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On the Stability of the K_1-Functor for Chevalley Groups of Type E_7

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This paper is devoted to the study of the surjective stability of the K_1-functor for Chevalley groups of type E_7. This is a particular case of the stability problem for Chevalley groups, which was posed by H. Bass. In this paper, surjective stability of the K_1-functor is proved under natural conditions on the dimension of the maximal spectrum of a ring and, independently, under a special condition which is meaningful from the point of view of equations defining a Chevalley group.

Key Words: Chevalley groups; K_1-functor; surjective stability.

INTRODUCTION

The problem of K_1-functor stability for semisimple algebraic groups was discussed by Bass [5, p. 278]. He proposed formulating stability hypotheses in terms of relations between the dimension of the maximal spectrum of a ring and the rank of a maximal split torus of a group.

In the framework of Chevalley groups, this problem was studied by Stein [27, 28], who developed an approach that considers, in the same way, both classical and exceptional cases. Despite this, however, the problem is still open. It consists of two, in fact, independent tasks. These are surjective and injective stability questions.

Traditionally, injective stability was considered as a more difficult question. However, recently it has been completely investigated. Vavilov has shown that *injective stability of the K_1-functor for all regular embeddings of*

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root systems holds under natural (*depending on embedding*) *conditions* on *the stable rank of a ring*. This result is based on the Suslin–Tulenbaev proof [32] of the Dennis–Vaserstein decomposition [13, 14, 36].

For surjective stability the situation is as follows. It is known [4, 6, 35] that for the embeddings $A_{n-1} \to A_n$, $C_{n-1} \to C_n$, surjective stability holds under stable rank conditions. However, the cases of the orthogonal groups $B_{n-1} \to B_n$, $D_{n-1} \to D_n$ require stronger conditions on a ring [27, 36]. These are either the absolute stable rank condition ASR_n [27], or some special condition V_n, introduced by Vaserstein [36] for orthogonal and unitary groups, or just the condition on the maximal spectrum of a ring $\dim \operatorname{Max}(R)$. The absolute stable rank condition turns out to be sufficient also for some other cases of Chevalley group embeddings ($A_1 \to G_2$, B_3, $C_3 \to F_4$, $D_5 \to E_6$, [22, 23, 27, 28]).

The aim of this paper is to prove surjective stability of the K_1-functor for the embedding of Chevalley groups of the type $\mathrm{E}_6 \to \mathrm{E}_7$ under some natural conditions on the ring. Then, for the embeddings of root systems of the same type, the only case that remains to be studied is $\mathrm{E}_7 \to \mathrm{E}_8$.

1. PRELIMINARIES

Let Φ be a reduced irreducible root system of rank l, let $G(\Phi, \ )$ be a simply connected Chevalley–Demazure group scheme over $\mathbb{Z}$ of type Φ (see [9, 12]), and let $T(\Phi, \ )$ be a split maximal torus in it. If R is a commutative ring with 1, the value of the functor $G(\Phi, \ )$ on R is called the simply connected Chevalley group of type Φ over R and is denoted by $G(\Phi, R)$. To each root $\alpha \in \Phi$ there correspond elementary (with respect to T) root unipotent elements $x_\alpha(\xi)$, $\xi \in R$. All the elementary unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, generate a group $E(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi, \xi \in R \rangle$, which is called the elementary subgroup of $G(\Phi, R)$.

It is well known that if Φ is an irreducible root system of rank $l \geq 2$, then $E(\Phi, R)$ is always *normal* in $G(\Phi, R)$ (see [30, 31], for the classical groups and [2, 33, 34, 37, 38] for the Chevalley groups).

Thus, the K_1-functor of Chevalley group $G(\Phi, R)$ is naturally defined as the quotient group

$$K_1(\Phi, R) = G(\Phi, R)/E(\Phi, R)$$

(see [1, 26–28]).

Any inclusion of root systems $\Delta \subset \Phi$ induces the homomorphisms of groups $G(\Delta, R) \to G(\Phi, R)$, $E(\Delta, R) \to E(\Phi, R)$ taking roots into roots, and the homomorphism of the corresponding K_1-functors $\nu\colon K_1(\Delta, R) \to K_1(\Phi, R)$.

The surjective stability question is *to find conditions on the ring R, depending on $\Delta \subset \Phi$, which provide surjectivity of the homomorphism ν.*

Since we consider surjective stability for not necessarily infinite series of Chevalley groups, it is better to reformulate this question as a question of the conditions on R which provide the existence of the decomposition $G(\Phi, R) = E(\Phi, R)G(\Delta, R)$.

Denote by $\langle X \rangle$ the ideal generated by X if $X \subset R$; the subgroup generated by X if X is a subset of a group; the minimal closed subsystem of roots containing X if $X \subset \Phi$.

Recall that a commutative ring R satisfies the absolute stable rank condition ASR_n if for any row $(r_1, \dots, r_n)$ with coordinates in R, there exist elements $t_1, \dots, t_{n-1} \in R$, such that every maximal ideal of R containing the ideal $\langle r_1 + t_1 r_n, \dots, r_{n-1} + t_{n-1} r_n \rangle$ also contains the ideal $\langle r_1, \dots, r_n \rangle$. This notion was introduced in [15] and used in [27, 28] and then in [22, 23] for stability problems. The description of various properties of ASR_n, as well as numerous applications (for not necessarily commutative rings), can be found in [19].

It was mentioned by Bak that the absolute stable rank condition ASR_n is equivalent to the following one. Let $(r_1, \dots, r_{n+1})$ be a left unimodular vector. Then there are elements $t_1, \dots, t_{n-1} \in R$, such that $(r_1 + t_1 r_n, \dots, r_{n-1} + t_{n-1} r_n, r_{n+1} + t r_n)$ is unimodular for any $t \in R$.

If we assume that a row $(r_1, \dots, r_n)$ is unimodular then the absolute stable rank condition is transformed into the stable rank condition [4, 35]. The absolute stable rank condition satisfies the usual properties, namely for every ideal $I \lhd R$ it may be lifted to R/I, and if $n \geq m$, then ASR_m implies ASR_n. Finally, it is well known that if the dimension of the maximal spectrum dimMax(R) is $n - 2$, then both the conditions ASR_n and SR_n are fulfilled [15, 19, 27].

2. WEIGHT DIAGRAMS AND BASIC REPRESENTATIONS

Let us fix an order on Φ, and let Φ^+, Φ^-, and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the sets of positive, negative, and fundamental roots, respectively. Our numbering of the fundamental roots follows that of [7]. By $\overline{\omega}_1, \dots, \overline{\omega}_l$ one denotes the corresponding fundamental weights. Let $W = W(\Phi)$ be the Weyl group of the root system Φ, i.e., the group generated by the set of fundamental reflections $w_{\alpha_1}, \dots, w_{\alpha_l}$.

