hom(W(X), H)$, while objects $LG^X_\Theta(H)$ of $LG_\Theta(H)$ are lattices of definable sets in $Hom(W(X), H)$. By definition, every algebraic set is a definable set.

Morphisms between $AG^X_\Theta(H)$ and $AG^Y_\Theta(H)$, as well as between $LG^X_\Theta(H)$ and $LG^Y_\Theta(H)$, are defined in terms of the maps $s : W(Y) \to W(X)$. We will describe these morphisms in more detail.

First of all, recall that objects in the categories Θ^0 and $\widetilde{\Phi}_\Theta$ are free algebras $W(X)$ and algebras of formulas $\Phi(X)$, respectively. Every homomorphism $s : W(Y) \to W(X)$ gives rise to a morphism $s_* : \Phi(Y) \to \Phi(X)$. In particular, s_* acts on equalities as follows: $s_*(w_1 \equiv w_2) = (s(w_1) \equiv s(w_2))$ (action of s_* is regulated by Definition 2.3). Note that equalities of the form $w \equiv w'$, w, w' in $W(X)$, can be treated as formulas in $\Phi(X)$. This correspondence $s \mapsto s_*$ allows us to define morphisms $\widetilde{s}$ and $\widetilde{s}_*$ in $AG_\Theta(H)$ and $LG_\Theta(H)$.

Given $s : W(Y) \to W(X)$, a morphism $\widetilde{s} : AG_\Theta^X(H) \to AG_\Theta^Y(H)$ is defined as follows. For an algebraic set A in $AG_\Theta^X(H)$ take all points ν in $Hom(W(Y), H)$ of the form $\nu = \mu s$, where $\mu \in A$. Define $B = \widetilde{s}A$ as the algebraic set determined by the set of all such ν. Then the object $AG_\Theta^Y(H)$ corresponding to $AG_\Theta^X(H)$ contains all B of such kind. So, morphisms in $AG_\Theta(H)$ are maps of posets, originated from homomorphisms of free algebras, that is maps of the form $\widetilde{s}$. Note, that all $\widetilde{s}$ preserve poset structure by definition.

Analogously, a morphism $\widetilde{s}_* : LG_\Theta^X(H) \to LG_\Theta^Y(H)$ is defined as follows: given $A \in LG_\Theta^X(H)$ and $s : W(Y) \to W(X)$, the set $B = \widetilde{s}_*A$ is the definable set determined by all points ν of the form $\nu = \mu s$, $\mu \in A$. The object $LG_\Theta^Y(H)$ corresponding to $LG_\Theta^X(H)$ contains all B of such form.

Now we define categories of algebras of formulas $C_\Theta(H)$ and $F_\Theta(H)$. Let us start with $C_\Theta(H)$. If $A \in AG_\Theta^X(H)$, then take $T = A'_H$. This is an H-closed congruence on $W(X)$, that is, $T'_H = A$. Denote by $C_\Theta^X(H)$ the poset of all such T, where A runs through $AG_\Theta^X(H)$. These $C_\Theta^X(H)$ are objects of $C_\Theta(H)$. They are in one-to-one correspondence with objects $AG_\Theta^X(H)$.

Let us describe morphisms of $C_\Theta(H)$. Let $s : W(Y) \to W(X)$ be a morphism in Θ^0. Recall that $s_*(w_1 \equiv w_2) = (s(w_1) \equiv s(w_2))$. Let T_2 be an H-closed congruence in $C_\Theta^Y(H)$. Define T_1 as the H-closed congruence in $C_\Theta^X(H)$ determined by the set of all equalities of the form $s_*(w \equiv w')$, where $w \equiv w'$ in T_2. So $T_1 = (s_*T_2)''$.

Consider the commutative diagram

$$\begin{tikzcd} T_2 \arrow[r, "s_*"] \arrow[d, "Val_H^Y"'] & T_1 \arrow[d, "Val_H^X"] \\ B & A, \arrow[l, "\widetilde{s}"'] \end{tikzcd} \qquad (\Diamond)$$

where $A'_H = T_1$, $T'_{1H} = A$, $B'_H = T_2$, $T'_{2H} = B$ (follows from Section 2.9). Here T_2 and T_1 are H-closed congruences in $W(Y)$ and $W(X)$, respectively. In particular, $(\Diamond)$ implies that $s_* : C_\Theta^Y(H) \to C_\Theta^X(H)$ is a map of posets.

This diagram gives rise to the category $C_\Theta(H)$ of all H-closed congruences.

It is important to get another look at the morphisms s_* in $C_\Theta(H)$. Let H-closed congruences T_2 in $C_\Theta^Y(H)$ and T_1 in $C_\Theta^X(H)$ be given. The morphism s_* takes T_2 to T_1 if and only if s_* satisfies the diagram $(\Diamond)$. So, s_* assigns T_1 to T_2 if and only if we have $(\Diamond)$. Moreover, if one knows s_* and T_1, then $(\Diamond)$ recovers T_2.

Proposition 3.14. *Let $Var(H) = \Theta$. The category $C_\Theta(H)$ of posets of H-closed congruences is anti-isomorphic to the category $AG_\Theta(H)$ of posets of algebraic sets.*

Proof. The correspondence $C_\Theta(H) \to AG_\Theta(H)$ is one-to-one. The condition $Var(H) = \Theta$ provides that the correspondence $s_* \to \widetilde{s}$ is also one-to-one (see Proposition 2.2). $\square$

The dual category $C^{-1}(H)$ is isomorphic to $AG_\Theta(H)$.

We shall repeat the similar construction using L-Galois correspondence. We have the diagram $(\Diamond\Diamond)$ (whose particular case is the diagram $(\Diamond)$) :

$$\begin{tikzcd} T_2 \arrow[r, "s_*"] \arrow[d, "Val_H^Y"'] & T_1 \arrow[d, "Val_H^X"] \\ B & A \arrow[l, "\widetilde{s}_*"'] \end{tikzcd} \qquad (\Diamond\Diamond)$$

where $A_H^L = T_1$, $T_{1H}^L = A$, $B_H^L = T_2$, $T_{2H}^L = B$. Here T_2 and T_1 are H-closed filters in $\Phi(X)$ and $\Phi(Y)$, respectively. It gives rise to the categories of H-closed filters $F_\Theta(H)$ and $F_\Theta^{-1}(H)$. Objects of $F_\Theta(H)$ are lattices of H-closed filters $F_\Theta^X(H)$. Let F_2 be an H-closed filter in $F_\Theta^Y(H)$. Define F_1 as the H-closed filter determined by the set of formulas of the form s_*v, where v in T_2. So, $F_1 = (s_*F_2)^{LL}$.

In other words, let H-closed filters T_2 and T_1 in $F_\Theta^Y(H)$ and $F_\Theta^X(H)$, respectively, be given. Take $T_{1H}^L = A$ and $T_{2H}^L = B$. The diagram $(\Diamond\Diamond)$ determines when s_* takes T_2 to T_1. In particular, T_1 defines uniquely T_2 by $T_2 = s_*^{-1}(T_1)$, that is, T_2 is the inverse image of T_1.

Proposition 3.15. *Let $Var(H) = \Theta$. The category $F_\Theta(H)$ of lattices of H-closed filters is anti-isomorphic to the category $LG_\Theta(H)$ of lattices of definable sets.*

The dual category $F_\Theta^{-1}(H)$ is isomorphic to the category of definable sets $LG_\Theta(H)$.

3.5. **Geometric and logical similarity of algebras.**

Definition 3.16. We call algebras H_1 and H_2 *geometrically similar* if the categories of algebraic sets $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ are isomorphic.

