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Journal of Algebra 300 (2006) 363–375

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Thompson-like characterizations of the solvable radical ☆

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Received 29 August 2005

Available online 27 March 2006

Communicated by Eamonn O'Brien

Dedicated to Charles Leedham-Green on the occasion of his 65th birthday

Abstract

We prove that the solvable radical of a finite group G coincides with the set of elements y having the following property: for any $x \in G$ the subgroup of G generated by x and y is solvable. This confirms a conjecture of Flavell. We present analogues of this result for finite-dimensional Lie algebras and some classes of infinite groups. We also consider a similar problem for pairs of elements.

☆ Guralnick was partially supported by the National Science Foundation Grant DMS 0140578. Kunyavskiĭ and Plotkin were partially supported by the Ministry of Absorption (Israel), the Israel Science Foundation founded by the Israel Academy of Sciences, Center of Excellence Program, the Minerva Foundation through the Emmy Noether Research Institute of Mathematics, and the EU networks HPRN-CT-2002-00287 and INTAS 00-566. Shalev was partially supported by the Israel Science Foundation and the Bi-National Science Foundation United States–Israel.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.03.001

1. Introduction

The first motivating result for this paper is the following famous theorem of J. Thompson [25] (see also [8]): a finite group G is solvable if and only if every 2-generated subgroup of G is solvable.

Our main aim is to prove the following extension of Thompson's theorem.

Theorem 1.1. *Let G be a finite group, and let $R(G)$ be the solvable radical of G (namely the maximal solvable normal subgroup of G). Then $R(G)$ coincides with the set of all elements $y \in G$ with the following property: for any $x \in G$ the subgroup generated by x and y is solvable.*

This result was conjectured by P. Flavell in 1997 [9] (see also [10]). This also trivially implies another conjecture from the same paper: a Sylow p-subgroup P of the finite group G is contained in the solvable radial of G if and only if $\langle P, x\rangle$ is solvable for all $x \in G$. Indeed the theorem above implies the same result for any subgroup of a finite group.

Our proof of Theorem 1.1 invokes the classification of finite simple groups, and uses the so-called "one and a half generation" of almost simple groups, proved by Guralnick and Kantor [13] using probabilistic arguments (see Theorem 3.3 below). Thus our result may be regarded as yet another demonstration of the power of probabilistic and counting methods in group theory (see the survey paper [23] for further background).

Theorem 1.1 can be extended to some classes of infinite groups (see Theorems 4.1 and 4.4 below). It has an obvious Lie-algebraic counterpart (Theorem 2.1). We call a criterion given in Theorem 1.1 the Thompson-like characterization of the solvable radical.

We also strengthen Theorem 1.1, showing that, if G is a finite group, and $y_1, y_2 \in G$ are two elements both outside $R(G)$, then there exists $x \in G$ such that $\langle x, y_i\rangle$, $i = 1, 2$ are both nonsolvable (see Theorem 6.4 below). It is intriguing that this result is best possible, in the sense that it cannot be extended for three elements.

Apart from the results, our paper contains various related questions and conjectures which we plan to consider in the future.

2. Lie algebras

We start with a Lie-algebraic counterpart of Theorem 1.1 which will give us important hints to its proof.

Theorem 2.1. *Let L be a finite-dimensional Lie algebra defined over a field k of characteristic zero, and let $R(L)$ be the solvable radical of L (namely the maximal solvable ideal of L). Then $R(L)$ coincides with the set of elements $y \in L$ with the following property: for any $x \in L$ the subalgebra generated by x and y is solvable.*

Proof. If $y \in R(L)$, then for any $x \in L$ the subalgebra generated by x and y contains a solvable ideal with one-dimensional quotient and is therefore solvable. We shall give three proofs for the reverse inclusion.

1st proof. Suppose $y \notin R(L)$. We have to prove that there is $x \in L$ such that the subalgebra generated by x and y is not solvable. After factoring out $R(L)$, we are reduced to proving this in the case where L is semisimple. Clearly, it is enough to consider the case where L is simple. In that case the result follows from [15] where it is proved that for any nonzero element y of a simple Lie algebra L there is x such that x and y generate L.