Recall that an irreducible representation π of the complex semisimple Lie algebra L is called *basic* [20] if the Weyl group $W = W(\Phi)$ acts transitively on the set $\Lambda(\pi)$ of nonzero weights of the representation π. This is equivalent to saying that if for any two nonzero weights λ, ν their difference is a fundamental root $\alpha = \lambda - \nu$, then $w_\alpha \lambda = \nu$ for the corresponding fundamental reflection $w_\alpha \in W$.

In this paper we can restrict ourselves to the basic representations without zero weight. In this case, $\Lambda(\pi)$ is the set of all weights of π, and the weights $\Lambda(\pi)$ form the one Weyl orbit. Such representations are called *microweight* or *minuscule* representations, and the list of these representations is very well known (see [8]).

To each complex representation π of a simple Lie algebra L of type Φ there corresponds a representation π of the Chevalley group $G = G(\Phi, R)$ on the free R-module $V = V_R = V_Z \otimes_{\mathbb{Z}} R$ [20, 29]. If π is faithful we can identify G with its image $\pi(G) = G_\pi(\Phi, R)$ under this representation and omit the symbol π in the action of G on V. Thus, for an $x \in G$ and $v \in V$ we write xv for $\pi(x)v$. If we want to specify that the group $G = G(\Phi, R)$ is considered in the basic representation π with the highest weight μ, the notation $(G(\Phi, R), \mu)$ is used. In the sequel, μ always stands for the highest weight of a representation.

Decompose the module V into the direct sum of its weight submodules

$$V = \bigoplus \sum V^\lambda, \qquad \lambda \in \Lambda(\pi).$$

It follows from the definition of microweight representation that all V^λ, $\lambda \in \Lambda(\pi)$, are one dimensional. Matsumoto [20, Lemma 2.3] has shown that there is a special base of weight vectors $v^\lambda \in V^\lambda$, $\lambda \in \Lambda(\pi)$, such that the action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is described by the following simple formulas:

$$\begin{array}{ll} \text{i.} & \text{if } \lambda \in \Lambda(\pi),\ \lambda + \alpha \notin \Lambda(\pi), \text{ then } x_\alpha(\xi)v^\lambda = v^\lambda; \\ \text{ii.} & \text{if } \lambda, \lambda + \alpha \in \Lambda(\pi), \text{ then } x_\alpha(\xi)v^\lambda = v^\lambda \pm \xi v^{\lambda+\alpha}. \end{array} \qquad (*)$$

For any $v \in V$ in the chosen base, we have $v = \sum c_\lambda v^\lambda$, $\lambda \in \Lambda(\pi)$, and Matsumoto's lemma provides explicit formulas for the action of $x_\alpha(\xi)$ on v and on its coordinates c_λ.

Now if $g \in G = G(\Phi, R)$, then the νth column $g_{*,\nu}$ of the corresponding matrix $\pi(g) \in (G(\Phi, R), \mu)$, where μ is the highest weight of π, consists of the coefficients in the expansion of $\pi(g)e^\nu$ with respect to e^λ, $\lambda \in \Lambda(\pi)$. We may conceive any element $g \in G$ as a *matrix* $g = g_{\lambda,\nu}$, where λ and ν range over all the weights of the representation π. Then the columns above are obtained by freezing the second index in such a matrix. Analogously, the rows $g_{\lambda,*}$ are obtained by freezing the first index and correspond to the vectors from the dual module V^*.

As a rule, in stability questions all the calculations use only one (say, the first) column of the matrix. We denote by $\lambda(g)$ the λth coordinate of the first column $g_{\lambda\mu}$, $\lambda \in \Lambda(\pi)$, of the matrix g. As we know from $(*)$, one can very efficiently perform calculations with such columns (and rows) and calculate $x_\alpha(\xi)g$.

Moreover, we can obtain further simplification of calculations, using the machinery of weight diagrams (see [16, 21, 24, 27, 38–40] for the detailed description of these diagrams and references). Let us recall here the corresponding definitions.

It is well known that a choice of a fundamental system Π defines a partial order of the weight lattice $P(\Phi)$ as follows: $\lambda \geq \nu$ if and only if $\lambda - \nu$ is a linear combination of the fundamental roots with nonnegative integral coefficients. Let us associate with a representation π a graph which is in fact the Hasse diagram of the set $\Lambda(\pi)$ of its weights with respect to the above order.

We construct a labeled graph in the following way. Its vertices correspond to the weights $\lambda \in \Lambda(\pi)$ of the representation π, and the vertex corresponding to λ is actually marked by λ (usually these labels are omitted).

We read the diagram from right to left and from bottom to top, which means that a larger weight tends to stand to the left of and higher than a smaller one. The leftmost vertex corresponds to the highest weight μ of a representation.

The vertices corresponding to the weights $\lambda, \nu \in \Lambda(\pi)$ are joined by a bond marked α_i (or simply i) if and only if their difference $\lambda - \nu = \alpha_i \in \Pi$ is a fundamental root. We draw the diagrams in such a way that the marks on the opposite sides of a parallelogram are equal (as a rule, at least one of them is omitted). Thus, all paths of minimal length connecting two vertices have the same sum of labels. This means that if there is a root, corresponding to a pair of vertices, it can be determined by any path of minimal length between these vertices.

Now one may conceive a vector $v \in V$ as such a weight diagram which has an element of R attached to every node. A standard weight vector e^λ has 1 in the λth node and zeros elsewhere; an arbitrary vector v has its λth coordinate v^λ with respect to this weight base as the label at the λth node. The above-mentioned Matsumoto's lemma gives a very simple rule describing what happens with such a vector v under the action of $x_\alpha(\xi)$. For a minuscule π and a fundamental root $\alpha = \alpha_i$, $x_\alpha(\xi)$ adds or subtracts (*always* adds for a clever choice of the weight base) ξv^λ to v^ν along each edge labeled with i. For other roots one merely has to trace all paths in the diagram which have the same labels at their edges as the root α in its linear expansion with respect to the fundamental roots. For example, if $\alpha = 2\alpha_1 + \alpha_2$, one has to look at the paths which have the labels $1, 1, 2$, in any order (the order of the labels on such a path starting in λ together with the structure constants of the Lie algebra is responsible for the *sign* with which $x_\alpha(\xi)$ acts on v^λ).

In this paper we use only microweight representations; that is, we do not care about weight diagrams with zero weights. The details of how to construct and operate with weight diagrams in case of the presence of zero weight can be found in [24, 38]. Weight diagrams for the aims of stability of the K_1- and K_2-functors were introduced by Stein in [27].

3. STABILITY THEOREMS FOR $E_6 \to E_7$

Stability of the K_1-functor is closely related to the fact known as the Chevalley–Matsumoto decomposition theorem (see [9, 20, 27]). Let us formulate the particular case of this theorem used for the stability problem.