Since the categories $AG_\Theta(H)$ and $C_\Theta(H)$ are dual, algebras H_1 and H_2 are geometrically similar if and only if the categories $C_\Theta(H_1)$ and $C_\Theta(H_2)$ are isomorphic. In view of Theorem 3.2, if algebras H_1 and H_2 are geometrically equivalent, then they are geometrically similar.

Definition 3.17. We call algebras H_1 and H_2 *logically similar*, if the categories of definable sets $LG_\Theta(H_1)$ and $LG_\Theta(H_2)$ are isomorphic.

Algebras H_1 and H_2 are logically similar if and only if the categories $F_\Theta(H_1)$ and $F_\Theta(H_2)$ are isomorphic.

By Theorem 3.3 if H_1 and H_2 are logically equivalent, then they are logically similar.

The following problems are our main target:

Problem 1. Find necessary and sufficient conditions on algebras H_1 and H_2 in Θ that provide algebraic similarity of these algebras.

Problem 2. Find necessary and sufficient conditions on algebras H_1 and H_2 in Θ that provide logical similarity of these algebras.

We start with examples of specific varieties, where necessary and sufficient conditions for isomorphism of the categories of algebraic sets can be formulated solely in terms of properties of algebras H_1 and H_2. Afterwards we will dwell on a general approach. In what follows, all fields and rings are assumed to be infinite.

Theorem 3.18. *Let $Var(H_1) = Var(H_2) = \Theta$.*

(1) *Let Θ be one of the following varieties*
 - *$\Theta = Grp$, the variety of groups,*
 - *$\Theta = Jord$, the variety of Jordan algebras,*
 - *$\Theta = Inv$, the variety of inverse semigroups,*
 - *$\Theta = \mathfrak{N}_d$, the variety of nilpotent groups of class d.*

 Categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ are isomorphic if and only if the algebras H_1 and H_2 are geometrically equivalent (see [7]*)* [17], [45], [43]*).*

(2) *Let $\Theta = Com-P$ or $Lie-P$ and $\sigma \in Aut(P)$. Define a new algebra H^σ. The multiplication $\circ$ on a scalar in H^σ is defined through the multiplication in H by the rule:*

$$\lambda \circ a = \lambda^\sigma \cdot a, \quad \lambda \in P, \quad a \in H.$$

 Categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ are isomorphic if and only if the algebras H_1^σ and H_2 are geometrically equivalent for some $\sigma \in Aut(P)$ (see [3], [29], [14], [15], [11], [39]*).*

(3) *Let $\Theta = Ass - P$. Denote by H^* the algebra with the multiplication $*$ defined as follows: $a * b = b \cdot a$. The algebra H^* is called opposite to H. The categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ are isomorphic if and only if for some $\sigma \in Aut(P)$ the algebras $(H_1^*)^\sigma$ and H_2 are geometrically equivalent, where $(H_1^*)^\sigma$ is opposite to either H_1 or to H_1^* (* [1], [2], [29]*).*

Remark 3.19. The list of varieties of Theorem 3.18 is not complete. Similar results are known for the varieties of semigroups [16], linear algebras [44], [39], power associative algebras, alternative algebras [45], non-commutative non-associative algebras, commutative non-associative algebras, color Lie superalgebras, Lie p-algebras , color Lie p-superalgebras, Poisson algebras [39], free R-modules [10], Nielsen-Schreier varieties [39], and for the varieties of some classes of representations [37], [47], [46].

3.6. **Similarity of algebras and isomorphism of functors.** We will make some preparations, basing on the idea of isomorphism of functors.

Definition 3.20. Let φ_1, φ_2 be two functors from category C_1 to a concrete category C_2. We say that an isomorphism of functors $S : \varphi_1 \to \varphi_2$ is defined if for any morphism $\nu : A \to B$ in C_1 the following commutative diagram takes place

$$\begin{tikzcd}
\varphi_1(A) \arrow[r, "S_A"] \arrow[d, "\varphi_1(\nu)"'] & \varphi_2(A) \arrow[d, "\varphi_2(\nu)"] \\
\varphi_1(B) \arrow[r, "S_B"] & \varphi_2(B).
\end{tikzcd}$$

Here S_A is the A-component of S, that is, a function which makes a bijective correspondence between $\varphi_1(A)$ and $\varphi_2(A)$. The same is valid for S_B.

Note that S_A and S_B are not necessarily morphisms in C_2. Thus, this definition is different from the standard one, where all S_A have to be morphisms in C_2. The commutative diagram above can be reformulated as

$$\varphi_1(\nu) = S_B^{-1}\varphi_2(\nu)S_A, \quad \varphi_2(\nu) = S_B\varphi_1(\nu)S_A^{-1}.$$

An invertible functor from a category to itself is an automorphism of the category. The notion of isomorphism of functors gives rise to the notion of an inner automorphism of a category. An automorphism φ of the category C is called inner (see [29]) if φ is isomorphic to the identity functor 1_C. This provides the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "s_A"] \arrow[d, "\nu"'] & \varphi(A) \arrow[d, "\varphi(\nu)"] \\
B \arrow[r, "s_B"] & \varphi(B),
\end{tikzcd}$$

that is, $\varphi(\nu) = s_B\nu s_A^{-1}$.

Following Proposition is the main tool in the proof of Theorem 3.18:

Proposition 3.21 ([26]). *If for the variety Θ every automorphism of the category Θ^0 is inner, then two algebras H_1 and H_2 are geometrically similar if and only if they are geometrically equivalent.*

So, studying automorphisms of Θ^0 plays a crucial role in Problem 1. The latter problem is treated by means of Reduction Theorem (see [29], [10], [15], [36]). This theorem reduces investigation of automorphisms of the whole category Θ^0 of free in Θ algebras to studying the group $Aut(End(W(X))$ associated with a single object $W(X)$ in Θ^0. Here, $W(X)$ is a finitely generated free in Θ hopfian algebra, which generates the whole variety Θ. In fact, if all automorphisms of the endomorphism semigroup of a free algebra $W(X)$ are close to being inner, then all automorphisms of Θ^0 possess the same property. More precisely, denote by $Inn(End(W(X))$ the group of inner automorphisms of $Aut(End(W(X))$. Then the group of outer automorphisms $Aut(End(W(X))/Inn(End(W(X))$ measures, in some sense, the difference between the notions of geometric similarity and geometric equivalence.

Now we will treat the general problem using the Galois-closure functors. For every algebra $H \in \Theta$ consider two functors

$$Cl_H^A : \Theta^0 \to PoSet,$$

$$Cl_H^L : \widetilde{\Phi}_\Theta \to Lat,$$

where A and L stand for the functors of algebraic and logical closures, respectively. We will suppress these indices in the sequel, assuming that the type of Cl-functor is clear in each particular case.

In fact, $PoSet$ is the category $C_\Theta(H)$ of partially ordered sets of H-closed congruences $C_\Theta^X(H)$, while Lat is the category $F_\Theta(H)$ of lattices of H-closed filters $F_\Theta^X(H)$.

So, Cl_H assigns the poset $C_\Theta^X(H)$ of all H-closed congruences on $W(X)$ to every object $W(X)$ in Θ^0. If $s : W(Y) \to W(X)$ is a morphism in Θ^0, then $Cl_H(s) = s_* : C_\Theta^Y(H) \to C_\Theta^X(H)$ is a morphism in $C_\Theta(H)$.

Analogously, in case of $\widetilde{\Phi}_\Theta \to Lat$, every $s : W(Y) \to W(X)$ gives rise to

$$s_* : \Phi(Y) \to \Phi(X),$$

and for $T_2 \subset \Phi(Y)$, $T_1 \subset \Phi(X)$ define $s_* : T_2 \to T_1$ by taking all $v \in T_2$ such that $s_* v = u \in T_1$. Using ($\Diamond\Diamond$) we extend s_* to

$$s_* : Cl_H(T_2) \to Cl_H(T_1).$$

The correspondence $s \to s_*$ gives rise to contravariant Cl_H-functors $\Theta^0 \to PoSet$ and $\widetilde{\Phi}_\Theta \to F_\Theta(H)$.

