WORD EQUATIONS IN SIMPLE GROUPS AND POLYNOMIAL EQUATIONS IN SIMPLE ALGEBRAS

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To Kolya Vavilov, friend and colleague, on the occasion of the 60th anniversary

ABSTRACT. We give a brief survey of recent results on word maps on simple groups and polynomial maps on simple associative and Lie algebras. Our focus is on parallelism between these theories, allowing one to state many new open problems and giving new ways for solving older ones.

There were different times: a time to throw stones, a time to divide and subtract. Now it is a time to add and multiply. Circumstances force us to focus on adding and multiplying.

From an interview of Mikhail Silin, first vice-rector on research of Gubkin Russian State University of Oil and Gas to the newspaper "Vestnik of Murmansk", December 4, 2009

1. INTRODUCTION

In this paper we discuss word maps

$$w \colon G^d \to G, \tag{1}$$

induced on any group G by a group word $w = w(x_1, \ldots, x_d)$ in $x_1, x_1^{-1}, \ldots, x_d, x_d^{-1}$ (= an element of the free group $\mathcal{F}_d$). We also consider polynomial maps

$$P \colon A^d \to A, \tag{2}$$

induced on any associative (or Lie) algebra over a field k (or a ring R) by an associative (or Lie) polynomial $P = P(X_1, \ldots, X_d)$ in d variables (= an element of the free associative (or Lie) algebra).

Both (1) and (2) are evaluation maps: one substitutes d-tuples of elements of the group G (algebra A) instead of the variables and computes the value by performing all group (algebra) operations. We are interested in surjectivity of maps (1) and (2), or, more generally, in description of their images. In lowbrow terms, we are interested in solvability of equations of the form

$$w(x_1, \ldots, x_d) = g, \tag{3}$$

or

$$P(X_1, \ldots, X_d) = M, \tag{4}$$

for every right-hand side, or for some "typical" right-hand side, or whether every element of the group (algebra) admits a representation as a product (sum) of finite number values of the word (polynomial) map, etc.

This setting is a particular case of the following one. Let Θ be a variety of algebras, H be an algebra in Θ, $W(X)$ be a free finitely generated algebra in Θ with generators $x_1, \ldots, x_d$. Fix $w \in W(X)$ and consider the word map $w \colon H^d \to H$. Varying Θ and $H \in \Theta$, we arrive at the problem of solvability of equations in different varieties over different algebras. In this note we restrict ourselves to the varieties of groups, associative algebras, and Lie algebras.

Whereas in the group case the theory has been intensely developing and led to several spectacular results, including answering some old-standing questions, on the ring-theoretic side much less is known, though some new approaches to not less old questions have been recently found. Our main goal in this survey, which does not pretend to be comprehensive, is to emphasize parallels between the two theories. We hope that this may

bring cross-fertilization effects in near future. With an eye towards such a unification, we put here more questions than answers.

The interested reader is referred to the monograph [Se] and surveys [Sh1], [Ni], [BGK] in what concerns the group case. Some references on the algebra case can be found in [KBMR], [BGKP]. We leave aside extremely interesting questions on the number of solutions of equations (3), (4) (or, in other words, on the structure of the fibres of maps (1), (2)). See [BGK] for an overview of some results in this direction.

2. Value sets

Given a word map (1), it is important to distinguish between the following objects:

- the value set in strict sense: $w(G) = \{g \in G : \exists (g_1, \ldots, g_d) \quad w(g_1, \ldots, g_d) = g\}$;
- the symmetrized value set $w(G)^{\pm}$ consisting of the elements of $w(G)$ and their inverses;
- the verbal subgroup $\langle w(G) \rangle$ of G generated by $w(G)$.

Respectively, given a polynomial map (2) of k-algebras, we distinguish between the value set $P(A) = \{a \in A : \exists (a_1, \ldots, a_d) \quad P(a_1, \ldots, a_d) = a\}$ and the vector space $\langle P(A) \rangle$ spanned by $P(A)$ over k.

It is usually much easier to describe $\langle w(G) \rangle$ and $\langle P(A) \rangle$ than the actual value sets.

2.1. **Surjectivity.** We start with the group case and ask whether word map (1) is surjective. More precisely,

- given a class of groups $\mathcal{G}$, we want to describe words w for which map (1) is surjective;
- given a class of words $\mathcal{W}$, we want to describe groups G for which map (1) is surjective.

In each of these set-ups arising problems range from very easy to extremely difficult, depending on the choice of the class. Here are some examples. Let us start with the first approach and take $\mathcal{G}$ to be the variety of all groups. Then the needed description is given by [Se, Lemma 3.1.1], where such words are called universal: they are all of the form $w = x^{e_1} \ldots x_d^{e_d} w'$ where w' is a product of commutators and $\gcd(e_1, \ldots, e_d) = 1$. Here is a parallel question for associative algebras:

Question 1. What are polynomials P such that the map $P \colon A^d \to A$ is surjective for all associative algebras A?

When we restrict the class $\mathcal{G}$, we arrive at more interesting and difficult questions. Answers heavily depend on this choice. For instance, a theorem of Rhemtulla (see [Se, Theorem 3.1.2]) says that if $\mathcal{G}$ consists of the free groups (and maybe also of some free products adjoined), then the behaviour of *any* non-universal word is very far from surjectivity: in the terminology explained below, any such word is of infinite width for any group of $\mathcal{G}$. That is why in this survey we prefer to stay away from equations in free (and close to free) groups; see, e.g., [CRK], particularly the introduction, for a comprehensive bibliographical survey of vast literature on this fascinating theory.

We mainly focus on another extreme case of *simple* groups which can also be viewed as a building block for some more general theory. Here is our first question.

Question 2. Let $\mathcal{G}$ be the class of simple groups G of the form $G = \mathbb{G}(k)$ where $k = \bar{k}$ is an algebraically closed field and $\mathbb{G}$ is a semisimple adjoint linear algebraic group. Is it true that for all $w \neq 1$ word map (1) is surjective?

Remarks 1.   (i) If we drop the assumption $k = \bar{k}$, or allow G to be not of adjoint type (i.e., consider quasisimple groups), there are easy counter-examples to surjectivity [My], [Bo]. On the other hand, if $G = SU(n)$ and $w(x, y)$ is any word not belonging to the second derived subgroup of $\mathcal{F}_2$, then the induced word map is surjective for infinitely many n [ET].