2nd proof. Suppose y has the property stated in the theorem. We have to prove that $y \in R(L)$. Consider the sequence of words $v_n(x, y)$ defined inductively by the following rule:

$$v_1(x, y) = x, \qquad v_{n+1}(x, y) = \bigl[v_n(x, y), [x, y]\bigr], \qquad \ldots .$$

This sequence can be used for characterization of $R(L)$: according to [3, Theorem 3.7], $y \in R(L)$ if and only if for any x there exists n such that $v_n(x, y) = 0$.

Let now x be an arbitrary element of L. Since the subalgebra generated by x and y is solvable, it satisfies the identity $v_n(x, y) \equiv 0$ for some n [3, Theorem 3.4], and we are done.

3rd proof. Assume that $y \notin R(L)$. Set $L(y) = \{x \in L \mid \langle x, y\rangle \text{ is solvable}\}$. Since the condition that $\langle x, y\rangle$ is solvable is given by a certain set of words being 0, $L(y)$ is closed in the Zariski topology. We induct on $\dim L$. Since L is dense in its Zariski closure (in some algebraic closure of k), we may assume that k is algebraically closed. By induction, L is simple (as in the first proof). If L has rank 1 (i.e. $L \cong sl(2)$), the result is straightforward (for any element is in only finitely many Borel subalgebras and any maximal subalgebra is a Borel). If the rank of L is larger than 1, it is straightforward to see that there is a maximal parabolic subalgebra P of L such that $y \in P \setminus R(P)$ (see [14, Lemma 2.1] for an easy proof of this for groups). Thus, by induction, there exists $x \in P$ with $\langle x, y\rangle$ not solvable. □

Let us now consider the case of finite-dimensional Lie algebras of positive characteristic p. However, the structure of modular Lie algebras, especially those in characteristics 2 and 3, is quite a bit more complicated than that of Lie algebras in characteristic zero. We pose the following natural problem:

Question 2.2. Is every simple finite-dimensional Lie algebra generated by two elements?

If the underlying field is algebraically closed of characteristic $p > 3$ one can use the available classification to check this case by case, and a positive answer seems very likely.

We point out that, though the problem above is open, at least solvability of finite-dimensional Lie algebras can be determined by pairs of elements. The key to this is the analog of Thompson's theorem on minimal simple groups (in characteristic zero, this is quite easy from the basic structure theorems—the modular case is quite a bit more difficult, but still not as difficult as Thompson's result).

We thank A. Premet for pointing out the following result of Schue [22]. Premet also pointed out that one can also prove this using [21].

Lemma 2.3. *Let $p > 3$ be a prime and let F be an algebraically closed field of characteristic p. Let L be a finite-dimensional Lie algebra over F such that every proper subalgebra of L is solvable. Then $L/R(L) \cong sl(2)$, where $R(L)$ is the solvable radical of L.*

There was a mistake in [22] but it does not affect the statement above. Also, Schue only concludes that $L/R(L)$ is 3-dimensional—but the only simple three-dimensional Lie algebra in characteristic larger than 3 is $sl(2)$.

This easily yields:

Theorem 2.4. *Let $p > 3$ be a prime and let F be an infinite field of characteristic p. Let L be a finite-dimensional Lie algebra over F. Assume that every pair of elements in L generate a solvable Lie algebra. Then L is solvable.*

Proof. Let E be the algebraic closure of F and set $M = E \otimes_F L$. Since the set of pairs $(x, y) \in L \times L$ that generate a solvable Lie algebra is closed (in the Zariski topology) and since $L \times L$ is dense in $M \times M$, it follows that M satisfies the same hypotheses. So we may assume that F is algebraically closed. Induct on $\dim L$. If $R(L) \neq 0$, we may pass to the quotient. So we may assume that $R(L) = 0$ and every proper subalgebra is solvable. By the lemma, this implies that $L \cong sl(2)$. This algebra is clearly generated by 2 elements and is not solvable, a contradiction. □

For $p > 5$, this can be deduced from [12, Theorem 3.1].

3. Finite groups

In this section we prove Theorem 1.1. For brevity, let us introduce the following notion.

Definition 3.1. Let G be a group. We say that $y \in G$ is a radical element if for any $x \in G$ the subgroup generated by x and y is solvable. Denote by $S(G)$ the set of radical elements of G.

Note that in any group G we have $R(G) \subseteq S(G)$. Indeed, if $y \in R(G)$, then for any x the subgroup generated by x and y contains a solvable normal subgroup with cyclic quotient and is therefore solvable.