Definition 3.22. Algebras H_1 and H_2 are called *weakly geometrically equivalent* if the geometric functors Cl_{H_1} and Cl_{H_2} are isomorphic.

Definition 3.23. Algebras H_1 and H_2 are called *weakly logically equivalent* if the logical functors Cl_{H_1} and Cl_{H_2} are isomorphic.

It is clear that if algebras H_1 and H_2 are geometrically (logically) equivalent, then they are weakly geometrically (logically) equivalent.

3.7. **Automorphic equivalence of algebras.** Apply these notions to Problem 1 and Problem 2. Consider a commutative diagram

$$\begin{tikzcd} \Theta^0 \arrow[rr, "\varphi"] \arrow[dr, "Cl_{H_1}"'] & & \Theta^0 \arrow[dl, "Cl_{H_2}"] \\ & PoSet_\Theta, & \end{tikzcd}$$

where φ is an automorphism of Θ^0. Commutativity of these diagrams means that there exists an isomorphism of functors

$$\alpha(\varphi) : Cl_{H_1} \to Cl_{H_2} \cdot \varphi.$$

In its turn, this isomorphism of functors means that the diagram

$$
\begin{tikzcd}
Cl_{H_1}(W(Y)) \arrow[r, "\alpha(\varphi)_{W(Y)}"] \arrow[d, "Cl_{H_1}(s)"'] & Cl_{H_2}(\varphi(W(Y))) \arrow[d, "Cl_{H_2}(\varphi(s))"] \\
Cl_{H_1}(W(X)) \arrow[r, "\alpha(\varphi)_{W(X)}"] & Cl_{H_2}(\varphi(W(X))),
\end{tikzcd}
$$

is commutative.

Definition 3.24. Algebras H_1 and H_2 are called *geometrically automorphically equivalent* if for some automorphism $\varphi : \Theta^0 \to \Theta^0$ the geometric functors Cl_{H_1} and $Cl_{H_2}\varphi$ are isomorphic by an isomorphism $\alpha(\varphi)$.

If the type of Cl-functors is specified we speak merely of automorphically equivalent algebras. Note that Definition 3.24 of automorphic equivalence is different from the one, previously used in the literature (see, for example [42]-[46]).

Our next aim is to get a special presentation of $\alpha(\varphi)$. We start from the semigroup of endomorphisms $End(W(Y))$, where $W = W(Y)$ is an object of the category Θ^0.

Assume that a binary relation ρ is defined on $End(W(Y))$. Given ρ, define an H-closed congruence $T = \tau(\rho)$ on $W(Y)$.

Let $\nu\rho\nu'$, where ν, ν' belong to $End(W(Y))$. Given $w \in W(Y)$, take the elements $w^\nu = w_1$ and $w^{\nu'} = w_2$. Consider the system of equations $w_1 = w_2$, assuming that w runs through $W(Y)$ and (ν, ν') runs through ρ. Denote by $T = \tau(\rho)$ the H-closed congruence on $W(Y)$ defined by the system of equations $w_1 = w_2$.

Define μ_T to be the homomorphism $\mu_T : W(Y) \to W(Y)/T$. Suppose that an H-closed congruence T on $W(Y)$ is given. Define $\rho = \rho(T)$ by $\nu\rho\nu'$ if and only if $\mu_T\nu = \mu_T\nu'$. So we have the correspondences $\rho \mapsto T = \tau(\rho)$ and $T \to \rho = \rho(T)$. One can check that if $\tau(\rho) = T$, then $\rho(T) = \rho$, that is, $\tau(\rho(T)) = T$ and, correspondingly, $\rho(\tau(\rho)) = \rho$.

Define the relation $\rho^* = \varphi(\rho)$ on $End(\varphi(W))$ by the rule: $\mu\varphi(\rho)\mu'$ where $\mu, \mu' \in End(\varphi(W))$, if there exist ν and $\nu' \in End(W)$ with $\varphi(\nu) = \mu, \varphi(\nu') = \mu'$ and $\nu\rho\nu'$. For the sake of simplicity we assume here that the cardinalities of X and $\varphi(Y)$ coincide. So, $\rho^* = \varphi(\rho)$ on $End(\varphi(W))$ is determined by ρ and φ. More precisely, if $T^* \in Cl_{H_2}(\varphi(W))$, then $\rho^*(T^*) = \varphi(\rho)(T^*) = \varphi(\rho(T))$.

In this setting the the isomorphism $\alpha(\varphi)$ is defined by the rule:

$$\alpha(\varphi)(T) = \tau_{\varphi(W)}(\varphi(\rho(T))),$$

where $T \in Cl_{H_1}(W)$, i.e., T is a H_1-closed congruence on W. Indeed, for $T \in Cl_{H_1}(W)$ we have $\alpha(\varphi)(T) = T^*$, where $T^* \in Cl_{H_2}(\varphi(W))$. Represent T^* as $T^* = \tau^*_{\varphi(W)}(\rho^*(T^*))$. Using $\rho^*(T^*) = \varphi(\rho(T))$, we get $T^* = \tau^*_{\varphi(W)}(\varphi(\rho(T))$. Hence, $\alpha(\varphi)_W(T) = \tau_{\varphi(W)}(\varphi(\rho_W T)))$.

We omit the proof of the following theorem.

Theorem 3.25. *Let $Var(H_1) = Var(H_2) = \Theta$. Suppose that the algebras H_1 and H_2 are geometrically automorphically equivalent. Then the algebras H_1 and H_2 are geometrically similar.*

Moreover, there is an order preserving isomorphism of the categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$.

In the particular case $\varphi = id_{\Theta^0}$ we come out with the isomorphism of Cl_{H_1} and Cl_{H_2} which means that the algebras H_1 and H_2 are weakly geometrically equivalent.

3.8. **Logically automorphically equivalent algebras.** Let us start from the following triangular diagram:

$$\begin{tikzcd} \tilde{\Phi}_\Theta \arrow[rr, "\varphi"] \arrow[dr, "Cl_{H_1}"'] & & \tilde{\Phi}_\Theta \arrow[dl, "Cl_{H_2}"] \\ & Lat_\Theta & \end{tikzcd}$$

Commutativity of this diagram means that there is an isomorphism of functors

$$\alpha\varphi : Cl_{H_1} \to Cl_{H_2}\varphi.$$

Let us represent this isomorphism of functors as a commutative diagram

$$\begin{tikzcd} Cl_{H_1}(\Phi(Y)) \arrow[r, "(\alpha\varphi)_{\Phi(Y)}"] \arrow[d, "Cl_{H_1}(s_*)"'] & Cl_{H_2}\varphi(\Phi(Y)) \arrow[d, "Cl_{H_2}\varphi(s_*)"] \\ Cl_{H_1}(\Phi(X)) \arrow[r, "(\alpha\varphi)_{\Phi(X)}"] & Cl_{H_2}\varphi(\Phi(X)). \end{tikzcd}$$

In both upper and lower rows we have many different mappings of sets. Vertical mappings are defined uniquely. They are determined by the homomorphism $s : W(Y) \to W(X)$ which implies $s_* : \Phi(Y) \to \Phi(X)$. In the sequel we will choose unique mappings for the upper and lower horizontal rows. Let us do it for the upper row.

Take the semigroup $End(\Phi(Y))$ of endomorphisms of the algebra of formulas $\Phi(Y)$. Let a binary relation ρ be defined on $End(\Phi(Y))$. Given ρ, define an H-closed filter $T = \tau(\rho)$ on $F(Y)$.