Consider a basic representation π of the group $G(\Phi, R)$ with the highest weight μ. Denote by $\alpha_k \in \Pi$ the fundamental root, such that $\mu - \alpha_k$ is a weight, and by Δ the subsystem in Φ generated by all fundamental roots except α_k. Further, let $\Sigma = \Phi \setminus \Delta$, $\Sigma^+ = \Phi^+ \cap \Sigma$, $\Sigma^- = \Phi^- \cap \Sigma$, and

$$U(\Sigma, R) = \langle x_\alpha(\xi), \alpha \in \Sigma^+, t \in R \rangle,$$

$$V(\Sigma, R) = \langle x_\alpha(\xi), \alpha \in \Sigma^-, t \in R \rangle.$$

Now take a matrix $g \in (G(\Phi, R), \mu)$ and suppose that $g_{\mu\mu}$ is an invertible element in R. Then the Chevalley–Matsumoto theorem states that g can be expressed in the form

$$g = v g_1 u,$$

where $v \in V(\Sigma, R)$, $u \in U(\Sigma, R)$, $g_1 \in T(\Phi, R)G(\Delta, R)$, and all the factors are uniquely determined. Moreover, if $g_{\mu\mu} = 1$ then $g_1 \in G(\Delta, R)$.

The set Φ is the disjoint union of sets:

$$\Phi = \Sigma^+ \cup \Delta \cup \Sigma^-,$$

and, in other words, the Chevalley–Matsumoto theorem says that if $g_{\mu\mu}$ is invertible, then an element g of Chevalley group $G(\Phi, R)$ can be expressed as a product of the element from a Levy factor of the proper parabolic subgroup $G(\Delta, R)$ and two factors from the unipotent radicals of this parabolic subgroup and its opposite [38].

Thus, the Chevalley–Matsumoto theorem yields, that getting by elementary transformations a unit of the ring in the left corner of the matrix $g \in (G(\Phi, R), \mu)$, we get the surjective stability of the K_1-functor for the embedding $\Delta \to \Phi$, where Δ and Φ are of the same type.

Therefore, the problem is to transform a matrix $g \in (G(\Phi, R), \mu)$ using multiplications by elements $e \in (E(\Phi, R), \mu)$ to a matrix with the invertible element $g_{\mu\mu}$.

Let us agree to use the notation $e \in E(\Delta \to \Phi, R)$ for the element $e \in E(\Delta, R) \subset G(\Phi, R)$ and the notation $e \in (E(\Delta \to \Phi, R), \mu)$ if we want to specify the representation.

Denote by $Eq(\Phi, \mu)$ a set of equations that determine the orbit $G(\Phi, R)v^{\mu}$ of the highest weight vector v^{μ}. It is known (see [18, 41]) that this set consists of quadratic equations.

Now we can define three types of vectors. Denote by $Um_{\mu}(R, \Phi)$ the set of unimodular columns (rows) of length $n = \dim \pi$; by $Um'_{\mu}(R, \Phi)$ the set of unimodular columns (rows) of length n which satisfy the set of equations $Eq(\Phi, \mu)$; and by $Um''_{\mu}(R, \mu)$ the set of unimodular columns (rows) of length n which can be completed up to a matrix $g \in (G(\Phi, R), \mu)$. It is clear that $Um''_{\mu}(R, \Phi) \subset Um'_{\mu}(R, \Phi) \subset Um_{\mu}(R, \Phi)$. The difference between $Um'_{\mu}(R, \Phi)$ and $Um''_{\mu}(R, \Phi)$ is measured by the corresponding K-functor.

The following result can be derived from the proof of the Chevalley–Matsumoto theorem.

PROPOSITION 1. *Let* $(g_1, \dots, g_n) \in Um'_{\mu}(R, \Phi)$ *and* $g_1 \in R^*$. *There exists* $e \in (E(\Phi, R), \mu)$, *such that* $(eg)_i = 0$, $i \neq 1$.

Consider now the weight diagram of the type (E_7, ω_7), i.e., the diagram of the microweight representation of $G(\mathrm{E}_7, R)$ with the highest weight ω_7 (see Figure 3). Let us number the weights of the representation according to Figure 4.

LEMMA 1. *Let* $(g_1, \dots, g_i, \dots, g_{-i}, \dots, g_{-1}) \in Um'_{\omega_7}(R, \mathrm{E}_7)$, $i = 1, \dots, 28$. *Suppose* $g_1 = 0$ *and* $(g_2, \dots, g_{28})$ *is unimodular. Then* $(g_2, \dots, g_{28}) \in Um'_{\omega_6}(R, \mathrm{E}_6)$.

Proof. In fact, we have to check that in the conditions of the lemma the elements of the column $(g_2, \dots, g_{28})$ satisfy $Eq(\mathrm{E}_6, \omega_6)$. This fact immediately follows from the full description of the sets $Eq(\mathrm{E}_6, \omega_6)$ and $Eq(\mathrm{E}_7, \omega_7)$ in [40, Theorem 3]; see also [3, 11, 40].

Here we sketch another approach which is based on the elementary recipe on how one can find equations from the sets $Eq(\mathrm{E}_6, \omega_6)$ and $Eq(\mathrm{E}_7, \omega_7)$ using the Chevalley–Matsumoto theorem.

Take an element $g \in G(\mathrm{E}_7, R)$ with the invertible entry $g_{\mu\mu}$. Then the column $(g)_{\lambda\mu}$, $\lambda \in \Lambda$, belongs to $Um'_{\omega_7}(R, \mathrm{E}_7)$ and its elements satisfy $Eq(\mathrm{E}_7, \omega_7)$. Let us find some of these equations.

By the Chevalley–Matsumoto theorem there is the element $e \in E(\mathrm{E}_7, R)$ of the form $e = \Pi x_{\alpha}(\xi)$, $\xi \in R$, α runs all the roots from $\Sigma = \mathrm{E}_7 \setminus \mathrm{E}_6$ taken in a fixed order, such that $eg \in G(\mathrm{E}_6, R)$. This means that $(eg)_{\lambda\mu} = 0$ for all $\lambda \neq \mu$. Let Γ be the set of all weights λ, such that $\mu - \lambda \in \Sigma$. It is clear that by choosing appropriate values of ξ in the element e one can obtain zeros on the entries $(eg)_{\lambda\mu}$, $\lambda \in \Gamma$. But all the entries $(eg)_{\lambda\mu}$, $\lambda \notin \Gamma$, should be zeros as well. It remains to calculate their values and equalize them to zero. This gives us 28 equations from $Eq(\mathrm{E}_7, \omega_7)$.

Using Figures 3 and 4, we see that we have to calculate $(eg)_{i\mu}$, $i = -28, \dots, -1$, or, suppressing μ in the notation, the elements $(eg)_i$, $i = -28, \dots, -1$.

