Nikolai Vavilov, mathematics and life *

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To Kolya

1 Nikolai Alexandrovich Vavilov

Each moment in time leaves imprints in our memory. These imprints remain there — unseen, imperishable, and beyond the control of anything or anyone — artifacts of eras and passions, waiting to resurface before closed eyes and bring back into the present a fragment of a shimmering, distant, yet infinitely continuous life.

The Mathematics Institute at Fontanka 27, seminar room 311, third floor. At the table sits Dmitry Konstantinovich Faddeev, bent forward, propping his slender face with his right hand, attentively listening to the speaker. Next to him is Zenon Ivanovich Borevich, bristling his mustache, writing something in a notebook, occasionally glancing at the blackboard before returning to his notes. By the window stands Anatoly Vladimirovich Yakovlev, brisk and sharp, his speech abrupt. He periodically looks out the window, lost in his thoughts. Occasionally, Andrei Alexandrovich Suslin would drop by, rarely but still. He would arrive late, pause at the door, glance at the blackboard, instantly grasp everything, and then sit wherever there was space, waiting for the talk to end. Among them now is Nikolai Alexandrovich Vavilov. Forever... eternally. To many, he was just Kolya — scientist, teacher, colleague, friend. It's impossible to believe...

Professor Nikolai Alexandrovich Vavilov passed away suddenly on September 14, 2023, at his home in Saint Petersburg, just three days short of his 71st birthday. He departed at the peak of his creative and active life, brimming with scientific ideas and organizational plans. With him, an entire world vanished — a world that supported his family and everyone fortunate enough to know and work with him. Beyond proving numerous theorems and developing new theories, Nikolai Alexandrovich achieved something that few do: he built a large, vibrant, and multifaceted scientific school in the structural theory of Chevalley groups over rings. Students and colleagues of N. A. Vavilov work in universities across Russia, Israel, Poland, Germany, and China. Under his supervision, over 20 Ph.D. candidates and five doctoral researchers defended their theses.

Nikolai Vavilov was born on September 17, 1952, in Leningrad, into a family of LETI professors — Natalia Nikolaevna Sozina and Alexander Alexandrovich Vavilov. Natalia Nikolaevna was a talented physicist, an associate professor in the Department of Electronic Engineering. One of her students was future Nobel laureate Zhores Alferov. She was undoubtedly the keeper of the family hearth and the source of warmth in the Vavilov home. His father, A. A. Vavilov, became a professor at an early age, head of the Department of Automation and Control Processes, later rector of LETI, and chairman

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Figure 1: LOMI, 1981. E. Dybkova, G. Maloletkin, M. Bashmakov, A. Semenov, D. Faddeev, N. Vavilov, Z. Borevich, A. Merkurjev, A. Yakovlev.

of the Council of Rectors of Leningrad's universities. He was always busy with work, decisions, and responsibilities. Such was the atmosphere in the family where Nikolai grew up.

Kolya attended the famous Leningrad School No. 30 in the late 1960s, the same school that still stands today on the corner of Sredny Prospekt and 7th Line of Vasilievsky Island. At that time, the name of teacher Iosif Yakovlevich Verebeychik resounded throughout Leningrad. A teacher by the grace of God, he personally selected capable students for the class he led. Among them were Kolya's future friends and renowned scientists: V. Kreinovich, M. Zakharevich, I. Vidensky, and E. Gluskin. However, N. Vavilov and N. Gordeev found themselves in a class where Verebeychik only taught mathematics, while the homeroom teacher was an experienced physicist, M. L. Shifman. Initially, Kolya Vavilov was deeply interested in chemistry and foreign languages. The idea of a professional career in mathematics hadn't yet arisen. But gradually, Verebeychik instilled in his students a genuine love for the subject. The magic of his personality and teaching talent fundamentally changed Kolya's interests, and by the tenth grade, the decision to enter the Mathematics and Mechanics Faculty (MatMech) was made.

Nikolai Vavilov enrolled at the Faculty of Mathematics and Mechanics of Leningrad University in 1969. Among all the subjects, algebra immediately stood out as his favorite. Early on, he established connections with Anatoly Nikolaevich Andrianov, a professor at LOMI and an expert in the theory of modular forms. However, their collaboration didn't fully materialize, and Nikolai wrote his diploma thesis under Anatoly Vladimirovich Yakovlev on Galois theory. In 1975, he entered graduate school under Zenon Ivanovich Borevich.