For a given ρ take the elements $(u^\nu \to u^{\nu'}) \wedge (u^{\nu'} \to u^\nu)$ for any $u \in \Phi(Y)$ and all $\nu\rho\nu'$. Generate an H-closed filter T by all elements of such kind. Denote $T = \tau_\Phi(\rho)$. So, $\rho \mapsto T = \tau(\rho)$.

Conversely, let an H-closed Boolean filter $T \in \Phi(Y)$ be given. Consider the homomorphism of Boolean algebras

$$\mu_T : \Phi(Y) \to \Phi(Y)/T.$$

Take two elements ν and ν' in $End(\Phi(Y))$. We set: $\nu\rho\nu'$ if and only if $\mu_T\nu = \mu_T\nu'$. This means that u^ν and $u^{\nu'}$ are the same in $\Phi(X)/T$ for any $u \in \Phi(X)$. In other words, $(u^\nu \to u^{\nu'}) \wedge (u^{\nu'} \to u^\nu) \in T$ for any $u \in \Phi(X)$. Thus, $T \mapsto \rho(T) = \rho$.

We have

$$\tau_\Phi(\rho(T)) = T; \qquad \rho(\tau_\Phi(\rho)) = \rho.$$

We considered the relation ρ for the algebra $\Phi(Y)$. We now intend to study the relation $\varphi(\rho)$ for the algebra $\varphi(\Phi(Y))$. The relation $\varphi(\rho)$ is defined in a standard way. Let μ and μ' be endomorphisms of the algebra $\varphi(\Phi(Y))$. We set: $\mu\varphi(\rho)\mu'$ holds if and only if $\nu\rho\nu'$ holds for $\varphi(\nu) = \mu$ and $\varphi(\nu') = \mu'$. Let us apply the latter to the diagram defining isomorphism of functors Cl_{H_1} and $Cl_{H_2}\varphi$. Take $T \in Cl_{H_1}(\Phi(Y))$ and denote $(\alpha\varphi)_{\Phi((Y)}(T)$ by T^*. This T^* lies in $Cl_{H_2}\varphi(\Phi(Y))$. Here $T = \tau_{\Phi(Y)}(\rho(T))$. Correspondingly, $T^* = \tau_{\varphi(\Phi(Y))}(\varphi(\rho(T)))$. Hence, T^* is uniquely determined by the filter T.

We apply the passage from T to T^* to the upper and lower horizontal rows of the diagram.

Definition 3.26. Two algebras H_1 and H_2 of the variety Θ are called *logically automorphically equivalent* if for some automorphism φ of the category $\widetilde{\Phi}_\Theta$ there is an isomorphism of functors $\alpha\varphi : Cl_{H_1} \to Cl_{H_2}\varphi$.

The following theorem holds true:

Theorem 3.27. *If the algebras H_1 and H_2 of the variety Θ are logically automorphically equivalent, then they are logically similar.*

Moreover, there exists an isomorphism of the categories $LG_\Theta(H_1)$ and $LG_\Theta(H_2)$ which preserves the order relation and correlates with the lattices of definable sets.

In the particular case $\varphi = id_{\Theta^0}$ the algebras H_1 and H_2 are weakly logically equivalent.

Remark 3.28. We considered a special transition from the filter T to another filter T^*, based on the relation ρ on the set $End(\Phi(X))$, and we wrote $(\alpha\varphi)_{\Phi(X)}(T) = T^*$. Other transitions are possible as well.

Let us sketch one of the possible transitions from T to T^*. Consider a constraint for affine spaces $Hom(W(X), H)$. The algebra $W(X)$ cannot be represented as a subalgebra in the algebra H. This means, that for any point $\mu : W(X) \to H$ there is a nontrivial kernel $Ker(\mu)$. The point μ satisfies the equality $w \equiv w',\ w, w' \in W(X)$. Then we have $w \equiv w' \in LKer(\mu)$. This implies $Cl_H(w \equiv w') = (w \equiv w')^{LL}_H \subset LKer(\mu)$. Denote $T = (w \equiv w')^{LL}_H$. Since T is a filter, then $\nu\rho\nu'$ implies $(u^\nu \to u^{\nu'}) \wedge (u^{\nu'} \to u^\nu) \in T$ for any $u \in \Phi(X)$ and the given ρ. The initial relation ρ determines the filter T and the equality $w \equiv w'$ determines the same T. This hints to correlate the transitions from T to T^* with equalities in the situation of special affine spaces. Besides, we keep in mind that equalities generate the algebra $\widetilde{\Phi}_\Theta$.

Now we shall formulate several problems related to logical geometry. Some of them are relevant also for the $AG-$case. Let us start with the variety $\Theta = Grp$.

Problem 3. It is known [38], [48], that any group H which is LG-equivalent to a free group $W(X)$, is isomorphic to it. What is the situation, if H is logically automorphically equivalent to $W(X)$?

Problem 4. What can be said about a group H which is logically similar to a free group $W(X)$?

Problem 5. If two groups are LG-equivalent, then they are isotypic and, hence, elementary equivalent. What is the relation between the elementary equivalence of groups and their logical similarity?

Problem 6. Are there logically similar groups H_1 and H_2, such that the functors Cl_{H_1} and $Cl_{H_2}\varphi$ are not isomorphic for any automorphism φ?

Similar questions makes sense for algebras

Problem 7. Whether it is true that if the algebras H_1 and H_2 of the variety Θ are logically similar, then for some automorphism φ they are logically automorphically equivalent.

Problem 8. Propositions 3.5 and 3.6 provide implicative and disjunctive criteria for algebras to be logically equivalent. Find criteria which provide automorphical equivalence of algebras.

As it was said above, the group of automorphisms of the category Θ^0 plays an exceptional role in problems related to geometrical similarity. The following problems are directed to find out what is the situation in the case of logical geometry.

Problem 9. Study the group of automorphisms of the category $\widetilde{\Phi}_\Theta$.

Problem 10. Study the group of automorphisms $Aut(End(\Phi(X)))$.

3.9. **Logically perfect and logically regular varieties.** Up to now we assumed that the variety Θ is arbitrary. Further on we distinguish classes of varieties which are characterized by specific logical properties.

Let H be an algebra in Θ.

Definition 3.29. Algebra H is called *logically homogeneous* if for every two points $\mu : W(X) \to H$ and $\nu : W(X) \to H$ the equality $LKer(\mu) = LKer(\nu)$ holds if and only if there exists an automorphism σ of the algebra H such that $\mu = \nu\sigma$.

Definition 3.30. A variety of algebras Θ is called *logically perfect* if every finitely generated free in Θ algebra $W(X)$, $X \in \Gamma$ is logically homogeneous.

Definition 3.31. An algebra H in Θ is called *logically separable*, if every algebra $H' \in \Theta$ which is LG-equivalent to H is isomorphic to H.

Definition 3.32. A variety Θ is called *logically regular* if every free in Θ algebra $W(X)$, $X \in \Gamma$ is logically separable.

The following theorem is valid:

Theorem 3.33. *If the variety Θ is logically perfect, then it is logically regular.*

Proof. Let the variety Θ be logically perfect and $W = W(X)$ be a free in Θ algebra of rank n, $X = \{x_1, \ldots, x_n\}$. Rewrite $W = H =< a_1, \ldots, a_n >$, where $a_1, \ldots, a_n$ are free generators in H. Let H and $G \in \Theta$ be isotypic.

Take $\mu : W(X) \to H$ with $\mu(x_i) = a_i$. We have $\nu : W(X) \to G$ with $T_P^H(\mu) = T_P^G(\nu)$, $\nu(x_i) = b_i$, $B =< b_1, \ldots, b_n >$. The algebras H and B are isomorphic by the isomorphism $a_i \to b_i$, $i = 1, \ldots, n$.