Straightforward calculations of the entries $(eg)_{i\mu}$, $i = -28, \dots, -1$, by formula $(*)$ and Figure 3 yield that up to the choice of signs, which is immaterial in this case (and which can be determined using the algorithm described in [40]), there is the following set of equations from $Eq(\mathrm{E}_7, \omega_7)$:

$$g_1g_{-28} \pm g_8g_5 \pm g_9g_4 \pm g_{10}g_3 \pm g_{11}g_2 \pm g_7g_6 = 0,$$
$$g_1g_{-27} \pm g_{12}g_6 \pm g_{13}g_5 \pm g_{14}g_4 \pm g_{15}g_3 \pm g_{16}g_2 = 0,$$
$$g_1g_{-26} \pm g_{12}g_8 \pm g_{13}g_7 \pm g_{17}g_4 \pm g_{18}g_3 \pm g_{19}g_2 = 0,$$
$$g_1g_{-25} \pm g_{12}g_9 \pm g_{14}g_7 \pm g_{20}g_3 \pm g_{17}g_5 \pm g_{21}g_2 = 0,$$
$$g_1g_{-24} \pm g_{12}g_{10} \pm g_{15}g_7 \pm g_{18}g_5 \pm g_{20}g_4 \pm g_{22}g_2 = 0,$$
$$g_1g_{-23} \pm g_{12}g_{11} \pm g_{16}g_7 \pm g_{19}g_5 \pm g_{21}g_4 \pm g_{22}g_3 = 0,$$
$$g_1g_{-22} \pm g_{13}g_9 \pm g_{14}g_8 \pm g_{17}g_6 \pm g_{23}g_3 \pm g_{24}g_2 = 0,$$
$$g_1g_{-21} \pm g_{13}g_{10} \pm g_{15}g_8 \pm g_{18}g_6 \pm g_{23}g_4 \pm g_{25}g_2 = 0,$$
$$g_1g_{-20} \pm g_{13}g_{11} \pm g_{16}g_8 \pm g_{24}g_4 \pm g_{25}g_3 \pm g_{19}g_6 = 0,$$
$$g_1g_{-19} \pm g_{14}g_{10} \pm g_{15}g_9 \pm g_{20}g_6 \pm g_{23}g_5 \pm g_{26}g_2 = 0,$$
$$g_1g_{-18} \pm g_{14}g_{11} \pm g_{16}g_9 \pm g_{21}g_6 \pm g_{24}g_5 \pm g_{26}g_3 = 0,$$
$$g_1g_{-17} \pm g_{15}g_{11} \pm g_{16}g_{10} \pm g_{22}g_6 \pm g_{25}g_5 \pm g_{26}g_4 = 0,$$
$$g_1g_{-16} \pm g_{17}g_{10} \pm g_{18}g_9 \pm g_{20}g_8 \pm g_{23}g_7 \pm g_{27}g_2 = 0,$$
$$g_1g_{-15} \pm g_{17}g_{11} \pm g_{19}g_9 \pm g_{21}g_8 \pm g_{24}g_7 \pm g_{27}g_3 = 0,$$
$$g_1g_{-14} \pm g_{18}g_{11} \pm g_{19}g_{10} \pm g_{22}g_8 \pm g_{25}g_7 \pm g_{27}g_4 = 0,$$
$$g_1g_{-13} \pm g_{20}g_{11} \pm g_{21}g_{10} \pm g_{22}g_9 \pm g_{26}g_7 \pm g_{27}g_5 = 0,$$
$$g_1g_{-12} \pm g_{23}g_{11} \pm g_{24}g_{10} \pm g_{25}g_9 \pm g_{26}g_8 \pm g_{27}g_6 = 0,$$
$$g_1g_{-11} \pm g_{17}g_{15} \pm g_{18}g_{14} \pm g_{20}g_{13} \pm g_{28}g_2 \pm g_{23}g_{12} = 0,$$
$$g_1g_{-10} \pm g_{16}g_{17} \pm g_{19}g_{14} \pm g_{21}g_{13} \pm g_{24}g_{12} \pm g_{28}g_3 = 0,$$
$$g_1g_{-9} \pm g_{18}g_{16} \pm g_{19}g_{15} \pm g_{22}g_{13} \pm g_{25}g_{12} \pm g_{28}g_4 = 0,$$
$$g_1g_{-8} \pm g_{20}g_{16} \pm g_{21}g_{15} \pm g_{22}g_{14} \pm g_{26}g_{12} \pm g_{28}g_5 = 0,$$
$$g_1g_{-7} \pm g_{23}g_{16} \pm g_{24}g_{15} \pm g_{25}g_{14} \pm g_{26}g_{13} \pm g_{28}g_6 = 0,$$
$$g_1g_{-6} \pm g_{20}g_{19} \pm g_{21}g_{18} \pm g_{22}g_{17} \pm g_{27}g_{12} \pm g_{28}g_7 = 0,$$
$$g_1g_{-5} \pm g_{23}g_{19} \pm g_{24}g_{18} \pm g_{25}g_{17} \pm g_{27}g_{13} \pm g_{28}g_8 = 0,$$
$$g_1g_{-4} \pm g_{23}g_{21} \pm g_{24}g_{20} \pm g_{26}g_{17} \pm g_{27}g_{14} \pm g_{28}g_9 = 0,$$
$$g_1g_{-3} \pm g_{23}g_{22} \pm g_{25}g_{20} \pm g_{26}g_{18} \pm g_{28}g_{10} \pm g_{27}g_{15} = 0,$$
$$g_1g_{-2} \pm g_{24}g_{22} \pm g_{25}g_{21} \pm g_{26}g_{19} \pm g_{27}g_{16} \pm g_{28}g_{11} = 0,$$

plus one more complicated equation, which corresponds to the angle $2\pi/3$ between roots—in the sense of [40, Theorem 3], where the technique of internal modules is used for the description of the sets $Eq(\mathrm{E}_6, \omega_6)$ and $Eq(\mathrm{E}_7, \omega_7)$. (Let us mention that the form of the set of equalities above agrees with the results of [17, 25], that all equations of microweight representations come from the systems of type D_n.)

Modulo g_1 the first 27 equations above include g_i, $i = 2, \dots, 28$ and coincide with the set $Eq(\mathrm{E}_6, \omega_6)$—see [10, 40, 41]—they are exactly 27 quadratic equations defining the orbit of the highest weight vector for the minimal representation of E_6. This implies that in the conditions of Lemma 1 the unimodular vector $(g_2, \dots, g_{28})$ satisfies $Eq(\mathrm{E}_6, \omega_6)$ and, therefore, $(g_2, \dots, g_{28}) \in Um'_{\omega_6}(R, \mathrm{E}_6)$. ∎

Consider Figure 1, which depicts the diagram of microweight representation (E_6, ω_6) with the numbering of weights as shown in Figure 2.

LEMMA 2. *Let the ring R satisfy the condition* $\dim\mathrm{Max}(R) \leq 4$. *Then for any row $a = (a_{\lambda_1}, \dots, \alpha_{\lambda_{27}}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$ there exists $e \in (E(\mathrm{E}_6, R), \omega_6)$, such that $\langle (ae)_{\lambda_1}, (ae)_{\lambda_{18}} \rangle = R$.*

Proof. Denote by Γ the set of weights λ_i, such that $\mu - \lambda_i \notin \Phi$. It is easy to see that these are the weights λ_i, $i = 18, \dots, 27$ (the notation $i, \dots, j$ means all numbers between i and j).