   (ii) If $\mathcal{G}$ is some infinite family of finite simple groups, then any power word $w = x^n$ gives rise to the word map which is not surjective for infinitely many groups (those whose order is not prime to n). A conjecture of Shalev, asserting that such phenomenon may arise only for power maps, turned out to be over-optimistic, see [JLO] for a counter-example.

  (iii) Going over to analogues of Question 2 for finite-dimensional simple associative and Lie algebras over an algebraically closed field, one immediately gets obvious obstructions: there are nontrivial (associative and Lie) polynomials that vanish identically; in addition, an associative polynomial map of matrix algebras may take only central or trace-zero values; even apart from these obvious obstructions, there are counter-examples to surjectivity, see [BGKP] and [KBMR]; if the ground field

is not algebraically closed, the situation is even more complicated (for example, it is an interesting question which of the surjectivity results of [KBMR], obtained over quadratically closed fields, survive over $\mathbb{R}$). Nevertheless, some of techniques developed for associative polynomials may turn out to be useful for attacking Question 2, see Remarks 3 and 4 below.

The opposite direction for studying surjectivity is more tricky, even if the class of words $\mathcal{W}$ consists of only one word. The first non-universal word to be considered is commutator.

2.1.1. *Commutator: Ore and beyond.* Let $w = xyx^{-1}y^{-1} \in \mathcal{F}_2$. The obvious necessary condition for the surjectivity of map (1) is $\langle w(G) \rangle = G$, i.e., G must be perfect. It is very far from being sufficient, even within the class of finite groups, see, e.g., [Is]. Making the further assumption that G is simple, one can say something much more positive, namely that map (1) is surjective in each of the following cases:

(i) G is finite (Ore's problem, whose long history has been finished in [LOST1]);
(ii) $G = S_\infty$, the infinite symmetric group [Or], [DR] (more generally, *any* non-power word is surjective on S_∞ [Ly]);
(iii) $G = \mathbb{G}(k)$, where $\mathbb{G}$ is a semisimple adjoint linear algebraic group over an algebraically closed field k [Ree] (and for many other similar cases, such as semisimple Lie groups, etc., see [VW]);
(iv) G is the automorphism group of some "nice" topological or combinatorial object (e.g., the Cantor set), as listed in [DR]; see also [BM].

In connection with (iv), it is worth quoting I. Rivin (see `http://mathoverflow.net/questions/77398/how-did-ores-conjecture-become-a-conjecture`): "It is a conjecture I attribute to myself, but probably goes back to the ancients, that in every reasonable simple group every element is a commutator". Of course, the whole point here is in the word "reasonable": as any meaningful principle, Rivin's principle is subject to breaking, see Section 3.4 for counter-examples. In a positive direction, we suggest the following question:

Question 3. For each of the infinite simple groups listed in [DR], is it true that any word map is surjective (or, at least, has a "large" image)? (In other words, is it true that in each such group one can solve word equations with "generic" right-hand side?)

We also note that the Ore property is not just a standing alone phenomenon in group theory. It has many interesting applications in algebraic topology (see [DR] and references therein) and birational geometry [Ku].

Remarks 2. There are several ways to go beyond Ore.

(i) First, representing $1 \neq g \in G$ in the form $g = [x, y]$, one may require to make the choice of x as "uniform" as possible. A typical result of such kind is a theorem of Gow [Gow]: every finite simple group G of Lie type contains a regular semisimple conjugacy class C such that each semisimple $g \in G$ can be represented in the form $g = [x, y]$ with $x \in C$. A similar result in the case where G is a split semisimple Chevalley group over an infinite field follows from the prescribed Gauss decomposition [EG]. The case where G is not split seems open.
(ii) One can require that x, y satisfy some additional properties. For example, in [DR] Ore's theorem on the infinite symmetric group has been strengthened by choosing x and y so that they generate a transitive subgroup of S_∞. It would be interesting to get similar results in the other cases listed above. For instance, one can ask how big can the subgroup $\langle x, y \rangle \subset G$ be. Note that we cannot guarantee $\langle x, y \rangle = G$, there are counter-examples among alternating groups (we thank A. Shalev for this remark).
(iii) Another well-known generalization is Thompson's conjecture asserting the existence of a conjugacy class C in every simple group G such that $C^2 = G$. This problem has a positive solution for the split semisimple algebraic groups [EG]. However, it is still open for finite simple groups, in spite of the major breakthrough made in [EG] (see also [YCW] and some more recent results (see, e.g., [BGK] for relevant references)). The case of the infinite symmetric group has been settled in [Gr]. It is a natural question whether there is a "gap" between Ore's and Thompson's conjectures, i.e., whether there is a simple group where Ore's conjecture holds and Thompson's does not. N. Gordeev pointed out that such a group exists: take $G = A_\infty$, the finitary alternating group. Then evidently each element of G is a commutator but in every fixed conjugacy class C of G any element moves a fixed number N of points, hence any product of two elements of C moves at most $2N$ points. Therefore $C^2 \neq G$.

(iv) In the case where L is a "classical" Lie algebra (i.e., the Lie algebra of a Chevalley group G over a sufficiently large field), most of the statements discussed above admit easy analogues when the bracket stands for the Lie operation. Say, the analogue of Ore's conjecture has been established in [Br], and an analogue of Gow's theorem can easily be obtained looking at the linear map $\varphi_x \colon L \to L$, $y \mapsto [x,y]$: its kernel is the centralizer of x, and one can conclude that in the representation $g = [x,y]$ the element x can be chosen from a fixed G-orbit (with respect to the adjoint action of G on L). Further, for the Lie algebras of such kind the analogue of Thompson's conjecture also holds true [Gorde]. However, all such generalizations seem unknown for non-split simple classical Lie algebras, simple Lie algebras of Cartan type in positive characteristic, and for infinite-dimensional Lie algebras. The first case to be explored is that of Kac–Moody algebras (as well as Kac–Moody groups, where the prescribed Gauss decomposition is known [MP]).