With this terminology, Theorem 1.1 says that if G is finite, then $R(G) = S(G)$.

Proof. It suffices to prove that $S(G) \subseteq R(G)$. One could try to mimic one of three proofs of Theorem 2.1. Thus, taking into account the second proof, Theorem 1.1 would immediately follow from the following conjecture.

Conjecture 3.2. [3, Conjecture 2.12] *For any finite group G there exists a sequence $u_n(x, y)$ such that $R(G)$ coincides with the set of elements y having the following property: for any $x \in G$ there is n such that $u_n(x, y) = 1$.*

However, [3] contains only partial results towards Conjecture 3.2.

The reduction argument used in the third proof also encounters some problems in the group case (say, for groups of Lie type over small fields).

Fortunately, the "one and a half generation" theorem of Guralnick and Kantor [13, Corollary of Theorem I on p. 745] allows us to imitate the first proof of Theorem 2.1. We state this theorem as follows.

Theorem 3.3. *Let G be a finite almost simple group with socle T. Then for every nonidentity element $y \in G$ there exists an element $x \in G$ such that $\langle x, y\rangle$ contains T.*

First, we use this to prove an auxiliary result.

Lemma 3.4. *Let G be a finite group. Suppose that N is a minimal normal subgroup of G and G/N is generated by the coset yN with y not in $C_G(N)$. Then there exists $x \in N$ with $G = \langle x, y\rangle$.*

Proof. We prove this by induction on $|G|$.

If N is abelian, then as N is an irreducible module for $\langle y\rangle$, $G = \langle x, y\rangle$ for any nontrivial $x \in N$.

So we may assume that $N = T^n$ where T is a nonabelian simple group and y transitively permutes the n copies of T. Set $C := C_G(N)$. Since $C \cap N = 1$ and N is the derived subgroup of G, $C = Z(G)$. If $C \neq 1$, then by minimality, $G/C = \langle xC, yC\rangle$ for some $x \in N$. Thus, $\langle x, y\rangle C$ contains CN and so $\langle x, y\rangle$ contains N, the derived subgroup of CN, and so $G = \langle x, y\rangle$. So by induction, $C = 1$.

We claim that we may assume that y has prime order. Suppose y has order pq where p is prime and $q > 1$. Set $w = y^q$ and let M be a normal subgroup of N minimal with respect to being normalized by w. By induction, $\langle M, w\rangle = \langle x, w\rangle$ for some $x \in M$. In particular, $\langle x, w\rangle$ contains a simple direct factor of N. Since y is transitive on the simple direct factors, it follows that $\langle y, x\rangle$ contains $\langle w, x\rangle$ and so contains all components of N, whence it contains N and $\langle y, x\rangle = \langle N, y\rangle$ as required.

If $n = 1$, the statement of the lemma is an immediate corollary of Theorem 3.3.

If $n > 1$, then n must be prime and y must permute the n copies of T. So we may assume that $y(t_1, \ldots, t_n)y^{-1} = (t_n, t_1, \ldots, t_{n-1})$. Choose $t_1, t_n \in T$ such that t_n is an involution and $T = \langle t_1, t_n\rangle$ (which is possible by another application of [13]). Take $x = (t_1, 1, \ldots, 1, t_n)$. Consider $J := \langle x, x^y\rangle$. Let π denote the projection of N onto the first copy of T. Clearly $\pi(J) = T$ (since $\pi(x) = t_1$ and $\pi(x^y) = t_2$). Also, x^2 is a nontrivial element of T (since t_n is an involution and t_1 is not). Thus, $[x^2, J] = T \leqslant J$. Since y permutes the copies of T transitively, this implies that $N \leqslant \langle x, y\rangle$, whence $G = \langle x, y\rangle$. $\square$

We are now able to prove that $S(G) \subseteq R(G)$. It is easy to see that $S(G/R(G)) = S(G)/R(G)$ (more precisely $gR(G) \in S(G/R(G))$ if and only if $g \in S(G)$). Factoring out $R(G)$, we may assume that G is semisimple (i.e. $R(G) = 1$). We have to prove that $S(G) = 1$.