Consider $a = (a_{\lambda_1}, \dots, \alpha_{\lambda_n}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$. Let us choose in each irreducible component A_i of the space $\mathrm{MaxSpec}(R)$ a maximal ideal $\mathfrak{u}_i$ which does not belong to other components, and set $\mathfrak{u} = \prod \mathfrak{u}_i$. The ring $R/\mathfrak{u}$ is semilocal and we can find an element $e \in (E(\mathrm{E}_6, R), \omega_6)$, such that $(ae)_{\lambda_1} \equiv 1 \pmod{\mathfrak{u}}$. Since $a = (a_{\lambda_1}, \dots, \alpha_{\lambda_{27}}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$, using Proposition 1, we can choose e, such that $(ae)_{\lambda_i} \equiv 0 \pmod{\mathfrak{u}}$, $i \neq 1$. Let $ae = a_1$. Applying the condition ASR_6 to the subrow $((a_1)_{\lambda_1}, \dots, (a_1)_{\lambda_4}, (a_1)_{\lambda_6}, (a_1)_{\lambda_{10}})$ of the row a_1, we can find $e_1 \in (E(A_5 \to \mathrm{E}_6, R), \omega_1)$, such that the row $((a_1 e_1)_{\lambda_2}, \dots, (a_1 e_1)_{\lambda_{27}})$ is unimodular. Since $\alpha_2 \notin A_5 \to \mathrm{E}_6$ we have

$$(a_1 e_1)_{\lambda_i} \equiv 0 \pmod{\mathfrak{u}}, \qquad i = 5, 7, 8, 9, 11, \dots, 27.$$

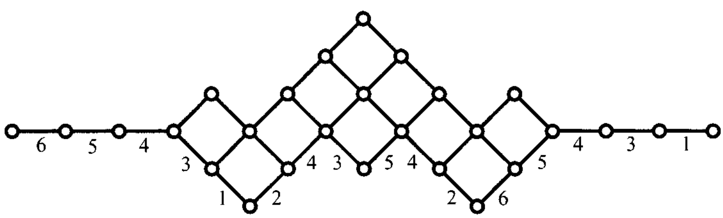

FIG. 1. (E_6, ω_6).

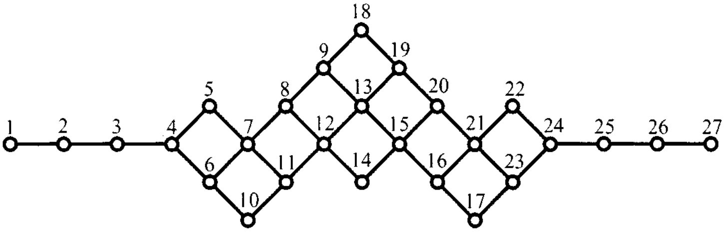

FIG. 2. (E_6, ω_6).

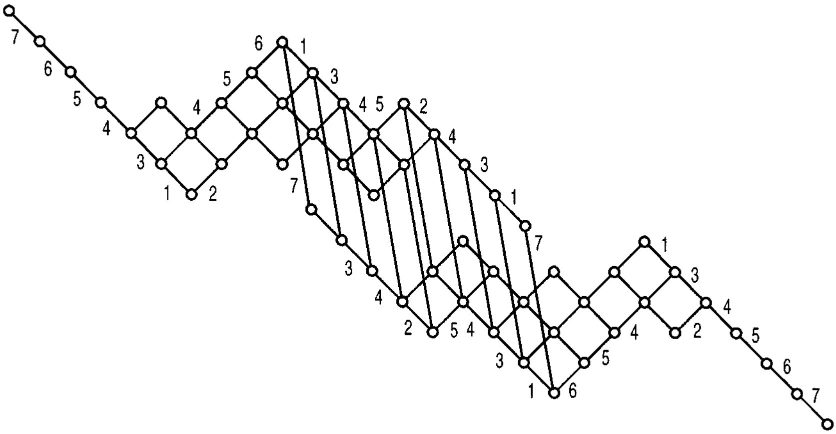

FIG. 3. (E_7, ω_7).

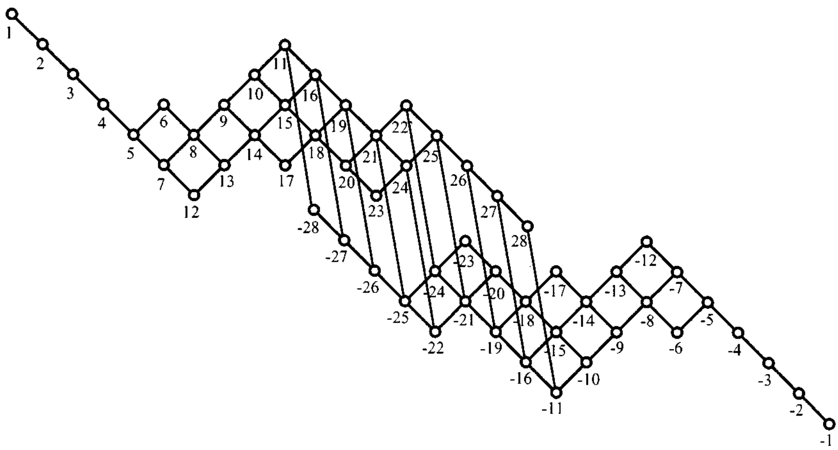

FIG. 4. (E_7, ω_7).

Let $a_1 e_1 = a_2$. Set $\mathfrak{a} = \langle (a_2)_{\lambda_i} \rangle$, $\lambda_i \in \Gamma$. The row $((a_2)_{\lambda_2}, \dots, (a_2)_{\lambda_{17}})$ is unimodular modulo ideal $\mathfrak{a}$. Thus, there exist $t_2, \dots, t_{17} \in R$, such that $t_2(a_2)_{\lambda_2} + \cdots + t_{17}(a_2)_{\lambda_{17}} \equiv 1 - (a_2)_{\lambda_1} \pmod{\mathfrak{a}}$. Since $\mu - \lambda_i \in \Phi$, $i = 2, \dots, 17$, we can find $e_2 \in (E(\mathrm{E}_6, R), \omega_6)$, such that $(a_2 e_2)_{\lambda_1} \equiv 1 \pmod{\mathfrak{a}}$ and $\langle (a_2 e_2)_{\lambda_i} \rangle = \mathfrak{a}$, $\lambda_i \in \Gamma$. Therefore, the row $((a_2 e_2)_{\lambda_1}, (a_2 e_2)_{\lambda_{18}}, \dots, (a_2 e_2)_{\lambda_{27}})$ is unimodular. Besides, $(a_2 e_2)_{\lambda_i} \equiv 0 \pmod{\mathfrak{u}}$, $i = 18, \dots, 27$. Set $\langle (a_2 e_2)_{\lambda_1} \rangle = \mathfrak{n}$ and consider $R/\mathfrak{n}$. Since $\mathfrak{n} + \mathfrak{a} = R$ where $\mathfrak{a} \subset \mathfrak{u}$, standard argumentation with the multiplicative system $S = R \setminus \bigcup \mathfrak{u}_i$ [4, 35, 36] shows that $\dim\mathrm{Max}(R)/\mathfrak{n} \le 3$. Let $a_2 e_2 = a_3$. The row $((a_3)_{\lambda_{18}}, \dots, (a_3)_{\lambda_{27}})$ is unimodular modulo $\mathfrak{n}$. It remains to mention that Theorem 2.1 of [27] (case $\mathrm{D}_{n-1} \to \mathrm{D}_n$) does not require that a row $(a_1, \dots, a_n)$ belong to $Um'_{\omega_1}(\mathrm{D}_n, R)$, but only that the group $G(\mathrm{D}_n, R)$ acts on $Um_{\omega_1}(\mathrm{D}_n, R)$. Since $\dim\mathrm{Max}(R) = 3$ implies ASR_5, there exists $e_3 \in (E(\mathrm{D}_5 \to \mathrm{E}_6, R), \omega_1)$, such that $(a_3 e_3)_{\lambda_{18}} \equiv 1 \pmod{\mathfrak{n}}$. At last, obviously, $(a_3 e_3)_{\lambda_1} = (a_3)_{\lambda_1}$. Therefore, $\langle (a_3 e_3)_{\lambda_1}, (a_3 e_3)_{\lambda_{18}} \rangle = R$.