(v) Let now A be an associative algebra, and the polynomial under consideration be the additive commutator $P(X,Y) = XY - YX$. Let $[A,A]$ denote the subalgebra additively generated by the values of P. If $A \neq [A,A]$, there is no hope for the surjectivity of the map $P \colon A^2 \to A$. This happens whenever A is finitely generated over $\mathbb{Q}$ or, more generally, if A is any (not necessarily finitely generated) PI-algebra [Be], [Mes]. So reasonably interesting questions only arise for the kernel of the reduced trace map (trace zero matrices, for brevity). If $A = M_n(D)$, where D is a division algebra over a field and $n \geq 2$, then every trace zero matrix is an additive commutator [AR]. If, however, $n = 1$, the question is open (except for the special case of a central division algebra of prime degree over a local field [Ros]), and it is hard to believe in an easy affirmative answer (that would imply the affirmative solution of an old open problem on the cyclicity of any central simple algebra of prime degree p over a field of characteristic p, see [Ros] for details). From discussions with L. Rowen we got an impression that one should rather expect a negative answer, and a counter-example could arise in the algebra of generic matrices (see below). This shows that the questions posed in (iv) for non-split simple Lie algebras are probably answered in the negative. Note that a too straightforward attempt to extend the theorem of Amitsur–Rowen to $M_n(R)$, where R is an arbitrary ring, fails in general [RR2], [Ros], [Mes]. As to infinite-dimensional simple algebras, the questions seem to be totally unexplored, to the best of our knowledge.

2.1.2. *Engel words.* Apart from the commutator, there are some other interesting words for which the surjectivity is known in several cases; see, e.g., [BGK] for some relevant references. Here we only want to mention the most natural generalization of the commutator, the family of Engel words $e_n(x,y) = [[x,y],y]\ldots y]$. Their surjectivity has been only established for groups $G = \mathrm{PSL}(2,q)$ (under some assumptions on n and q) [BGG], $G = \mathrm{SU}(n)$ [ET], $G = \mathrm{PSL}_2(\mathbb{C})$ [Kl], [BGG] and for simple classical Lie algebras [BGKP]. An answer to the following question would be a natural continuation of [Ree].

Question 4. Let G be a connected semisimple adjoint linear algebraic group over an algebraically closed field. Is it true that every Engel word induces a surjective map $G^2 \to G$?

2.2. **Dominance: the Deligne–Sullivan trick and Amitsur's theorem.** We start with a theorem of Borel [Bo] providing a sketch of a somewhat new approach (the proof given in [La] is essentially the same as in the original paper). Our proof is based on using the generic division algebra (see [Fo2] for details on this important object, including the history of its creation). Let F be a field (for simplicity, assumed of characteristic zero), let n be a positive integer, let x_{ij}^k ($1 \leq i, j \leq n$, $k \in \mathbb{N}$) be independent commuting indeterminates. The F-subalgebra of $M_n(F[x_{ij}^k])$ generated by the matrices $X_k = (x_{ij}^k)$ is called a ring of generic matrices. Denote it by $R = F\{X\}$. It is a domain [Am2], and its ring of fractions $Q(R)$ is a central division algebra of dimension n^2 over its centre $Z(Q(R))$ [Am1]. The centre $Z(R)$ of R consists of the central polynomials (and is hence nontrivial for every n [Fo1],[Ra]). The centre $C(Q(R))$ is the field of quotients of $C(R)$. It is a long-standing open problem whether the field $C(Q(R))$ is a purely transcendental extension of F, see the surveys [Fo2], [LB], [ABGV].

Theorem 1. [Bo] *Let $w \in \mathcal{F}^d$ ($d \geq 2$) be a nontrivial word, and let G be a connected semisimple algebraic group over a field F. Then the induced morphism $w \colon G^d \to G$ of underlying algebraic varieties is dominant (i.e., its image is Zariski dense).*

Proof. We present a sketch of proof in the case where F is of characteristic zero. Several parts of the proof are exactly the same as in the original one. First, as dominance is compatible with any extension of the ground field, we may and will assume F to be algebraically closed of infinite transcendence degree. Next, we may assume that G is simple of type A_n (the reduction to the simple case and the passage from SL_n to the

other types are as in [Bo]). Further, it is enough to prove the dominance for the case where w is a product of commutators. Indeed, suppose that the theorem is proven for such words, and let $w \colon G^d \to G$ be an arbitrary word map. Consider the map $\tilde{w} \colon G^{2d} \to G$ defined as follows: $\tilde{w}(x_1, y_1, \dots, x_d, y_d) := w([x_1, y_1], \dots, [x_d, y_d])$. Then the image of $\tilde{w}$ is dense, hence so is the image of w. Thus henceforth we assume $w \in [\mathcal{F}_d, \mathcal{F}_d]$. Furthermore, it is enough to prove the dominance for the map

$$w \colon (\mathrm{GL}_n)^d \to \mathrm{SL}_n. \tag{5}$$

Indeed, the image of this map coincides with $w(\mathrm{SL}_n)$ because every $g \in \mathrm{GL}_n$ can be replaced with $g/\det(g)$.

Let us now argue by induction on the rank. The case of rank 1 is treated exactly as in [Bo]. The key point is the induction step. Assume that map (5) is dominant for the rank $n - 1$ and prove that it is dominant for the rank n. As in the original proof, it is enough to prove the existence of a matrix C in the image of w such that none of its eigenvalues equals 1.

To this end, it suffices to prove that the image contains a *generic* matrix with this property. Indeed, as soon as this is established, by specialization arguments (which are legitimate because of the assumption on the transcendence degree of F) we obtain a non-empty Zariski open set of needed matrices.

To prove the existence of a generic matrix with the required property, assume the contrary. Let C denote a generic matrix from the image of w. Since it has an eigenvalue 1, the characteristic polynomial $p_C(t)$ is divisible by $t - 1$, so $p_C(t) = (t - 1)p_1(t)$. The Cayley–Hamilton theorem then gives $(C - I_n)p_1(C) = 0$, which is in a contradiction with Amitsur's theorem stating that the ring of generic matrices is a domain. $\square$

Remark 3. In the proof presented above, the core induction argument is based on Amitsur's theorem, instead of the Deligne–Sullivan argument used in the original proof (which is based on going over to an anisotropic form of G and dates back to the unitary trick of Weyl). We believe that this interrelation between matrix groups and algebras is very important, and its potential is not exhausted. In particular, such an approach may be useful for proving the surjectivity of the word map for semisimple groups over algebraically closed fields and, more generally, for getting a more precise description of the image of the word map.