We use the notion of the generalized Fitting subgroup $F^*(G)$. See [1] for the basic definitions and results. The important facts about $F^*(G)$ for us are that if $R(G) = 1$, then $F^*(G)$ is a direct product of nonabelian simple groups (the components of G) and the centralizer of $F^*(G)$ is trivial.

Since $R(G) = 1$, if $1 \neq y \in G$, then there is some component L of G such that y does not centralize L. Let $\Delta = \{L^g \mid g \in \langle y \rangle\}$ and set N to be the subgroup generated by the $L^g \in \Delta$. Then N is a direct product of these components. Consider $J := \langle N, y \rangle$. Observe that N is the unique minimal normal subgroup of J. Applying Lemma 3.4 shows that $J = \langle x, y \rangle$ for some $x \in N$. In particular, J is not solvable and so $S(G) = 1$.

The theorem is proved. $\square$

Since y being a radical element is equivalent to $\langle y^{\langle x \rangle} \rangle$ being solvable for all x, we have the following:

Corollary 3.5. *Let G be a finite group, let $y \in G$, and let $\langle y^G \rangle$ denote the minimal normal subgroup of G containing y. Then $\langle y^G \rangle$ is solvable if and only if the subgroup $\langle y^{\langle x \rangle} \rangle$ is solvable for all $x \in G$.*

4. Linear groups and PI-groups

In this section, we obtain extensions of our earlier results for some classes of infinite groups. We start with linear groups.

Theorem 4.1. *Let K be a field. If $G \leqslant \mathrm{GL}(n, K)$, then $R(G) = S(G)$.*

Proof. As noted earlier we have $R(G) \subseteq S(G)$, and therefore it suffices to prove that the subgroup H generated by the set $S(G)$ is solvable (and thus coincides with the radical).

Let $H_1 = \langle g_1, \dots, g_s \rangle$ be a finitely generated subgroup of H where all g_i's are radical elements. Then H_1 is approximated by finite linear groups $G_\alpha = H_1/N_\alpha$, $\bigcap N_\alpha = 1$ in dimension n [16]. Each G_α is finite and is generated by the images of radical elements which are radical as well, and thus G_α is solvable. Since all G_α's are linear in dimension n, their derived length is bounded, say, by $k = k(n)$. Thus the group H_1 has derived length at most k. Each finitely generated subgroup of H lies in some H_1. Thus H is locally solvable. Since H is linear, it is solvable [26]. $\square$

For the case of PI-groups we use some facts from [17,19,20].

Definition 4.2. A group G is called a PI-group (PI-representable in terms of [17]) if G is a subgroup of the group of invertible elements of an associative PI-algebra over a field.

Linear groups are a particular case of PI-groups. It is known that every PI-group G has a unique maximal locally solvable normal subgroup $R(G)$ called the locally solvable radical of G and that the locally solvable radical of a finitely generated PI-group is solvable [17]. (For arbitrary groups the locally solvable radical may not exist, and for arbitrary PI- groups the locally solvable radical is not necessarily solvable.)

PI-groups have the following invariant series: $1 \lhd H_0 \lhd H \lhd G$ where H_0 is a locally nilpotent normal subgroup, H/H_0 is nilpotent and G/H is a linear group over a cartesian sum of fields [20].

We want to show that in a PI-group G the locally solvable radical coincides with $S(G)$. Let us introduce a useful notion of oversolvable group.

Definition 4.3. (cf. [19]) A group G is called oversolvable if it has an ascending normal series with locally nilpotent factors.

An arbitrary group G has the oversolvable radical $\widetilde{HP}(G) = \tilde{\eta}(G)$ (that is the unique maximal normal oversolvable subgroup). The quotient group $G/\tilde{\eta}(G)$ is semisimple with respect to the property of being locally nilpotent, i.e. $\eta(G/\tilde{\eta}(G)) = 1$ where $\eta(G)$ is the locally nilpotent radical of G (see [19] for the above facts). If G is finite, noetherian, or linear, $\tilde{\eta}(G)$ coincides with the solvable radical $R(G)$ [24].

Theorem 4.4. *If G is a PI-group, then $R(G) = \tilde{\eta}(G) = S(G)$.*

Proof. We consider three cases. (1) $G \leqslant \mathrm{GL}_n(P)$ where P is a field. (2) $G \leqslant \mathrm{GL}_n(K)$ where K is a cartesian sum of fields. (3) General case.