THEOREM 1. *Let* $\dim\mathrm{Max}(R) \le 4$. *Then the homomorphism*

$$\nu \colon K_1(\mathrm{E}_6, R) \to K_1(\mathrm{E}_7, R)$$

is surjective.

Proof. Let $g \in (G(\mathrm{E}_7, R), \omega_7)$. Then $(\lambda_1(g), \dots, \lambda_{-1}(g)) \in Um''_{\omega_7} \cdot (\mathrm{E}_7, R)$.

Let

$$\mathfrak{a} = \langle (\lambda_6(g), \lambda_8(g), \dots, \lambda_{11}(g), \lambda_{13}(g), \dots, \lambda_{28}(g), \\ \lambda_{-28}(g), \dots, \lambda_{-1}(g) \rangle.$$

The row $(\lambda_1(g), \dots, \lambda_5(g), \lambda_7(g), \lambda_{12}(g))$ is unimodular modulo $\mathfrak{a}$, and, using SR_7, we can find $e \in (E(\mathrm{A}_6 \to \mathrm{E}_7, R), \omega_1)$, such that $(\lambda_2(eg), \dots, \lambda_5(eg), \lambda_7(eg))$ is unimodular modulo $\mathfrak{a}$. Since

$$\langle (\lambda_6(eg), \lambda_8(eg), \dots, \lambda_{11}(eg), \lambda_{13}(eg), \dots, \lambda_{28}(eg), \\ \lambda_{-28}(eg), \dots, \lambda_{-1}(eg) \rangle = \mathfrak{a},$$

the row $(\lambda_2(eg), \dots, \lambda_{-1}(eg))$ is unimodular.

Set $eg = g_1$. Now let $\mathfrak{a} = \langle \lambda_{-28}(g_1), \dots, \lambda_{-1}(g_1) \rangle$. Then $\langle \lambda_2(g_1), \dots, \lambda_{28}(g_1) \rangle + \mathfrak{a} = R$. Thus, there exist $t_2, \dots, t_{28} \in R$, such that $\sum \lambda_i(g_1) t_i \equiv 1 - \lambda_1(g_1) \pmod{\mathfrak{a}}$, $i = 2, \dots, 28$. Denote by Γ the set of weights λ_i, $i = 2, \dots, 28$. Let us set

$$e_1 = \prod x_{\beta_i}(t_i), \qquad \beta_i = \mu - \lambda_i, \qquad \lambda_i \in \Gamma.$$

Applying $(*)$, we have

$$\lambda_1(e_1g_1) = \lambda_1(g_1) + \sum_{i=2}^{28} \lambda_i(g_1)t_i + u,$$

where $u \in \mathfrak{a}$. Thus, $\lambda_1(e_1g_1) \equiv 1 \pmod{\mathfrak{a}}$ and the row

$$(\lambda_1(e_1g_1), \lambda_{-28}(e_1g_1), \dots, \lambda_{-1}(e_1g_1))$$

is unimodular.

Let $g_2 = e_1g_1$ and $\mathfrak{a} = \langle \lambda_1(g_2), \lambda_{-1}(g_2)\rangle$. The row $(\lambda_{-28}(g_2), \dots, \lambda_{-2}(g_2))$ is unimodular modulo $\mathfrak{a}$. From the proof of surjective stability of the K_1-functor for the embedding $\mathrm{D}_5 \to \mathrm{E}_6$ (see [22]), it follows that there exists $e_2 \in (E(\mathrm{E}_6 \to \mathrm{E}_7, R), \omega_6)$, such that

$$(\lambda_{-28}(e_2g_2), \lambda_{-11}(e_2g_2), \dots, \lambda_{-2}(e_2g_2))$$

is unimodular modulo $\mathfrak{a}$. Since $\langle \lambda_1(e_2g_2), \lambda_{-1}(e_2g_2)\rangle = \mathfrak{a}$, the row

$$(\lambda_1(e_2g_2), \lambda_{-28}(e_2g_2), \lambda_{-11}(e_2g_2), \dots, \lambda_{-2}(e_2g_2), \lambda_{-1}(e_2g_2))$$

is unimodular. Then the row

$$(\lambda_1(e_2g_2), \dots, \lambda_{11}(e_2g_2), \lambda_{28}(e_2g_2), \lambda_{-28}(e_2g_2), \lambda_{-11}(e_2g_2), \dots, \lambda_{-1}(e_2g_2))$$

is also unimodular.

Let $g_3 = e_2g_2$ and $\mathfrak{a} = \langle \lambda_1(g_3), \dots, \lambda_{11}(g_3), \lambda_{-28}(g_3)\rangle$. Take the row

$$(\lambda_{28}(g_3), \lambda_{-11}(g_3), \dots, \lambda_{-1}(g_3)).$$

Arguing as in [27] and using the condition ASR_6, we can obtain $e_3 \in (E(\mathrm{D}_6 \to \mathrm{E}_7, R), \omega_1)$, such that

$$\lambda_{28}(e_3g_3) \equiv 1 \pmod{\mathfrak{a}},$$

$$\lambda_{-11}(e_3g_3) \equiv 0 \pmod{\mathfrak{a}}, \dots, \lambda_{-2}(e_3g_3) \equiv 0 \pmod{\mathfrak{a}}.$$

Since $\langle \lambda_1(e_3g_3), \dots, \lambda_{11}(e_3g_3), \lambda_{-28}(e_3g_3)\rangle = \mathfrak{a}$, the row

$$(\lambda_1(e_3g_3), \dots, \lambda_{11}(e_3g_3), \lambda_{-28}(e_3g_3), \lambda_{28}(e_3g_3))$$

is unimodular.