Let $w = \prod_j a_{i(j)}^{\pm 1}$. Consider the values of w on invertible matrices, which will be denoted by the letters a_i, $i = 1, \dots, k$. The matrix a_i satisfies the Cayley–Hamilton equation

$$a_i^n - \xi_1(a_i)a_i^{n-1} + \cdots + (-1)^l \xi_l(a_i)a_i^{n-l} + \cdots + (-1)^n \xi_n(a_i) = 0 \tag{6}$$

where ξ_l is the lth characteristic coefficient of the matrix a_i. In particular, $\xi_1 = \mathrm{tr}$, $\xi_n = \det$. In the zero characteristic case, the ξ_l are expressed via traces of powers. Equation (6) can be rewritten in the form

$$\left(\sum_{l=1}^{n-1} (-1)^{n-l} a_i^{n-l} \xi_{l-1}(a_i)\right) \det(a_i)^{-1} = a_i^{-1}.$$

Set $\psi(a_i) = \sum_{l=1}^{n-1} (-1)^{n-l} a_i^{n-l} \xi_{l-1}(a_i)$ and rewrite the product $w = \prod_j a_{i(j)}^{\pm 1}$, replacing $a_{i(j)}^{-1}$ with $\psi(a_{i(j)})$ and leaving the factors $a_{i(j)}$ unchanged. We will obtain a polynomial (in the signature enlarged by the characteristic coefficients, in other words, a polynomial with forms), which is a product of polynomials in one variable.

There are some optimistic considerations regarding the study of images of such polynomials. Let us look at the construction of homogeneous polynomials whose image does not coincide with the whole space [KBMR]. Let P be an arbitrary matrix polynomial, and let $c \in k$. Consider the product

$$F = (\lambda_1(P) - c\lambda_2(P))(\lambda_2(P) - c\lambda_1(P))P = ((1 + c^2 + 2c)\det(P) - c\,\mathrm{tr}(P)^2)P.$$

It vanishes if the ratio of the eigenvalues $\lambda_i(P)$ equals $c^{\pm 1}$ or they are both zero. Therefore the image of F does not contain nonzero matrices with such a ratio of eigenvalues, and this construction is essential for the nature of the problem. If we are given a nontrivial product of polynomials in one variable in which several variables enter, then a requirement on the resulting relation between eigenvalues cannot be determined by one factor (if its determinant is not equal to zero). Of course, ψ is obtained by canceling the determinant, but nevertheless there is some ground for optimism regarding making Borel's theorem more precise.

Remark 4. Let us give an example of a more complicated argument showing the existence of a unipotent element in the image of a multilinear polynomial P in the space of matrices of second order (assuming it has a noncentral value with nonzero trace).

So let $P(X_1, \dots, X_n)$ be a multilinear polynomial, and assume the contrary, i.e., P has no unipotent values. Fix all variables except for X_1. Then $P_{X_2, \dots, X_n}(X_1) = P(X_1, \dots, X_n)$ is a linear in X_1 function,

meeting the discriminant surface at points of total multiplicity 2. Since we assume the absence of unipotent values, each such point is either scalar or nilpotent. If one point, X_1^0, is a nonzero scalar, and the other, X_1^1, is nilpotent, then at the point $X_1^0 + X_1^1$ we obtain a matrix proportional to a unipotent one with some coefficient $\lambda \neq 0$, and then at the point $(X_1^0 + X_1^1)/\lambda$ we shall obtain a unipotent value.

Further, if at both points we have a scalar value, then the original value of P is scalar, and if both values are nilpotent, then the original value has zero trace. Therefore, if the value of $P(X_1, \ldots, X_n)$ is not scalar and has nonzero trace, then while moving any coordinate we observe coincidence of the intersection points with the discriminant surface, i.e., tangency to the discriminant surface.

Then one can extract square root of the discriminant $(\lambda_1 - \lambda_2)^2$. In other words, $q = \lambda_1(P(X_1, \ldots, X_n)) - \lambda_2(P(X_1, \ldots, X_n))$ is a polynomial in entries of $X_1, \ldots, X_n$. The group SL_2 acts by simultaneous conjugations on the X_i and is connected. Then, under such an action, q is mapped to $\pm q$, and connectedness guarantees the invariance of q, by the first fundamental theorem (established in [Pr] in characteristic zero and in [Do] in positive characteristic).

Since q is a polynomial in characteristic values of the products of the X_i, so are $\lambda_1(P), \lambda_2(P)$ because $\lambda_1(P) + \lambda_2(P) = \mathrm{tr}(P)$.

By the Cayley–Hamilton theorem, $(P - \lambda_1(P))(P - \lambda_2(P)) = 0$ which also contradicts Amitsur's theorem on zero divisors. Hence P does have unipotent values.

We hope that such kind of reasoning can be helpful in getting a more precise description of the image of the word map in the set-up of Borel's theorem.

Remark 5. Note that an idea of using generic matrices has been recently used, in a slightly different context of "universal localization", in [HSVZ1] and [Step], for getting subtle information on commutators in Chevalley groups over rings.

Remark 6. There are results of Borel's flavour, stating that for some infinite groups G any "generic" element $g \in G$ falls into the value set of any non-power word w [Ma], [DT].

Remark 7. In the case of Lie algebras, an analogue of Borel's theorem for Lie polynomials P which are not identically zero, was established in [BGKP] for semisimple Chevalley algebras (modulo several exceptions over fields of small characteristic), under the additional assumption that P is not an identity of $\mathfrak{sl}(2)$. It is not clear whether this assumption can be removed. The answer heavily depends on whether one can extend a construction of so-called 3-central polynomials (see, e.g., [Row, Theorem 3.2.21]) to the Lie case. (Recall that a polynomial P is called n-central is P^n is central but P is not.) If such polynomials exist, they provide an example of a map which is not dominant. Probably, multilinear Lie polynomials with such a property do not exist, and in this case one can drop the assumption mentioned above.

Remark 8. The case of associative algebras is much more tricky. First, one has to take into account the obvious obstructions to dominance, and assume that the polynomial P is non-central and contains at least one value with nonzero trace. Even under these assumptions, there are counter-examples to dominance for maps on $M_2(K)$; to avoid them, one has to make additional assumptions on P (say, to assume that it is semi-homogeneous). This assumption is not enough for 3×3-matrices. One can show that if P is a nonscalar polynomial such that not all its values on $A = M_3(K)$ are 3-scalar or traceless, then its value set $P(A)$ is dense in A (moreover, if N is the set of non-diagonalizable matrices, then $P(A) \cap N$ is dense in N). If $P(A)$ lies in $\mathfrak{sl}_3(K)$ and contains a matrix which is not 3-scalar, then $P(A)$ is dense in $\mathfrak{sl}_3(K)$.