Case 1. If $G \leqslant \mathrm{GL}_n(P)$, where P is a field, then $R(G) = S(G)$ by Theorem 4.1.

Case 2. Suppose $G \leqslant \mathrm{GL}_n(K)$, where $K = \bigoplus_s P_s$ is a cartesian sum of fields. Consider the set of congruence subgroups U_s of $\mathrm{GL}_n(K)$ such that $\mathrm{GL}_n(K)/U_s \cong \mathrm{GL}_n(P_s)$. Since $\bigcap_s U_s = 1$, the group G lies in the cartesian product $\prod_s \mathrm{GL}_n(P_s)$. We have $G \subset \mathrm{GL}_n(K) \subset \prod_s \mathrm{GL}_n(P_s) \to \mathrm{GL}_n(P_s)$. Let H be the subgroup in G generated by the set $S(G)$. It is enough to show that H is solvable. Set $U'_s = H \cap U_s$. Then $\bigcap_s U'_s = 1$. Each H/U'_s can be viewed as a subgroup in $\mathrm{GL}_n(P_s)$ and is therefore generated by the radical elements. Thus they are all solvable of bounded derived length. Therefore H is solvable and $R(G) = S(G)$. Similar arguments give $R(G) = \tilde{\eta}(G)$.

Case 3. Let us first show that $R(G) \subseteq S(G)$. Let $g \in R(G)$, $h \in G$. Consider the subgroup $G_0 = \langle g, h \rangle$. We have $g \in R(G) \cap G_0$ and, consequently, $g \in R(G_0)$. By [17], the locally solvable radical of a 2-generated group is solvable. So the group G_0 is solvable as a cyclic extension of a solvable group.

Now we want to show that $R(G) = \tilde{\eta}(G)$. Let us first prove that $\tilde{\eta}(G) \subseteq R(G)$. We have to prove that the group $\tilde{\eta}(G)$ is locally solvable. We take a finitely generated subgroup G_0 in $\tilde{\eta}(G)$ and show that G_0 is solvable. Consider the locally solvable radical $R(G_0)$. Since G_0 is finitely generated, the radical $R(G_0)$ is solvable [17]. So it is enough to prove that the group $G_0/R(G_0)$ is solvable. We use the following result about the structure of PI-groups [20]: in every PI-group G the quotient group $\tilde{\eta}(G)/\eta(G)$ is solvable. We have $\eta(G) \subseteq R(G)$ as a locally nilpotent subgroup. Apply this to G_0. Since $G_0 \subseteq \tilde{\eta}(G)$, we have $\tilde{\eta}(G_0) = G_0$. Since $\tilde{\eta}(G_0)/\eta(G_0)$ is solvable and $\eta(G_0) \subseteq R(G_0)$, the group $\tilde{\eta}(G_0)/R(G_0)$ is solvable. Thus $G_0/R(G_0)$ is solvable, and hence so is G_0. The inclusion $\tilde{\eta}(G) \subseteq R(G)$ is proved.

Let us prove the opposite inclusion $R(G) \subseteq \tilde{\eta}(G)$. Recall that in every PI-group G there is a normal subgroup H which is an extension of a locally nilpotent group by a nilpotent group and such that G/H lies in $\mathrm{GL}_n(K)$ where K is a cartesian sum of fields. Then H is oversolvable. Therefore $H \subseteq \tilde{\eta}(G)$, and thus $H \subseteq R(G)$. Consider the group G/H and its subgroup $R(G)/H$. This is a locally solvable normal subgroup in G/H and thus lies in $R(G/H)$. The group G/H is linear and hence $R(G/H) = \tilde{\eta}(G/H)$. Then $R(G)/H \subseteq \tilde{\eta}(G)/H = \tilde{\eta}(G/H)$. Thus $R(G) \subseteq \tilde{\eta}(G)$ and $R(G) = \tilde{\eta}(G)$.