Indeed, $T_P^H(\mu) = T_P^G(\nu)$ implies $LKer(\mu) = LKer(\nu)$ and we have $Ker(\mu) = Ker(\nu)$. This gives the needed isomorphism $H \to B$.

Let us prove that $B = G$. Let $B \neq G$ and there is $b \in G$ which doesn't lie in B.

Take a subalgebra $B' =< b, b_1, \ldots, b_n >$ in G and a collection of variables $Y = \{y, x_1, \ldots, x_n\}$ with $\nu' : W(Y) \to G$, $\nu'(y) = b$, $\nu'(x_i) = \nu(x_i) = b_i$, $i = 1, \ldots, n$.

We have $\mu' : W(Y) \to H$ with $T_P^H(\mu') = T_P^G(\nu')$. Let $\mu'(y) = a'$, $\mu'(x_i) = {a'}_i$, $i = 1, \ldots, n$. Let the algebras $H' =< a', {a'}_1, \ldots, {a'}_n >$ and $B' =< b, b_1, \ldots, b_n >$ be isomorphic.

Further we work with the equality $LKer(\mu') = LKer(\nu')$. Take a formula $u \in LKer(\mu)$ and pass to a formula $u' = (y \equiv y) \wedge u$. The point $(b_1, \ldots, b_n)$ satisfies the formula u and, hence, the point ν' satisfies u'. Therefore, the point μ' satisfies u' as well, and $u' \in LKer(\mu')$.

Take now a point $\mu'' : W(X) \to H$ setting $\mu''(x_i) = {a'}_i$, $i = 1, \ldots, n$. The point μ' satisfies the formula u' if and only if the point μ'' satisfies u. Hence, $LKer(\mu) = LKer(\mu'')$. Therefore, the point μ'' is conjugated with the point μ by some isomorphism σ. Thus, the point $< {a'}_1, \ldots, {a'}_n >$ is a basis in H and $a' \in< {a'}_1, \ldots, {a'}_n >$. This contradicts with $b \notin< b_1, \ldots, b_n >$. So, $B = G$ and H and G are isomorphic. ☐

Problem 11. Is the converse statement true? That is, whether every logically regular algebra is logically perfect.

It seems to us that the answer may be negative and the logical regularity of a variety Θ doesn't imply its logical perfectness. This leads to the problem

Problem 12. Find a logically regular but not logically perfect variety Θ. In particular, consider this problem for different varieties of groups and varieties of semigroups.

Let us give some examples of perfectness and regularity for varieties of groups and semigroups (see [19], [20], [21], [38], [48]).

- The variety of all groups is logically perfect, and, hence, is logically regular.
- The variety of abelian groups is logically perfect, and, hence, is logically regular.
- The variety of all nilpotent groups of class at most n is logically perfect, and, hence, is logically regular.

- The variety of all semigroups is logically regular.
- The variety of all inverse semigroups is logically regular.

Now we can specify Problem 12 to the case of semigroups.

Problem 13. Check whether the varieties of all semigroups and of all inverse semigroups are logically perfect.

We shall emphasize two following problems regarding solvable groups.

Problem 14. What can be said about logical regularity and logical perfectness for the variety of all solvable groups of the derived length at most n.

Problem 15. Is the variety of metabelian groups logically perfect? Is the variety of metabelian groups logically regular?

The situation with logical regularity and logical perfectness of other varieties of algebras is not clear. Let us point out some questions which appear by varying the variety Θ. First of all:

Problem 16. Let Θ be a classical variety $Com - P$, the variety of commutative and associative algebras with unit over a field P. The problem is to verify its logical regularity and logical perfectness.

The same question stands with respect to some other well-known varieties. So, are the following varieties logically perfect or logically regular?

Problem 17. The variety $Ass - P$ of associative algebras over a field P.

Problem 18. The variety $Lee - P$ of Lee algebras over a field P.

Problem 19. The variety of n-nilpotent associative algebras.

Problem 20. The variety of n-nilpotent Lee algebras.

Problem 21. The varieties of solvable Lee/associative algebras of derived length at most n .

It is also important to find out how the passage from a semigroup/group to a semigroup/group algebra behaves with respect to logical regularity and logical perfectness. This leads to the problem:

Problem 22. Let S be a semigroup/group and P a field, both logically homogeneous. Whether it is true that the semigroup/group algebra PS is logically homogeneous as well.

3.10. **Logically noetherian and saturated algebras.**

Definition 3.34. An algebra H is called *logically noetherian* if for any set of formulas $T \subset \Phi(X)$, $X \in \Gamma$ there is a finite subset T_0 in T determining the same set of points A that is determined by the set T.

Definition 3.35. An algebra $H \in \Theta$ is called *LG-saturated* if for every $X \in \Gamma$ each ultrafilter T in $\Phi(X)$ containing $Th^X(h)$ has the form $T = LKer(\mu)$ for some $u : W(X) \to H$.

Theorem 3.36. *If an algebra H is logically noetherian then H is LG-saturated.*

Proof. We start from the homomorphism:

$$Val_H^X : \Phi(X) \to Hal_\Theta^X(H).$$

Here $Ker(Val_H^X) = Th^X(H)$. Consider the quotient algebra $\Phi(X)/Th^X(H)$ which is isomorphic to a subalgebra in $Hal_\Theta^X(H)$. For every $u \in \Phi(X)$ denote by $[u]$ the image of u in the quotient algebra. By definition $[u] = 0$ means that $Val_H^X(u)$ is the empty subset in $Hom(W(X), H)$. Analogously $[u] = 1$ means that $Val_H^X(u)$ is the whole space $Hom(W(X), H)$ and, thus, $u \in Th^X(H)$.

Denote by T an ultrafilter in $\Phi(X)$, containing the theory $Th^X(H)$. We need to check that there is a point $\mu : W(X) \to H$ such that $T = LKer(\mu)$. Let $[u] = 0$. Then $[\neg u] = 1$, which means that $\neg u \in Th^X(H) \subset T$. Hence $\neg u \in T$. Then u does not belong to $Th^X(H)$, since T cannot contain both u and $\neg u$. So $u \notin T$. Thus, if $[u] = 0$ then $u \notin T$. If $u \in T$, then $[u] \neq 0$. This means that $Val_H^X(u)$ is not empty. Thus, we have a point $\mu : W(X) \to H$ which satisfies u, that is $u \in LKer(\mu)$. Since H is logically noetherian, then there exists a finite subset $T_0 = \{u_1, \ldots, u_n\}$ such that $T_H^L = (T_0)_H^L$. Take $u = u_1 \wedge u_2 \wedge \ldots u_n$. Since all $u_i \in T$, then $u \in T$ and there exists μ satisfying formula u. The same point μ satisfies every u_i. Thus, $\mu \in (T_0)^L(H)$=$T^L(H)$ and T lies in $LKer(\mu)$. Therefore $T = LKer(\mu)$. □

Each finite algebra H is logically noetherian. Hence, every finite H is LG-saturated. This holds for every Θ.

3.11. **Automorphically finitary algebras.** We have already mentioned that the group $Aut(H)$ acts in each space $Hom(W(X), H)$, $X \in \Gamma$.

Definition 3.37. Let us call an algebra H *automorphically finitary* if in each such action there is only a finite number of $Aut(H)$-orbits.

It is easy to show that if algebra H is automorphically finitary, then it is logically noetherian. The example of abelian groups of exponent p shows that there exist infinite automorphically finitary algebras and, thus, there are infinite saturated algebras.

Problem 23. Describe all automorphically finitary abelian groups.

Problem 24. Construct examples of non-commutative automorphically finitary groups.

Problem 25. Classify abelian groups by LG-equivalence relation.