If P is assumed multilinear, the main open question is the validity of the Kaplansky–L'vov conjecture which states that the image of P is either 0, or K, or $\mathfrak{sl}_n(K)$, or $M_n(K)$. As a first step, we would suggest to look at a downgraded version of this conjecture where we weaken it by allowing the image to be a dense subset of either $\mathfrak{sl}_n(K)$, or $M_n(K)$. As above, one of the key problems is the existence of n-central multilinear polynomials. This problem is related to subtle properties of division algebras and is not discussed in this survey. See [KBMR] for some details.

To sum up, we present a list of questions which seem to be a good starting point.

Question 5. Let $P \colon M_2(K)^d \to M_2(K)$.

  (i) If P is multilinear, what are its possible images for $K = \mathbb{R}$? In particular, does the Kaplansky–L'vov conjecture hold in this case?
  (ii) Let K be quadratically closed and P be an arbitrary (not necessarily homogeneous) polynomial. Is it true that its value set is either the set $\{F([x, y])\}$ of values of a traceless polynomial, or the whole $M_2(K)$ (up to Zariski closure)?

For arbitrary $n \geq 3$ we ask weaker questions.

Question 6. Let $P\colon M_n(K)^d \to M_n(K)$ be a multilinear polynomial.

(i) Can it be n-central?
(ii) Suppose that P is not n-central. Is it true that its image is then dense provided it contains a matrix with nonzero trace? Can it contain non-diagonalizable matrices?

Similar questions can be asked for traceless polynomials.

Remark 9. Being even more modest, one can start with multilinear Lie polynomials, asking similar questions. Even the case of algebras of small rank is open. Let us mention an analogue of the Kaplansky–L'vov conjecture.

Question 7. Let L be a Chevalley Lie algebra, and let P be a multilinear Lie polynomial. Is it true that the image of P on L is either 0, or L?

Remark 10. Even a further downgrading might be of interest, in view of eventual generalizations of Makar-Limanov's Freiheitsatz [ML] to the case of positive characteristic: given a polynomial map $P\colon M_n(K)^d \to M_n(K)$ such that $n \gg \deg P$, can one bound from above the codimension of its image by a "reasonable" function in n?

2.3. **Words with small image.** If G is a finite simple group, the verbal subgroup $\langle w(G) \rangle$ equals either 1 or G. However, for any G one can expect the existence of w such that the actual image $w(G)$ is fairly small (though different from 1). Such examples were recently constructed in [KN] and [Le].

Question 8. (N. Nikolov) Let G be a finite simple group, and let $X \subset G$ be a union of conjugacy classes. Does there exist w such that $w(G) = X$?

Note that a similar question for polynomials on matrix algebras over finite rings was answered in the affirmative in [Ch].

3. Width

3.1. **General words and polynomials.** In the situation where map (1) is not surjective (or this is not known), but $\langle w(G) \rangle = G$, one can ask how many elements of $w(G)$ (or of $w(G)^{\pm}$) are needed to represent every element of G. (In the case where $\langle w(G) \rangle \neq G$, one represents every element of $\langle w(G) \rangle$.) The smallest such number is called the w-width of G. We denote it by $\mathrm{wd}_w(G)$. If G does not have finite w-width, it is said to be of infinite w-width. The reader can find comprehensive discussions of this notion in [Se], [Ni] (see also [BGK] for a survey of some more recent developments in the case where G is a finite simple group). We focus here, as above, on parallels with the case of associative algebras.

First note that as soon as we establish, for some topological group G, any kind of Borel's dominance theorem, in the sense that the image of w contains a dense open subset, we immediately conclude by a standard argument that $\mathrm{wd}_w(G) \leq 2$. Thus this is the case if G is (the group of points of) a connected semisimple algebraic group over an algebraically closed field. For arbitrary algebraic groups over fields, one can still prove their finite width with respect to any nontrivial word and any element of the verbal subgroup [Mer]. If, however, we go over to algebraic groups over arbitrary rings, the situation changes dramatically, see the next section.

In the class of finite simple groups, dominance arguments do not make much sense. However, the observation made in [La] that word maps still have, in a sense, a large image when viewed within an infinite family of finite simple groups, gave rise to a series of wonderful results on uniform word width, christened by Shalev "Waring type properties". Making the long story short, we just mention the papers [LS], where the existence of such a uniform bound was established, and [LST1], [LST2], where, as a culmination of efforts of many people, the uniform bounds $\mathrm{wd}_w(G) \leq 2$ (resp. $\mathrm{wd}_w(G) \leq 3$) have been established for all sufficiently large finite simple (resp. quasisimple) groups and all words $w \neq 1$. See [BGK] for more references and details. Note that quite recently Shalev with his collaborators extended many results of this type to some simple algebraic groups over p-adic integers, see [Sh2] for a list of relevant questions. In the case of polynomial maps of algebras, we propose a notion parallel to word width in the group case.

Definition 1. Let $P(X_1, \ldots, X_d)$ be an associative (resp. Lie) polynomial, let A (resp. L) be an associative (resp. Lie) algebra over a ring R, let $P \colon A^d \to A$ (resp. $P \colon L^d \to L$) denote the induced map, and let V_P be the R-module spanned by the image of P. If there exists a positive integer m such that every element $v \in V_P$ can be represented as a sum of m values of P, we call the least m with such property the P-width of the algebra and denote it $\mathrm{wd}_P(A)$ (resp. $\mathrm{wd}_P(L)$). Otherwise, we say that the algebra is of infinite P-width.

We are not aware of any general results concerning this notion, beyond those that follow from the surjectivity or dominance (see, however, next sections for some particular cases). The following question seems conceptually important.

Question 9. Let k be an infinite field. Let $\mathcal{A}$ denote the class of finite-dimensional central simple k-algebras. Let $\mathcal{P}$ denote the class of associative polynomials such that none of them is central for some $A \in \mathcal{A}$. Is it true that all $A \in \mathcal{A}$ are of finite P-width for all $P \in \mathcal{P}$? If yes, is it true that $\sup_{P \in \mathcal{P}, A \in \mathcal{A}} \mathrm{wd}_P(A) < \infty$?