We are now able to prove that $S(G) \subseteq R(G)$. Let $g \in S(G)$. Denote by $\bar{g} \in G/H$ its image under the natural projection. Then $\bar{g} \in R(G/H)$ and thus $\bar{g} \in \tilde{\eta}(G)/H$. Then $g \in \tilde{\eta}(G) = R(G)$. $\square$

The above theorem has an obvious consequence which can be viewed as a natural generalization of Thompson's theorem:

Corollary 4.5. *A PI-group G is locally solvable if and only if every two-generated subgroup of G is solvable.*

Remark 4.6. In linear groups the locally solvable radical is solvable. From the above theorem it follows that in PI-groups the locally solvable radical is solvable modulo the locally nilpotent radical. Indeed, $\tilde{\eta}(G)/\eta(G)$ is solvable [20], and $\tilde{\eta}(G) = R(G)$ by Theorem 4.4.

Corollary 4.7. *If G is a PI-group then the normal subgroup $\langle y^G \rangle$ is locally solvable if and only if the element y is radical. In particular, if G is a finitely generated PI-group or a linear group, then $\langle y^G \rangle$ is solvable if and only if the element y is radical.*

5. Residually finite groups and Burnside-type problems

In this section, we consider residually finite groups and present some results and open problems. A trivial consequence of Theorem 1.1 is the following:

Corollary 5.1. *Let G be a residually finite group. Then $\langle y^G \rangle$ is residually finite solvable if and only if $\langle x, y \rangle$ is residually finite solvable for all $x \in G$.*

In infinite groups, there need not be a maximal solvable normal subgroup. However, it is still clear that the solvability of $\langle y^G \rangle$ implies that y is a radical element. So the following question naturally arises:

Question 5.2. Which groups G have the property that $\langle y^G \rangle$ is solvable for any radical element $y \in G$?

By our earlier results, finite and linear groups have this property. It is easy to construct examples of infinitely generated residually finite groups for which the property fails.

For instance, let G_i be a finite solvable group of derived length i generated by two elements x_i, y_i. Let P be the direct product of the G_i and $S < P$ the direct sum. Let

$y = (y_1, y_2, \ldots) \in P \setminus S$ and set $G = \langle S, y \rangle$. Clearly, P (and so G) is residually finite. Suppose that $x \in G$. Then $x = sy^j$ for some $s \in S$ and integer j. Setting $H = \langle x, y \rangle$, we see that $[H, H]$ is contained in a finite direct product of the G_i and so is solvable. Hence so is H, and thus y is a radical element of G. On the other hand, the normal closure of y in G is not solvable (since it contains $[G, G]$ and G is not solvable).

Indeed, the property in 5.2 fails even for finitely generated residually finite groups.

Proposition 5.3. *There exists a finitely generated residually finite group G such that for some radical element $y \in G$ the group $\langle y^G \rangle$ is not solvable.*

Proof. Let G be a three generated residually finite group that is not nilpotent and in which every two generated subgroup is a nilpotent group. Such groups exist due to Golod–Shafarevich [11].

Clearly every element of G is Engel and radical. Take any element $y \in G$ which does not belong to the locally nilpotent radical of G. We claim that the normal subgroup $\langle y^G \rangle$ is not solvable. For a solvable normal subgroup consisting of Engel elements should be locally nilpotent, and thus should be contained in the locally nilpotent radical, contradicting the choice of y. We conclude that y is a radical element of G, but its normal closure $\langle y^G \rangle$ is not solvable. □

Note that we have a stronger necessary condition for $\langle y^G \rangle$ to be solvable. If $\langle y^G \rangle$ is solvable of derived length d, then $\langle x, y \rangle$ is solvable of derived length at most $d + 1$ for every $x \in G$.

Question 5.4. Let G be a residually finite group generated by c elements. Fix a positive integer d and an element $y \in G$. Suppose that for all $x \in G$, $\langle x, y \rangle$ is solvable of derived length at most d. Does it follow that $\langle y^G \rangle$ is solvable of derived length at most $f(c, d)$ for some function f?

Note that this reduces to solving the problem for finite groups. An affirmative answer would give a characterization of the set of elements in a residually finite group whose normal closure is solvable.

The Golod example shows that every 2-generated subgroup being solvable does not imply that G is solvable for G a finitely generated residually finite group. So we ask:

Question 5.5. Let G be a residually finite group generated by c elements and d a positive integer. Suppose that $\langle x, y \rangle$ is solvable of derived length at most d for every pair of elements $x, y \in G$. Does it follow that G is solvable of derived length at most $f(c, d)$ for some function f?