Let us make some comments regarding Problem 25. According to Theorem 3.9, LG-equivalent abelian groups are isotypic. As we know (Corollary 3.10), isotypeness of algebras implies their elementary equivalence. Classification of abelian groups with respect to elementary equivalence was obtained by W. Szmielew in her classical paper [41]. So, Problem 25 asks how one should modify the list from [41] in order to obtain the isotypic abelian groups.

We considered two important characteristics of varieties of algebras, namely, their logical perfectness and logical regularity. Let us introduce one more characteristic.

We call a variety Θ *exceptional* if

- any two distinct free in Θ algebras $W(X)$ and $W(Y)$ of a finite rank, generating the whole Θ, are elementarily equivalent, and
- if $W(X)$ and $W(Y)$ are isotypic then they are isomorphic.

Problem 26. Find examples of non-trivial exceptional varieties. Is it true that the *Burnside variety* B_n of all groups of exponent n, where n is big enough, is exceptional? Is it true that the *Engel variety* E_n of all groups with the identity $e_n(x,y) = [[[x,y],y],\dots,y] \equiv 1$, where n is big enough, is exceptional? Here $[x,y] = xyx^{-1}y^{-1}$, and the commutator in $e_n(x,y)$ is taken n-times.

4. Model theoretical types and logically geometric types

4.1. **Definitions of types.** The notion of a type is one of the key notions of Model Theory. In what follows we will distinguish between model theoretical types (MT-types) and logically geometric types (LG-types). Both kinds of types are oriented towards some algebra $H \in \Theta$, where Θ is a fixed variety of algebras.

Generally speaking, a type of a point $\mu : W(X) \to H$ is a logical characteristic of the point μ. Model-theoretical idea of a type and its definition is described in many sources, see, in particular, [9], [13]. We consider this idea from the perspective of algebraic logic (cf., [33]) and give all the definitions in the corresponding terms.

Proceed from the algebra of formulas $\Phi(X^0)$, where X^0 is an infinite set of variables. It is obtained from the algebra of pure first-order formulas with equalities $w \equiv w'$, $w, w' \in W(X^0)$ by Lindenbaum-Tarski algebraization approach (cf. Section 2.8). $\Phi(X^0)$ is an X^0-extended Boolean algebra, which means that $\Phi(X^0)$ is a Boolean algebra with quantifiers $\exists x$, $x \in X^0$ and equalities $w \equiv w'$, where $w, w' \in W(X^0)$. Here, $W(X^0)$ is the free over X^0 algebra in Θ. All these equalities generate the algebra $\Phi(X^0)$. Besides, the semigroup $End(W(X^0))$ acts on the Boolean algebra $\Phi(X^0)$ and we can speak of a *polyadic algebra* $\Phi(X^0)$ [8]. However, the elements $s \in End(W(X^0))$ and the corresponding s_* are not included in the signature of the algebra $\Phi(X^0)$.

Since $\Phi(X^0)$ is a one-sorted algebra, one can speak, as usual, about free and bound occurrences of the variables in the formulas $u \in \Phi(X^0)$.

Define further X-special formulas in $\Phi(X^0)$, $X = \{x_1, \dots, x_n\}$. Take $X^0 \backslash X = Y^0$. A formula $u \in \Phi(X^0)$ is X*-special* if each of its free variables occurs in X and each bound variable belongs to Y^0. A formula $u \in \Phi(X^0)$

is *closed* if it does not have free variables. Only finite number of variables occur in each formula.

Denoting an X-special formula u as $u = u(x_1, \dots, x_n;\ y_1, \dots, y_m)$ we solely mean that the set X consists of variables x_i, $i = 1, \dots n$, and those of them who occur in u, occur freely.

Definition 4.1. Let H be an algebra from Θ. An X-type (over H) is a set of X-special formulas in $\Phi(X^0)$, consistent with the elementary theory of the algebra H.

We call such type an X-MT-type (Model Theoretic type) over H. An X-MT-type is called *complete* if it is maximal with respect to inclusion. Any complete X-MT-type is a Boolean ultrafilter in the algebra $\Phi(X^0)$. Hence, for every X-special formula $u \in \Phi(X^0)$, either u or its negation belongs to a complete type.

Definition 4.2. An X-LG-type (Logically Geometric type) (over H) is a Boolean ultrafilter in the corresponding $\Phi(X)$, which contains the elementary theory $Th^X(H)$.

So, any X-MT-type lies in the one-sorted algebra $\Phi(X^0)$. Any X-LG-type lies in the domain $\Phi(X)$ of the multi-sorted algebra $\widetilde{\Phi}$.

We denote the MT-type of a point $\mu : W(X) \to H$ by $Tp^H(\mu)$, while the LG-type of the same point is, by definition, its logical kernel $LKer(\mu)$.

Definition 4.3. Let a point $\mu : W(X) \to H$, with $a_i = \mu(x_i)$, be given. An X-special formula $u = u(x_1, \dots, x_n; y_1, \dots, y_m)$ belongs to the type $Tp^H(\mu)$ if the formula $u(a_1, \dots, a_n; y_1, \dots, y_m)$ is satisfied in the algebra H.

The type $Tp^H(\mu)$ consists of all X-special formulas satisfied on μ. It is a complete X-MT-type over H.

By definition, the formula $v = u(a_1, \dots, a_n; y_1, \dots, y_m)$ is closed. Thus, if it is satisfied on a point, then its value set $Val^H_X(v)$ is the whole affine space $Hom(W(X), H)$.

Note that in our definition of an X-MT-type the set of free variables in the formula u is not necessarily the whole $X = \{x_1, \dots, x_n\}$ and can be a part of it. In particular, the set of free variables can be empty. In this case the formula u belongs to the type if it is satisfied in H.

In the previous sections the algebra $\widetilde{\Phi}$ was built basing on the set Γ of all finite subsets of the set Γ. In fact, one can take the system $\Gamma^* = \Gamma \bigcup X^0$ instead of Γ and construct the corresponding multi-sorted algebra. Then, to each homomorphism $s : W(X^0) \to W(X)$ it corresponds a morphism $s_* : \Phi(X^0) \to \Phi(X)$ and, vice versa, $s : W(X) \to W(X^0)$ induces $s_* : \Phi(X) \to \Phi(X^0)$. In this setting the extended Boolean algebra $Hal^{X^0}_\Theta(H)$ and the homomorphism $Val^{X^0}_H : \Phi(X^0) \to Hal^{X^0}_\Theta(H)$ are defined in the usual way. A point $\mu : W(X^0) \to H$ satisfies $u \in \Phi(X^0)$ if $\mu \in Val^{X^0}_H(u)$.

One more remark. $\Phi(X^0)$ is generated by equalities. Hence, when we say that a variable occurs in a formula $u \in \Phi(X^0)$, this means that it occurs in one of the equalities $w = w'$, participating in u. The set of variables occurring in u determines a subalgebra $\Phi(X \cup Y)$ in $\Phi(X^0)$, such that $u \in \Phi(X \cup Y)$.

If we stay in one-sorted logic, this is a subalgebra in the signature of the one-sorted algebra $\Phi(X^0)$.

On the other hand, we can view algebra $\Phi(X \cup Y)$ as an object in the multi-sorted logic. Here, to every homomorphism $s : W(X \cup Y) \to W(X' \cup Y')$ it corresponds a morphism $s_* : \Phi(X \cup Y) \to \Phi(X' \cup Y')$. For $u \in \Phi(X \cup Y)$ we have $s_* u \in \Phi(X' \cup Y')$. Let u be an X-special formula. It is important to know for which s the formula $s_* u$ is X'-special.

4.2. **Another characteristic of the type $Tp^H(\mu)$.** We would like to relate the MT-type of a point to its LG-type.