This question makes sense in more restrictive set-ups, when either some class of polynomials or some class of algebras, or both, are fixed (and may be narrow enough, even consisting of one element, see some examples below). Similar questions may be posed for Lie polynomials on finite-dimensional simple Lie algebras. Finally, in both cases (associative and Lie) one can also consider infinite-dimensional simple algebras, finitely or infinitely generated. All this seems completely unexplored. Below we consider a couple of more concrete settings.

3.2. **Commutator width.** The case of the width of various groups G with respect to $w = xyx^{-1}y^{-1}$ got much attention in the literature. Without pretending to giving a comprehensive survey (various aspects are reflected in [Se], [Ni], [HSVZ2], [BGK]), we only present some references.

- As mentioned above, the finite simple groups have commutator width 1. As to finite quasi-simple groups, it is at most 2, and all the cases when it actually equals 2 are listed [LOST2].
- As mentioned above, if G is a Chevalley group over an infinite field, its commutator width is at most 2, in view of Borel's theorem. However, this breaks down for Chevalley groups over rings, which tend to have very few commutators (the type of behaviour called "anti-Ore" in [HSVZ1]). In view of examples of groups of infinite commutator width [DV], we can also call it "anti-Waring". The situation improves if we go over to infinite matrices: most Chevalley groups of "infinite rank" return to Waring (if not to Ore) behaviour: it is known that their commutator width is at most 2 [HS], [DV]; similar results were recently obtained for other groups of infinite matrices, see [GH] and references therein. This gives rise to a natural question:

Question 10. Is it true that the groups of commutator width at most 2 listed in [DV] are of finite word width with respect to any nontrivial word w? Is the image of w "large" (i.e., do we have any analogue of Borel's theorem)?

Here are some parallel results and questions for the additive commutator $P(X, Y) = XY - YX$ in associative algebras.

- The commutator width of a matrix algebra $A = M_n(R)$ over any ring is at most two ([AR] for the case where R is a division algebra, [Ros] for an arbitrary commutative ring R, [Mes] in general).
- Moreover, in the representation $M = [X, Y] + [Z, T]$ one can fix X and Z which are good for every $M \in A$ (in [AR] only Z was fixed, and this was strengthened in [RR1], [Ros] and [Mes] using genericity arguments).

This gives rise to the following notion, slightly resembling the notion of one-and-a-half-generation of simple groups [Stei], [GK].

Definition 2. Let A be an associative algebra of commutator width m. If there exist $X_1, \ldots, X_s \in A$ such that every $M \in A$ can be represented in a form

$$M = [X_1, Y_1] + \cdots + [X_s, Y_s] + [Z_{s+1}, Y_{s+1}] + \cdots + [Z_m, Y_m],$$

and s is maximal with this property, we say that A is of fractional commutator width $m - \frac{s}{2m}$.

Thus the matrix algebras are of fractional commutator width one and a half (the result of [AR] gave only one and three-quarters).

One can define fractional commutator width for groups in a similar fashion and ask the following question.

Question 11. Which of the groups G having commutator width 2 are of fractional commutator width one and a half? one and three-quarters?

One can start with Chevalley groups of infinite rank [DV] and 14 finite quasisimple groups of width 2 from the list of [LOST2].

One can define fractional width for more general words and polynomials. We leave this to the interested reader.

3.3. Power width.
Again, we only briefly quote some results on power width in groups. For finite simple groups, the story which started with establishing the existence of uniform finite width in [MZ], [SW] led to almost conclusive results in [GM] where for many powers it was proved that this width equals 2. The uniform width for general finite groups was established in [NS]. For infinite groups (e.g., for Chevalley groups over rings), the problem is almost unexplored. As in the commutator case, some pathologies have been discovered (see the next section).

For matrix algebras $M_n(R)$, we only mention the pioneering paper [Va] where the problem of representing a matrix over a commutative ring as a sum of dth powers was considered, and some general results of finite power width flavour were obtained (see [KG] for the history of the problem and more references), and [Pu2] for some results for matrices over noncommutative rings, in particular, for central simple algebras.

3.4. Monsters.
It was unknown for a long time whether there exists a simple group of commutator width greater than 1. The first counter-example [BG] appeared in the context of symplectic geometry and gave a simple (infinitely generated) group of infinite commutator width. Later on, using various contexts, there were constructed simple groups of infinite commutator width which are finitely generated ([Mu1], using small cancellation theory) or finitely presented ([CF], among quotients of Kac–Moody groups); among groups appearing in [Mu1], there are those of arbitrary large finite commutator width. In [Mu2], there were constructed simple groups of infinite square width (and hence infinite commutator width, because of the equality $[x, y] = x^2(x^{-1}y)^2(y^{-1})^2$, showing that finite commutator width implies finite square width). We refer the interested reader to [GG] for a walk along the zoo of monsters with such anti-Ore and anti-Waring behaviour (and some conceptual explanations of such phenomena), which grew up in differential-geometric environment, following the spirit of [BG]; they are fed up using some advanced techniques, including quasi-morphisms and quantum cohomology.

It is an interesting question whether such monsters exist among Lie algebras. The case of finite-dimensional algebras of Cartan type over fields of positive characteristic was already mentioned above. As to infinite-dimensional algebras in characteristic zero, discussions with E. Zelmanov give a hint that there are infinitely generated simple Lie algebras of infinite width and finitely generated ones of arbitrary large finite width. Elaborating such examples could give a clue to understanding the situation with associative algebras.

As a final remark, we should mention that apart from Lie algebras, little is known on the problems discussed in this paper in the case of non-associative algebras; see, however, [Gordo] and [Pu1] for some particular results of this flavour. In our opinion, it would be interesting to study the image of more general polynomial maps for some classical examples, such as Cayley octonions, simple quadratic Jordan algebras, simple exceptional Jordan algebras HC_3, and the like.