This reduces to considering finite solvable groups. In fact, some of the questions above may be even posed in greater generality, namely for arbitrary groups. The obvious analogous problems for Lie algebras are also of interest. In particular:

Question 5.6. Let L be a Lie algebra in which every two elements generate a solvable subalgebra of derived length at most d. Does it follow that L is locally solvable?

The problems posed in this section may be regarded as Burnside-type problems; it would be interesting to find out whether Burnside-type techniques can help tackling them.

6. Pairs of elements

We can extend some of these results to pairs of elements not in the solvable radical.

We first state an even stronger version for Lie algebras.

Theorem 6.1. *Let L be a finite-dimensional Lie algebra over a field k of characteristic zero. Let X be a finite set of elements of $L \setminus R(L)$. Then there is some $y \in L$ such that for each $x \in X$, $\langle x, y\rangle$ is not solvable.*

Proof. Fix $x \in X$ and recall that $L(x) = \{y \in L \mid \langle x, y\rangle \text{ is solvable}\}$ is closed in the Zariski topology on L. By Theorem 2.1, $L(x)$ is a proper subvariety. An irreducible variety over an infinite field cannot be written as a finite union of proper closed subsets. Thus $\bigcup_{x\in X} L(x) \neq L$, whence the result. □

In fact, the theorem holds for any subset X of cardinality strictly less than the cardinality of k. We have a similar result for connected algebraic groups (and for connected Lie groups).

Theorem 6.2. *Let G be a connected algebraic group over an algebraically closed field k. Let X be any finite set of elements outside $R(G)$. Then there exists $y \in G(k)$ such that for each $x \in X$, $\langle x, y\rangle$ is not solvable.*

The proof is identical (for a fixed $x \in G(k)$, the set of $y \in G(k)$ with $\langle x, y\rangle$ solvable is a closed subvariety; if x is not in $R(G)$, it is proper by earlier results and so the union is also proper).

We have a weaker (but harder) result for finite groups. Clearly, we cannot take any finite subset as above (for example, if X is the set of all nontrivial elements of G). See [7] for references on this general problem.

Here is an example to show that even for subsets of size 3, there can be a problem. Consider $G = A_5$. Let $x_1 = (23)(45)$, $x_2 = (13)(45)$ and $x_3 = (12)(45)$. Note that no two of the x_i normalize a common Sylow 5-subgroup (since the product of any two distinct x_i has order 3 and the normalizer of a Sylow 5-subgroup is dihedral of order 10). Since there are six Sylow 5-subgroups, it follows that if y has order 5, $\langle x_i, y\rangle$ is dihedral of order 10 for exactly one x_i. If y has order 3, then either y fixes i for some $i \leqslant 3$ and so $\langle x_i, y\rangle$ is contained in the stabilizer of i (i.e. A_4) or y fixes 4 and 5 and so $\langle x_i, y\rangle$ is contained in an S_3 for each i. If y has order 2, then $\langle x_i, y\rangle$ is a dihedral group for each i. So we have shown that for any $y \in G$, for at least one i, $\langle x_i, y\rangle$ is solvable.

In order to prove the analogous result for finite groups (with $|X| = 2$), we need Theorem 1.4 from [7]. This extends the result of [13] (see Theorem 3.3 above). Both results are proved using probabilistic methods.

Theorem 6.3. [7, Theorem 1.4] *Let G be a finite almost simple group with socle S. If x, y are nontrivial elements of G, then there exists $s \in S$ such that S is contained in $\langle x, s\rangle$ and $\langle y, s\rangle$.*

The key point that we will use is that the subgroups $\langle x, s\rangle$ and $\langle y, s\rangle$ are not solvable.

We can now prove the main result of this section.

Theorem 6.4. *Let G be a finite group. Suppose that x and y are not in $R(G)$. Then there exists $s \in G$ such that $\langle x, s\rangle$ and $\langle y, s\rangle$ are not solvable.*

Proof. Suppose that G is a counterexample of minimal order. Thus, $R(G) = 1$ (or we could pass to $G/R(G)$). We can replace x and y by powers and so assume that they each have prime order (this is not essential to the proof given below—it does make it a bit easier to see the possibilities for the action of x or y).