Given an infinite set X^0 and a finite subset $X = \{x_1, \dots, x_n\}$, consider a special homomorphism $s : W(X^0) \to W(X)$ such that $s(x) = x$ for each $x \in X$, i.e., s is identical on the set X. According to the transition from s to s_*, we obtain

$$s_* : \Phi(X^0) \to \Phi(X).$$

Theorem 4.4. *For each special homomorphism s, each special formula $u = u(x_1, \dots, x_n; y_1, \dots, y_m)$ in $\Phi(X^0)$ and every point $\mu : W(X) \to H$, we have $u \in Tp^H(\mu)$ if and only if $s_* u \in LKer(\mu)$. Here, in the first case u is considered in one-sorted algebra $\Phi(X^0)$, while in the second case $s_* u$ lies in the domain $\Phi(X)$ of the multi-sorted $\widetilde{\Phi} = (\Phi(X),\ X \in \Gamma^*)$.*

Proof. Given a point μ, consider a set $A_\mu : W(X) \to H$ of the points $\eta : W(X^0) \to H$ defined by the rule $\eta(x_i) = \mu(x_i) = a_i$ for $x_i \in X$ and, $\eta(y)$ is an arbitrary element in H for $y \in Y^0$. Denote

$$T_\mu = \bigcap_{\eta \in A_\mu} LKer(\eta).$$

Here, as usual, $LKer(\eta)$ is the ultrafilter in $\Phi(X^0)$, consisting of formulas u valid on a point η. It is proved [33], that a special formula u belongs to the type $Tp^H(\mu)$ if and only if $u \in T_\mu$, which is equivalent to $Val_H^{X^0}(u) \supset A_\mu$.

Note that the formula u of the kind

$$x_1 \equiv x_1 \wedge \ldots \wedge x_n \equiv x_n \wedge v(y_1, \dots, y_m)$$

belongs to each $LKer(\eta)$ if the closed formula $v(y_1, \dots, y_m)$ is satisfied in the algebra H. This means also that T_μ is not empty for every μ.

Return to the special homomorphism $s : W(X^0) \to W(X)$ and consider the point $\mu s : W(X^0) \to H$. For $x_i \in X$ we have $\mu s(x_i) = \mu(x_i) = a_i$. Hence, the point μs belongs to A_μ.

Observe that for the formula $u = u(x_1, \dots, x_n; y_1, \dots, y_m)$, the formula $u(a_1, \dots, a_n; y_1, \dots, y_m)$ is satisfied in the algebra H if the set A_μ lies in $Val_H^{X^0}(u)$. Thus, μs belongs to $Val_H^{X^0}(u)$. By definition of s_* we have that

μ lies in $s_*Val_H^{X^0}(u) = Val_H^X(s_*u)$, which means that

$$s_*u \in LKer(\mu).$$

We proved the statement in one direction.

Conversely, let $s_*u \in LKer(\mu)$. Then

$$\mu \in Val_H^X(s_*u) = s_*Val_H^{X^0}(u)$$

and $\mu s \subset Val_H^{X^0}(u)$. Since the formula $u(a_1,\dots,a_n;y_1,\dots,y_m)$ is satisfied in H, then every point from the set A_μ belongs to $Val_H^{X^0}(u)$ (see also [6]). This means that the formula u belongs to $Tp^H(\mu)$.

□

We have mentioned the notion of LG-saturated algebra (see Definition 3.35). The standard notion of saturation defined in Model Theory will be called MT-saturation. MT-saturation of an algebra H means that for any X-type T there is a point $\mu : W(X) \to H$ such that $T \subset Tp^H(\mu)$.

Theorem 4.5. *If algebra H is LG-saturated, then H is MT-saturated.*

Proof. Let H be an LG-saturated algebra and T be an X-MT-type correlated with $Th^{X^0}(H)$. We can assume that the theory $Th^{X^0}(H)$ is contained in the set of formulas T.

Take a special homomorphism $s : W(X^0) \to W(X)$ and pass to $s_* : \Phi(X^0) \to \Phi(X)$. Given formula $u \in T$, take a formula $s_*u \in \Phi(X)$. Denote the set of all such s_*u by s_*T. Since if $u \in Th^{X^0}(H)$ then $s_*u \in Th^X(H)$, the set s_*T is a filter in $\Phi(X)$ containing the elementary theory $Th^X(H)$.

We embed the filter s_*T into the ultrafilter T_0 in $\Phi(X)$ which contains the theory $Th^X(H)$. By the LG-saturation of the algebra H condition, $T_0 = LKer(\mu)$ for some point $\mu : W(X) \to H$. Thus, $s_*u \in LKer(\mu)$ for each formula $u \in T$. Hence (Theorem 4.4), $u \in Tp^H(\mu)$ for each $u \in T$, and $T \subset Tp^H(\mu)$. This gives MT-saturation of the algebra H. □

We do not know whether MT-saturation implies LG-saturation.

4.3. Correspondence between $u \in \Phi(X)$ and $\widetilde{u} \in \Phi(X^0)$.

Definition 4.6. A formula $u \in \Phi(X)$ is called *correct*, if there exists an X-special formula $\widetilde{u}$ in $\Phi(X^0)$ such that for every point $\mu : W(X) \to H$ we have $u \in LKer(\mu)$ if and only if $\widetilde{u} \in Tp^H(\mu)$.

Now, we shall formulate the principal Theorem of G. Zhitomiskii (see [48]).This fact will be essentially used in Theorem 4.8 and Theorem 4.12. It reveals ties between two approaches to the idea of a type of a point: the one-sorted model theoretic approach and the multi-sorted logically geometric approach.

Theorem 4.7. [48] *For every $X = \{x_1,\dots,x_n\}$, every formula $u \in \Phi(X)$ is correct.*

4.4. **LG- and MT-isotypeness of algebras.** The following theorem helps to clarify the notion of isotypeness of algebras.

Theorem 4.8. [48] *Let the points $\mu : W(X) \to H_1$ and $\nu : W(X) \to H_2$ be given. Then*

$$Tp^{H_1}(\mu) = Tp^{H_2}(\nu)$$

if and only if

$$LKer(\mu) = LKer(\nu).$$

Proof. Let the points $\mu : W(X) \to H_1$ and $\nu : W(X) \to H_2$ be given and let $Tp^{H_1}(\mu) = Tp^{H_2}(\nu)$. Take $u \in LKer(\mu)$. Then $\widetilde{u} \in Tp^{H_1}(\mu)$ and, thus, $\widetilde{u} \in Tp^{H_2}(\nu)$. Hence, $u \in LKer(\nu)$. The same is true in the opposite direction.

Let, conversely, $LKer(\mu) = LKer(\nu)$. Take an arbitrary X-special formula u in $Tp^{H_1}(\mu)$. Take a special homomorphism from $s : W(X^0) \to W(X)$. The morphism $s_* : \Phi(X^0) \to \Phi(X)$ corresponds to s. Then, using Theorem 4.4, the formula $u \in Tp^H(\mu)$ is valid if and only if $s_*u \in LKer(\mu)$. Then $s_*u \in LKer(\nu)$. Then $u \in Tp^H(\nu)$. □

Definition 4.9. Given X, denote by $S^X(H)$ the set of MT-types of an algebra H, implemented (realized) by points in H. Algebras H_1 and H_2 are called MT-*isotypic* if $S^X(H_1) = S^X(H_2)$ for any $X \in \Gamma$.

Theorem 4.8 implies

Corollary 4.10. *Algebras H_1 and H_2 in the variety Θ are MT-isotypic if and only if they are LG-isotypic.*

So, it doesn't matter which type (LG-type or MT-type) is used in the definition of isotypeness. Hence, by Theorem 3.9, algebras H_1 and H_2 in the variety Θ are MT-isotypic if and only if they are LG-equivalent.