REFERENCES

[Am1]   S. A. Amitsur, *On rings with identities*, J. London Math. Soc. 20 (1955), 464–470.
[Am2]   S. A. Amitsur, *The T-ideals of the free ring*, J. London Math. Soc. 20 (1955), 470–475.
[AR]   S. A. Amitsur, L. H. Rowen, *Elements of reduced trace 0*, Israel J. Math. 87 (1994), 161–179.
[ABGV]   A. Auel, E. Brussel, S. Garibaldi, U. Vishne, *Open problems on central simple algebras*, Transform. Groups 16 (2011), 219–264.
[BGG]   T. Bandman, S. Garion, F. Grunewald, *On the surjectivity of Engel words on PSL(2, q)*, Groups Geom. Dyn., 6 (2012), 409–439.
[BGK]   T. Bandman, S. Garion, B. Kunyavskiĭ, *Equations in simple matrix groups: algebra, geometry, arithmetic, dynamics*, preprint, 2012.
[BGKP]   T. Bandman, N. Gordeev, B. Kunyavskiĭ, E. Plotkin, *Equations in simple Lie algebras*, J. Algebra 355 (2012), 67–79.

[BG]   J. Barge, E. Ghys, *Cocycles d'Euler et de Maslov*, Math. Ann. 294 (1992), 235–265.
[BM]   A. Basmajan, B. Maskit, *Space form isometries as commutators and products of involutions*, Trans. Amer. Math. Soc. 364, (2012), 5015–5033.
[Be]   A. Ya. Belov, *No associative PI-algebra coincides with its commutant*, Sibirsk. Mat. Zh. 44 (2003), 1239–1254; English transl. in Siberian Math. J. 44 (2003), 969–980.
[Bo]   A. Borel, *On free subgroups of semisimple groups*, Enseign. Math. 29 (1983), 151–164; reproduced in Œuvres - Collected Papers, vol. IV, Springer-Verlag, Berlin–Heidelberg, 2001, pp. 41–54.
[Br]   G. Brown, *On commutators in a simple Lie algebra*, Proc. Amer. Math. Soc. 14 (1963), 763–767.
[CF]   P.-E. Caprace, K. Fujiwara, *Rank-one isometries of buildings and quasi-morphisms of Kac–Moody groups*, Geom. Funct. Anal. 19 (2010), 1296–1319.
[CRK]   M. Casals-Ruiz, I. Kazachkov, *On systems of equations over free partially commutative groups*, Mem. Amer. Math. Soc. 212 (2011), no. 999.
[Ch]   C.-L. Chuang, *On ranges of polynomials in finite matrix rings*, Proc. Amer. Math. Soc. 110 (1990), 293–302.
[DS]   P. Deligne, D. Sullivan, *Division algebras and the Hausdorff–Banach–Tarski paradox*, Enseign. Math. 29 (1983), 145–150.
[DV]   R. K. Dennis, L. N. Vaserstein, *Commutators in linear groups*, K-Theory 2 (1989), 761–767.
[Do]   S. Donkin, *Invariants of several matrices*, Invent. Math. 110 (1992), 389–401.
[DR]   M. Droste, I. Rivin, *On extension of coverings*, Bull. Lond. Math. Soc. 42 (2010), 1044–1054.
[DT]   M. Droste, J. K. Truss, *On representing words in the automorphism group of the random graph*, J. Group Theory 9 (2006), 815–836.
[ET]   A. Elkasapy, A. Thom, *About Gotô's method showing surjectivity of word maps*, `arXiv:1207.5596`.
[EG]   E. W. Ellers, N. Gordeev, *On the conjectures of J. Thompson and O. Ore*, Trans. Amer. Math. Soc. 350 (1998), 3657–3671.
[Fo1]   E. Formanek, *Central polynomials for matrix rings*, J. Algebra 23 (1972), 129–132.
[Fo2]   E. Formanek, *The ring of generic matrices*, J. Algebra 258 (2002), 310–320.
[GG]   J.-M. Gambaudo, E. Ghys, *Commutators and diffeomorphisms of surfaces*, Ergodic Theory Dynam. Systems 24 (2004), 1591–1617.
[Gorde]   N. Gordeev, *Sums of orbits of algebraic groups*, I, J. Algebra 295 (2006), 62–80.
[Gordo]   S. R. Gordon, *Associators in simple algebras*, Pacific J. Math. 51 (1974), 131–141.
[Gow]   R. Gow, *Commutators in finite simple groups of Lie type*, Bull. London Math. Soc. 32 (2000), 311–315.
[Gr]   A. B. Gray, Jr., *Infinite symmetric groups and monomial groups*, Ph.D. Thesis, New Mexico State Univ., 1960.
[GH]   C. K. Gupta, W. Hołubowski, *Commutator subgroup of Vershik–Kerov group*, Linear Algebra Appl. 436 (2012), 4279–4284.
[GK]   R. M. Guralnick, W. M. Kantor, *The probability of generating a simple group*, J. Algebra 234 (2000), 743–792.
[GM]   R. Guralnick, G. Malle, *Products of conjugacy classes and fixed point spaces*, J. Amer. Math. Soc. 25 (2012), 77–121.
[HS]   P. de la Harpe, G. Skandalis, *Sur la simplicité essentielle du groupe des inversibles et du groupe unitaire dans une C^*-algèbre simple*, J. Funct. Anal. 62 (1985), 354–378.
[HSVZ1]   R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang, *The yoga of commutators*, Zap. Nauch. Sem. POMI 387 (2011), 53–82; English transl. in J. Math. Sci. (New York) 179 (2011), 662–678.
[HSVZ2]   R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang, *Commutator width in Chevalley groups*, `arXiv:1206.2128`.
[Is]   I. M. Isaacs, *Commutators and the commutator subgroup*, Amer. Math. Monthly 84 (1977), 720–722.
[JLO]   S. Jambor, M. W. Liebeck, E. A. O′Brien, *Some word maps that are non-surjective on infinitely many finite simple groups*, `arXiv:1205.1952`.
[KBMR]   A. Kanel-Belov, S. Malev, L. Rowen, *The images of non-commutative polynomials evaluated on 2×2 matrices*, Proc. Amer. Math. Soc. 140 (2012), 465–478.
[KN]   M. Kassabov, N. Nikolov, *Words with few values in finite simple groups*, `arXiv:1112.5484`.
[KG]   S. A. Katre, A. S. Garge, *Matrices over commutative rings as sums of k-th powers*, Proc. Amer. Math. Soc., to appear.
[Kl]   E. Klimenko, Engel maps for $PSL_2(\mathbb{C})$, Preprint.
[Ku]   B. Kunyavskii, *The Bogomolov multiplier of finite simple groups*, in: "Cohomological and Geometric Approaches to Rationality Problems" (F. Bogomolov, Yu. Tschinkel, Eds.), Progr. Math., vol. 282, Birkhäuser, Boston, MA, 2010, pp. 209–217.
[La]   M. Larsen, *Word maps have large image*, Israel J. Math. 139 (2004), 149–156.
[LST1]   M. Larsen, A. Shalev, P. H. Tiep, *Waring problem for finite simple groups*, Ann. Math. 174 (2011), 1885–1950.
[LST2]   M. Larsen, A. Shalev, P. H. Tiep, *Waring problem for finite quasisimple groups*, `arXiv:1107.3341`.
[LB]   L. Le Bruyn, *Centers of generic division algebras, the rationality problem 1965–1990*, Israel J. Math. 76 (1991), 97–111.
[Le]   M. Levy, *Word maps with small image in simple groups*, `arXiv:1206.1206`.
[LOST1]   M. W. Liebeck, E. A. O′Brien, A. Shalev, P. H. Tiep, *The Ore conjecture*, J. Europ. Math. Soc. 12 (2010), 939–1008.
[LOST2]   M. W. Liebeck, E. A. O′Brien, A. Shalev, P. H. Tiep, *Commutators in finite quasisimple groups*, Bull. London Math. Soc. 43 (2011), 1079–1092.
[LS]   M. W. Liebeck, A. Shalev, *Diameters of finite simple groups: sharp bounds and applications*, Ann. Math. 154 (2001), 383–406.
[Ly]   R. C. Lyndon, *Words and infinite permutations*, in: "Mots. Langue, Raisonnement, Calcul", Hermès, Paris, 1990, pp. 143–152.
[ML]   L. Makar-Limanov, *Algebraically closed skew fields*, J. Algebra 93 (1985), 117–135.
[Ma]   J. A. Maroli, *Representation of tree permutations by words*, Proc. Amer. Math. Soc. 110 (1990), 859–869.
[MZ]   C. Martinez, E. Zelmanov, *Products of powers in finite simple groups*, Israel J. Math. 96 (1996), 469–479.