Let $g_4 = e_3g_3$. Let us apply condition ASR_6 to the elements

$$\lambda_1(g_4), \dots, \lambda_{11}(g_4), \lambda_{-28}(g_4).$$

There exist $t_7, \dots, t_{11} \in R$, such that every maximal ideal, containing the ideal

$$\langle \lambda_7(g_4) + t_7\lambda_{-28}(g_4), \dots, \lambda_{11}(g_4) + t_{11}\lambda_{-28}(g_4)\rangle,$$

also contains $\langle \lambda_7(g_4), \dots, \lambda_{-28}(g_4)\rangle$. Let $e_4 \in (E(A_5 \to E_7, R), \omega_1)$ be the corresponding element making these transformations. Since $\langle \lambda_1(g_4), \dots, \lambda_6(g_4)\rangle = \langle \lambda_1(e_4 g_4), \dots, \lambda_6(e_4 g_4)\rangle$, every maximal ideal $\mathfrak{n}$ containing $\langle \lambda_1(g_5), \dots, \lambda_4(g_5)\rangle$, where $g_5 = e_4 g_4$, also contains the ideal $\mathfrak{a} = \langle \lambda_1(g_4), \dots, \lambda_{11}(g_4), \lambda_{-28}(g_4)\rangle$. Besides, $\lambda_{28}(g_5) \equiv 1 \pmod{\mathfrak{a}}$. Therefore, the row

$$(\lambda_1(g_5), \dots, \lambda_{11}(g_5), \lambda_{28}(g_5))$$

is unimodular. Thus, the row $(\lambda_1(g_5), \dots, \lambda_{28}(g_5))$ is unimodular.

Denote $\mathfrak{a} = \langle \lambda_1(g_5)\rangle$. By Lemma 1 we have

$$\lambda_2(\bar{g}_5), \dots, \lambda_{28}(\bar{g}_{28}) \in Um'_{\omega_6}(E_6, R/\mathfrak{a}),$$

where $\bar{g}_i \in R/\mathfrak{a}$. Lemma 2 provides that there exists $e_5 \in (E(E_6 \to E_7, R), \omega_6)$, such that the row $(\lambda_2(e_5 g_5), \lambda_{11}(e_5 g_5))$ is unimodular modulo $\mathfrak{a}$. Since $a_7 \notin E_6 \to E_7$, we have $\langle \lambda_1(e_5 g_5)\rangle = \mathfrak{a}$. Therefore, we have the unimodular row

$$(\lambda_1(e_5 g_5), \lambda_2(e_5 g_5), \lambda_{11}(e_5 g_5)).$$

Let $g_6 = e_5 g_5$. It is clear that there exists $e_6 \in (E(E_7, R), \omega_7)$, such that $\lambda_3(e_6 g_6) = 1$. ∎

Now we introduce the condition on a ring which generalizes Vaserstein's condition used in the proof of surjective stability of the K_1-functor for the case of the orthogonal groups $B_{n-1} \to B_n$, $D_{n-1} \to D_n$ (see [36]).

Recall how Vaserstein's condition looks for the particular case $\Phi = D_n$. A ring R satisfies the condition V_n if for any unimodular row

$$(a_1, \dots, a_n, a_{-n}, \dots, a_{-1}),$$

with elements in R there exist elements $t_1, \dots, t_n, t_{-n}, \dots, t_{-1}$ from R, such that

1. $\sum_{i=1}^{-1} t_i a_i = 1$,
2. $(t_1, \dots, t_n, t_{-n}, \dots, t_{-1}) \in Um'_{\omega_1}(D_n, R)$.

The second condition means that elements t_i, $i = 1, \dots, -1$, satisfy the equality $\sum_{i=1}^{n} t_i t_{-i} = 0$.

Suppose that a ring R satisfies both the conditions SR_n and V_{n-1}. Then the proof of surjective stability for $D_{n-1} \to D_n$ goes as follows (compare 39).

Consider the diagram of the representation (D_n, ω_1) with the natural numbering of weights (see Figures 5 and 6).

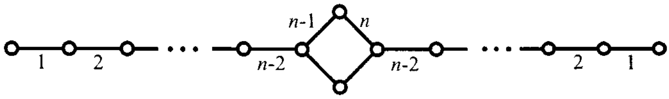

FIG. 5. (D_n, ω_1).

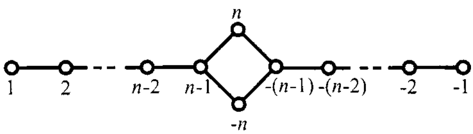

FIG. 6. (D_n, ω_1).

Let $g \in (G(D_n, R), \omega_1)$ and $(g_1, \dots, g_n, g_{-n}, \dots, g_{-1})$ be its first row. Using the condition SR_n, we can find elements $t_2, \dots, t_n \in R$, such that the row

$$(g'_2, \dots, g'_n, g'_{-n}, \dots, g'_{-1})$$

is unimodular, where

$$\begin{aligned} g'_i &= g_i + t_i g_1, \qquad & i &= 2, \dots, n, \\ g'_i &= g_i, & i &= -n, \dots, -2, \\ g'_{-1} &= g_{-1} + \sum_{i=2}^{n} t_i g_{-i}. \end{aligned}$$

Let $e \in (E(D_n, R), \omega_1)$ make these transformations and denote $g' = ge$. Analogously, using SR_n, one can choose $t'_2, \dots, t'_n \in R$, such that the row

$$(g''_2, \dots, g''_n, g''_{-n}, \dots, g''_{-2})$$

is unimodular, where

$$\begin{aligned} g''_i &= g'_i, \qquad & i &= 2, \dots, n, \\ g''_i &= g'_i + t'_i g'_{-1}, & i &= -n, \dots, -2. \end{aligned}$$

Let the element $e_1 \in (E(D_n, R), \omega_1)$ make these transformations and denote $g'' = g'e_1$. Applying V_{n-1} to the unimodular row

$$(g''_2, \dots, g''_n, g''_{-n}, \dots, g''_{-2}),$$

one can find $t''_2, \dots, t''_n, t''_{-n}, \dots, t''_{-2}$, such that

1. $\sum_{i=2}^{-2} t''_i g''_i = 1 - g''_1$,
2. $(t''_2, \dots, t''_n, t''_{-n}, \dots, t''_{-2}) \in Um'_{\omega_1}(D_{n-1}, R)$.

Let $\beta_i = \omega_1 - \lambda_i$, $i = 2, \dots, -2$, and set

$$e_2 = \prod_{i=2}^{-2} x_{-\beta_i}(t_i).$$

Denote $g''' = g'' e_2$. Calculating g'''_1, we have

$$\begin{aligned} g'''_1 &= g''_1 + \sum_{i=2}^{n} t''_i g''_i + \sum_{i=2}^{n} (g''_{-i} + t''_i g''_{-1}) t''_{-i} \\ &= g''_1 + \sum_{i=2}^{-2} t''_i g''_i + g''_1 \sum_{i=2}^{n} (t''_i t''_{-i}) = g''_1 + 1 - g''_1 = 1, \end{aligned}$$

which finishes the proof.

Let n be the dimension of the basic representation π with the highest weight μ.