If algebras H_1 and H_2 are isotypic then they are locally isomorphic. This means that if A is a finitely generated subalgebra in H, then there exists a subalgebra B in H_2 which is isomorphic to A. The same is true in the direction from H_2 to H_1.

On the other hand, local isomorphism of H_1 and H_2 does not imply their isotypeness: the groups F_n and F_m, $m, n > 1$ are locally isomorphic, but they are isotypic only for $n = m$.

Isotypeness implies elementary equivalence of algebras, but the same example with F_n and F_m shows that the converse is false.

In Section 2 we pointed out several problems related to isotypic algebras. Let us give some other problems:

Problem 27. Suppose that H_1 and H_2 are two finitely generated isotypic algebras. Are they always isomorphic?

In particular:

Problem 28. Let G_1 and G_2 be two finitely generated isotypic groups. Are they always isomorphic?

Problem 29. Let H_1 be a finitely generated algebra and H_2 be an isotypic to it algebra. Is H_2 also finitely generated?

The next problem is connected with the previously named problems on isotypeness and isomorphism of free algebras.

Problem 30. Let two isotypic finitely-generated free algebras H_1 and H_2 and two points $\mu : W(X) \to H_1$ and $\nu : W(X) \to H_2$ be given. Let $LKer(\mu) = LKer(\nu)$. Is it true that there exists an isomorphism $\sigma : H_1 \to H_2$ such that $\mu\sigma = \nu$?

4.5. **LG and MT-geometries.** Compare, first, different approaches to the notion of a definable set in the affine space $Hom(W(X), H)$.

Suppose that a variety Θ of algebras, an algebra $H \in \Theta$ and the finite set $X = \{x_1, \dots, x_n\}$ are fixed.

Consider subsets A in the affine space $Hom(W(X), H)$ whose points have the form $\mu : W(X) \to H$. Each point $\mu : W(X) \to H$ has a classical kernel $Ker(\mu)$, a logical kernel $LKer(\mu)$ and a type $(Tp^H(\mu))$. Correspondingly, we have three different geometries: algebraic geometry (AG), logical geometry (LG), and the model-theoretic geometry (MTG).

For AG consider a system T of equations $w \equiv w'$, $w, w' \in W(X)$. For LG we take a set of formulas T in the algebra of formulas $\Phi(X)$. For MTG we proceed from an X-type T. In all these cases the set can be infinite.

Now,

• A set A in $Hom(W(X), H)$ is definable in AG (i.e., A is *an algebraic set*) if there exists T in $W(X)$ such that $T'_H = A$, where

$$T'_H = \{\mu |\ T \subset Ker(\mu)\}.$$

• A set A in $Hom(W(X), H)$ is definable in LG (i.e., A *is LG-definable*) if there exists T in $\Phi(X)$ such that $T^L_H = A$, where

$$T^L_H = \{\mu |\ T \subset LKer(\mu)\} = \bigcap_{u \in T} Val^X_H(u).$$

• A set A in $Hom(W(X), H)$ is definable in MTG (i.e., A *is MT-definable*) if there exists an X-type T such that $T^{L_0}_H = A$, where

$$T^{L_0}_H = \{\mu |\ T \subset Tp^H(\mu)\} = \bigcap_{u \in T} Val^{X_0}_H(u).$$

Besides that, we have three closures: T''_H for AG, T^{LL}_H for LG, and $T^{L_0L_0}_H$ for MTG. In the reverse direction the Galois correspondence for each of the three cases above is as follows:

$$T = A'_H = \bigcap_{\mu \in A} Ker(\mu),$$

$$T = A^L_H = \bigcap_{\mu \in A} LKer(\mu),$$

$$T = A_H^{L_0} = \bigcap_{\mu \in A} Tp^H(\mu).$$

Correspondingly, we distinguish three types of equivalence relations on algebras from the variety Θ.
Algebras H_1 and H_2 are *algebraically equivalent* if

$$T''_{H_1} = T''_{H_2}.$$

Algebras H_1 and H_2 are *logically equivalent* if

$$T^{LL}_{H_1} = T^{LL}_{H_2}.$$

Algebras H_1 and H_2 are *MT-equivalent* if

$$T^{L_0 L_0}_{H_1} = T^{L_0 L_0}_{H_2}.$$

A natural question is

Problem 31. Whether the notions of LG-definable and MT-definable sets coincide?

First, we need to clarify some details. Take a special morphism $s : W(X^0) \to W(X)$ identical on the set $X \subset X^0$, $X \in \Gamma$. We have also $s_* : \Phi(X^0) \to \Phi(X)$. Define a set of formulas $s_*T = \{s_*u | u \in T\}$.

Theorem 4.11. *The equality $T_H^{L_0} = (s_*T)_H^L$ holds for every X-type T.*

Proof. Let $\mu \in T_H^{L_0}$. Then $T \subset Tp^H(\mu)$ and every formula $u \in T$ is contained in $Tp^H(\mu)$. Besides, $s_*u \in LKer(\mu)$ and $\mu \in Val_H^X(u)$. We have $\mu \in \bigcap_{u \in T} Val_H^X(u) = (s_*T)_H^L$.

Let now $\mu \in (s_*T)_H^L$. Then for every $u \in T$ we have $\mu \in Val_H^X(s_*u)$ and $s_*u \in LKer(\mu)$. Hence, $u \in Tp^H(\mu)$. This gives $T \subset Tp^H(\mu)$ and $\mu \in T_H^{L_0}$. $\square$

Moreover, the following theorem answers Problem 31 in the affirmative.

Theorem 4.12. *Let $A \subset Hom(W(X), H)$. The set A is LG-definable if and only if A is MT-definable.*

Proof. Theorem 4.11 implies that every MT-definable set is LG-definable. Consider the converse. We use Theorem 4.7: for every formula $u \in \Phi(X)$ there exists an X-special formula $\widetilde{u} \in \Phi(X^0)$ such that a point $\mu : W(X) \to H$ satisfies $\widetilde{u}$ if and only if it satisfies u. Let now the set $T_H^L = A$ be given. Every point μ from A satisfies every formula $u \in T$. Given T take T' consisting of all $\widetilde{u}$ which correspond to $u \in T$. The points $\mu \in A$ satisfy every formula from T'. This means that T' is a consistent set of X-special formulas. Thus T' is an X-type, such that $A \subset T'^{L_0}_H$.

Let now the point ν lie in $T'^{L_0}_H$. Then ν satisfies every formula $\widetilde{u}$. Hence, it satisfies every formula $u \in T$. Thus, ν lies in $T_H^L = A$. This means that

$$T'^{L_0}_H = A$$

and the theorem is proved. $\square$

Consider now the case when algebra H is logically homogeneous and A is an $Aut(H)$-orbit over the point $\mu : W(X) \to H$. We have $A = (LKer(\mu))^L_H$. The equality $LKer(\mu) = LKer(\nu)$ holds if and only if a point ν belongs to A. The same condition is needed for the equality $Tp^H(\mu) = Tp^H(\nu)$. Now, $\nu \in (Tp^H(\mu))^{L_0}_H$ by the definition of L_0. Thus, $A = (Tp^H(\mu))^{L_0}_H$. We proved that the orbit A is MT-definable and LG-definable.

Recall that we defined two full sub-categories $K_\Theta(H)$ and $LK_\Theta(H)$ in the category $Set_\Theta(H)$. Let us take one more sub-category denoted by $L_0K_\Theta(H)$. In each object (X, A) of this category the set A is an X-MT-type definable set. The category $L_0K_\Theta(H)$ is a full subcategory in $LK_\Theta(H)$. In view of Theorem 4.12 categories $LK_\Theta(H)$ and $L_0K_\Theta(H)$ coincide.

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