[Mer]   Ju. I. Merzljakov, *Algebraic linear groups as full groups of automorphisms and the closure of their verbal subgroups*, Algebra i Logika Sem. 6 (1967), no. 1, 83–94. (Russian.)

[Mes]   Z. Mesyan, *Commutator rings*, Bull. Austral. Math. Soc. 74 (2006), 279–288.

[MP]   J. Morita, E. Plotkin, *Gauss decompositions of Kac–Moody groups*, Comm. Algebra 27 (1999), 465–475.

[Mu1]   A. Muranov, *Finitely generated infinite simple groups of infinite commutator width*, Internat. J. Algebra Computation 17 (2007), 607–659.

[Mu2]   A. Muranov, *Finitely generated infinite simple groups of infinite square width and vanishing stable commutator length*, J. Topol. Anal. 2 (2010), 341–384.

[My]   J. Mycielski, *Can one solve equations in groups?*, Amer. Math. Monthly 84 (1977), 723–726.

[Ni]   N. Nikolov, *Algebraic properties of profinite groups*, `arXiv:1108.5130`.

[NS]   N. Nikolov, D. Segal, *Powers in finite groups*, Groups Geom. Dyn. 5 (2011), 501–507.

[Or]   O. Ore, *Some remarks on commutators*, Proc. Amer. Math. Soc. 2 (1951), 307–314.

[Pr]   C. Procesi, *The invariant theory of $n \times n$ matrices*, Adv. Math. 19 (1976), 306–381.

[Pu1]   S. Pumplün, *Sums of squares in octonion algebras*, Proc. Amer. Math. Soc. 133 (2005), 3143–3152.

[Pu2]   S. Pumplün, *Sums of d-th powers in non-commutative rings*, Beiträge Algebra Geom. 48 (2007), 291–301.

[Ra]   Yu. Razmyslov, *On a problem of Kaplansky*, Izv. Akad. Nauk SSSR. Ser. Mat. 73 (1973), 483–501; English transl. in: Math USSR Izv. 7 (1973), 479–496.

[Ree]   R. Ree, *Commutators in semi-simple algebraic groups*, Proc. Amer. Math. Soc. 15 (1964), 457–460.

[Ros]   M. Rosset, *Elements of trace zero and commutators*, Ph.D. Thesis, Bar-Ilan Univ., 1997.

[RR1]   M. Rosset, S. Rosset, *Elements of trace zero in central simple algebras*, in: "Rings, Extensions, and Cohomology (Evanston, IL, 1993)" (A. R. Magid, Ed.), Lecture Notes Pure Appl. Math., vol. 159, Marcel Dekker, New York, 1994, pp. 205–211.

[RR2]   M. Rosset, S. Rosset, *Elements of trace zero that are not commutators*, Comm. Algebra 28 (2000), 3059–3072.

[Row]   L. H. Rowen, *Polynomial identities in ring theory*, Academic Press, New York, 1980.

[SW]   J. Saxl, J. S. Wilson, *A note on powers in simple groups*, Math. Proc. Cambridge Phil. Soc. 122 (1997), 91–94.

[Se]   D. Segal, *Words: notes on verbal width in groups*, London Math. Soc. Lecture Notes Ser., vol. 361, Cambridge Univ. Press, Cambridge, 2009.

[Sh1]   A. Shalev, *Commutators, words, conjugacy classes and character methods*, Turkish J. Math. 31 (2007), 131–148.

[Sh2]   A. Shalev, *Applications of some zeta functions in group theory*, in: "Zeta Functions in Algebra and Geometry" (A. Campillo *et al.*, Eds.), Contemp. Math., vol. 566, Amer. Math. Soc., Providence, RI, 2012, pp. 331–344.

[Stei] A. Stein, $1\frac{1}{2}$-*generation of finite simple groups*, Beiträge Algebra Geom. 39 (1998), 349–358.

[Step] A. Stepanov, *Universal localization in algebraic groups*, preprint, 2010, available at `http://alexei.stepanov.spb.ru/papers/formal.pdf`.

[Va]   L. N. Vaserstein, *On the sum of powers of matrices*, Linear Multilin. Algebra 21 (1987), 261–270.

[VW]   L. N. Vaserstein, E. Wheland, *Commutators and companion matrices over rings of stable rank* 1, Linear Algebra Appl. 142 (1990), 263–277.

[YCW]   S.-K. Ye, S. Chen, C.-S. Wang, *Gauss decomposition with prescribed semisimple part in quadratic groups*, Comm. Algebra 37 (2009), 3054–3063.

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