DEFINITION. We say that the ring R satisfies the condition $V_\mu(\Phi, R)$ if for each unimodular vector $(r_1, \dots, r_n)$ there exist elements $t_1, \dots, t_n \in R$, such that

1. $\sum_{i=1}^{n} t_i r_i = 1$,
2. $(t_1, \dots, t_n) \in Um'_\mu(\Phi, R)$.

For the case $\Phi = D_n$ and $\mu = \omega_1$, this condition is converted to V_n. The following theorem combines the condition used by Stein and the generalized Vaserstein condition for the particular case of $E_6 \to E_7$ embedding.

THEOREM 2. *Let the ring R satisfy the conditions ASR_6 and $V_{\omega_6}(E_6, R)$. Then the homomorphism*

$$\nu: K_1(E_6, R) \to K_1(E_7, R)$$

is surjective.

Proof. Let $g \in (G(E_7, R), \omega_7)$. Then $(\lambda_1(g), \dots, \lambda_{-1}(g)) \in Um''_{\omega_7} \cdot (E_7, R)$. As in the proof of Theorem 1, one can get a unimodular row

$$(\lambda_1(g), \lambda_{-28}(g), \dots, \lambda_{-1}(g)),$$

using the condition SR_7. Moreover, the row

$$(\lambda_1(g), \dots, \lambda_6(g), \lambda_{-28}(g), \dots, \lambda_{-1}(g))$$

is also unimodular. Let $\mathfrak{a} = \langle \lambda_i(g) \rangle$, $i = 1, \dots, 6, -28, \dots, -7$. Applying SR_6 to the elements $\lambda_{-6}(g), \dots, \lambda_{-1}(g)$, one can find $e \in (E(\mathrm{A}_5 \to \mathrm{E}_7, R), \omega_1)$, such that $(\lambda_{-6}(eg), \dots, \lambda_{-2}(eg))$ is unimodular modulo $\mathfrak{a}$, and $\langle \lambda_i(eg) \rangle = \mathfrak{a}$, $i = 1, \dots, 6, -28, \dots, -7$. Then the row

$$(\lambda_1(eg), \dots, \lambda_6(eg), \lambda_{-28}(eg), \dots, \lambda_{-2}(eg))$$

is unimodular.

Set $eg = g_1$ and $\mathfrak{a} = \langle \lambda_1(g_1), \dots, \lambda_6(g_1) \rangle$. Using the condition $V_{\omega_6}(\mathrm{E}_6, R)$ with respect to the row $(\lambda_{-28}(g_1), \dots, \lambda_{-2}(g_1))$, one can find $(t_2, \dots, t_{28}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$, such that

$$\sum_{i=2}^{28} t_i r_i = 1 - \lambda_{-1}(g_1) \pmod{\mathfrak{a}}.$$

Let $\beta_i = \lambda_i - \lambda_1$, $i = -28, \dots, -1$, and

$$e_1 = \prod_{i=2}^{-28} x_{-\beta_i}(t_i).$$

Computing $\lambda_{-1}(g_2)$ where $g_2 = e_1 g_1$, we have

$$\lambda_{-1}(g_2) = \lambda_{-1}(g_1) + t_2 \lambda_{-2}(g_1) + \cdots + t_{28} \lambda_{-28}(g_1) + \sum_{i=2}^{28} u_i t_i + v_1 g_1,$$

where $u_i = u_i(t_i)$, $v_1 = v(t_i)$.

Straightforward computations using $(*)$ yield that $u_i = 0$, since

$$(t_2, \dots, t_{28}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$$

and $v_1 = 0$ as a linear combination of u_i.

Therefore,

$$\begin{aligned} \lambda_{-1}(g_2) &= \lambda_{-1}(g_1) + \sum_{i=2}^{28} t_i \lambda_{-i}(g_1) + a \\ &= 1 - \lambda_{-1}(g_1) + \lambda_{-1}(g_1) + a = 1 + a, \end{aligned}$$

where $a \in \mathfrak{a}$. Since the ideal $\langle \lambda_1(g_2), \dots, \lambda_6(g_2) \rangle$ is still $\mathfrak{a}$, then the row

$$(\lambda_1(g_2), \dots, \lambda_6(g_2), \lambda_{-1}(g_2))$$

is unimodular. It follows immediately, that the row

$$(\lambda_1(g_2), \dots, \lambda_{11}(g_2), \lambda_{-28}(g_2), \lambda_{-1}(g_2))$$

is unimodular. Replacing g_2 by appropriate $e_2 g_2 = g_3$, $e_2 \in (E(D_6 \to E_7, R), \omega_1)$, we find that the row

$$(\lambda_1(g_3), \dots, \lambda_{11}(g_3), \lambda_{-28}(g_3), \lambda_{28}(g_3))$$

is unimodular and

$$\lambda_i(g_3) \equiv 0 \pmod{\mathfrak{a}}, \qquad i = -11, \dots, -2,$$

where $\mathfrak{a} = \langle \lambda_1(g_3), \dots, \lambda_{11}(g_3), \lambda_{-28}(g_3) \rangle$. Now, arguing as in Theorem 1 and applying ASR_6, we can find $e_{e_3} \in (E(D_6 \to E_7, R), \omega_1)$, such that the row

$$(\lambda_1(e_3 g_3), \dots, \lambda_{11}(e_3 g_3), \lambda_{-28}(e_3 g_3))$$

is unimodular. Let $g_4 = e_3 g_3$. Using SR_6, one can get a unimodular row

$$(\lambda_2(g_5), \dots, \lambda_{11}(g_5), \lambda_{28}(g_5)),$$

where $g_5 = e_4 g_4$ and $e_4 \in (E(A_5 \to E_7, R), \omega_1)$.

Consider now the unimodular row $(\lambda_2(g_5), \dots, \lambda_{28}(g_5))$. Using the condition $V_{\omega_6}(E_6, R)$, we can find $(t_2, \dots, t_{28}) \in Um'_{\omega_6}(E_6, R)$, such that

$$\sum_{i=2}^{28} t_i \lambda_i(g_5) = 1 - \lambda_1(g_1).$$

Take the element $e_5 \in (E(E_7, R), \omega_7)$, which adds all the elements of

$$(\lambda_2(g_5), \dots, \lambda_{28}(g_5))$$

to the element $\lambda_1(g_5)$ with the coefficients t_i.

Then $g_1(e_5 g_5) = 1$ since $(t_2, \dots, t_{28}) \in Um'_{\omega_6}(E_6, R)$, and e_5 adds all the elements $\lambda_i(g_5)$, $i = -28, \dots, -1$, to $\lambda_1(g_5)$ with zero coefficients. ∎

To conclude, we formulate the theorem concerning the stability of the K_1-functor for the embedding $D_5 \to E_6$. The proof uses arguments similar to those in Theorem 2 and the description of the sets $Eq(D_5, \omega_5)$ and $Eq(E_6, \omega_6)$.

THEOREM 3. *Let the ring R satisfy the conditions SR_5 and $V_{\omega_5}(D_5, R)$. Then the homomorphism*

$$\nu \colon K_1(D_5, R) \to K_1(E_6, R)$$

is surjective.

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