Then G has a normal subgroup N (the generalized Fitting subgroup) that is the direct product of simple groups L_i, $1 \leqslant i \leqslant n$, and $C_G(N) = 1$. In particular, this implies that $\langle x^N\rangle$ is not solvable (for since x does not centralize N, $\langle x^N\rangle$ must intersect N in a nontrivial normal subgroup and any normal subgroup of N is a direct product of simple groups and in particular is not solvable). So already the hypotheses are satisfied in $\langle x, y, N\rangle$ and so the minimality hypothesis implies that $G = \langle x, y, N\rangle$.

The direct factors of N are the components of G. First suppose that there is some component L of G with neither x nor y in $C_G(L)$. Let M be the normal closure of L in G. Then we may assume that $M = N$ (otherwise, $C_N(M) \neq 1$ and we can pass to the smaller group $G/C_N(M)$ and arguing as above, we see that the normal closures of x and y are still not solvable).

Write $N = L_1 \times \cdots \times L_m$ with say $L = L_1$. If $m = 1$, then G is almost simple and we apply Theorem 6.3 to obtain the conclusion.

So assume that $m > 1$ and that $L_1^x = L_2$. If $L_1^y \neq L_1$ or L_2, choose $u, v \in L_1$ that generate L_1 and consider the element $s = uv^x v^y$. Note that $\langle s, s^{x^{-1}}\rangle$ projects onto L and so $\langle s, x\rangle$ is not solvable and similarly for $\langle s, y\rangle$.

If $L_1^y = L_2$, choose $u, v \in L$ such that $\langle u, v\rangle = \langle u, v^{xy^{-1}}\rangle = L$ (this can be done by [7, Theorem 1.2 or 1.4]). Then $s = uv^x$ satisfies the conclusion.

Finally suppose that $L_1^y = L_1$. So y induces a nontrivial automorphism of L_1. Then choose $u, v \in L$ (by Theorem 6.3) such that $L = \langle u, v\rangle = \langle u^{\langle y\rangle}, i = 0, 1, \ldots\rangle$ and again set $s = uv^x$.

The remaining case to consider is that for each component K of G, either x or y centralizes K (but not both by hypothesis).

Now write $N = N_x \times N_y$ where N_x is the direct product of the components not centralized by x and similarly for N_y. So $G = \langle N_x, x\rangle \times \langle N_y, y\rangle$. Now take $s = uv$ with $u \in N_x$ and $v \in N_y$ with $\langle x, u\rangle$ and $\langle y, v\rangle$ each nonsolvable. This completes the proof. □

Corollary 6.5. *Let G be a linear group over a field k. If x and y are not in the solvable radical of G, then there exists $s \in G$ with $\langle x, s\rangle$ and $\langle y, s\rangle$ not solvable.*

Proof. We may assume that G is finitely generated. By our main result on linear groups, there is a finite homomorphic image H of G in which the images of $\langle x^G\rangle$ and $\langle y^G\rangle$ are not solvable. So we may choose $\bar{s} \in H$ such that $\langle \bar{x}, \bar{s}\rangle$ and $\langle \bar{y}, \bar{s}\rangle$ are not solvable. Now take s to be any preimage of $\bar{s}$. □

7. Concluding remarks

Let us observe that certain important classes of groups and Lie algebras can be explicitly characterized in terms of two-variable identities: one can mention here classical results for finite-dimensional nilpotent Lie algebras (Engel) and finite nilpotent groups (Zorn [27]) and their recently obtained counterparts for finite-dimensional solvable Lie algebras [12] and finite (or linear) solvable groups [4,5] (see also [6]). Moreover, Engel identities were used by Baer to characterize explicitly the nilpotent radical of an arbitrary finite (and, more generally, noetherian) group [2]. Baer's theorem was extended to the locally nilpotent radical of linear groups and PI-groups [18,20] and to the nilpotent and the solvable radical of finite-dimensional Lie algebras [3]. These results give a certain hope for characterization of the solvable radical $R(G)$ of a finite group G in similar, Engel-like terms. However, the corresponding Conjecture 3.2 (see also [3]) is still far from being proved, and therefore less explicit descriptions of the solvable radical, such as the Thompson-like characterization of Theorem 1.1, are very useful.

Acknowledgments

The authors are very grateful to B. Plotkin and E. Zelmanov for inspiring suggestions and discussions, to V.D. Mazurov for pointing out Flavell's conjectures, and to two anonymous referees for a careful reading and valuable remarks.

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