At that time, Zenon Ivanovich had moved away from integral representations and local fields, becoming unexpectedly interested in linear groups. Around 1975, he conceived the idea of describing subgroups within the general linear group, and in 1976, he published his landmark article, "Description of Subgroups of the Full Linear Group Containing the Group of Diagonal Matrices" [7]. In this work, he introduced the concept of a net and used it to describe the structure of certain subgroups of the general linear group over a field, under natural constraints on the number of field elements. Notably, the description turned out to depend not on the base field but solely on the dimension of the general linear group.

Apparently, Borevich intuitively grasped the extraordinary significance of this elegant but specific result. He organized an algebraic seminar on the topic, involving participants such as E. V. Dybkova, R. A. Schmidt, and then-graduate students and undergraduates: N. Vavilov, L. Kolotilina, V. Koibaev, S.Krupetsky, E. Plotkin, and others. Occasionally, A. I. Skopin would also join. And then... perhaps the stars of distant constellations aligned just right, bringing together a multitude of circumstances — resulting in the birth of a field of mathematics that defined the destinies and lives of many mathematicians in St. Petersburg. From a scientific perspective, the emerging theory undoubtedly became an accomplished fact of contemporary mathematical knowledge.

The primary driving force behind this was the work of N. A. Vavilov in the seminar. Here's how it unfolded. Zenon Ivanovich would pace the room, meticulously writing formulas, pausing to reflect, and then slowly resuming his writing. He was always very precise, clear, and thoughtful. N. A. Vavilov absorbed everything on the fly. It felt as if Borevich's voice hadn't even faded, yet N. Vavilov had already picked up the thread. This continued for a relatively short time, and by 1977, the initiative decisively shifted to N. Vavilov. He soared in the seminar room, constantly stoking the fire with new ideas for proofs and, more importantly, uncovering paths for developing the theory. Both Borevich and Vavilov were incredibly fortunate to find each other — two individuals of different ages, characters, and energies, united by a single mathematical theme. Kolya was like an intellectual locomotive, gaining unstoppable momentum. At some point, he became the true engine of the whole endeavor. That's life...

In 1978, N. A. Vavilov defended his Ph.D. dissertation. The year before, another significant event occurred in his life. In 1977, Kolya married Olga Sergeevna Bychkova. Olga became the beacon and pillar of his life — the homemaker, the person who was always there to understand and help. For many years, Olga Vavilova was a lecturer — an associate professor in the Department of Physics at the Polytechnic Institute. In 1987, their son Sasha was born, who is now a top manager at a major communications firm. At home, all roles were carefully allocated and tailored to ensure Nikolai Alexandrovich could work and live comfortably.

It became evident within the first five minutes of meeting Nikolai Alexandrovich Vavilov that he was a bright and extraordinary individual. At times, it seemed as if Kolya were a Renaissance man, transplanted into the field of mathematics by sheer circumstance. His views and writings always carried a now-lost light of universal knowledge. Mathematics, history, linguistics, cultural studies, and dozens of quotes in the most unexpected languages created a sense of an invisible fraternity between N. A. Vavilov and the likes of Diderot, La Rochefoucauld, Buffon, Voltaire, and other illustrious minds. However, his relationship with the natural world was complicated and not particularly affectionate. The world of Confucius or Omar Khayyam in the original, on the other hand, was a different story. It was nothing for him to ask a companion for a quote from Laozi. "What, you don't remember Laozi?" Kolya would exclaim in surprise. "But of course, he said in such-and-such a treatise..., and Goethe wrote about the same, though in a Rhine context..." There was no need to verify him; everything was precise. Although, one could check — Kolya loved to shock and, at times, exaggerate.

Linguistics was the love of his childhood, and this passion remained throughout his life. He knew 10–12 languages, spoke fluently in 5–6 of them, and understood an indeterminate number more. He sensed linguistic structures with his fingertips, loved tracing Indo-European or Semitic roots, and drew parallels between distant languages. "How do you learn languages?" I once asked him. "Oh, it's very simple," he replied. "You take a dictionary with 30,000 words of a new language and read it twice, front to back. That's it, you've got about 20,000 words in your repertoire. Then, spend a week brushing up on the grammar — and that's all." To this day, I don't know whether he was joking, and if so, to what extent. In any case, aside from his rich English, he lectured in Italian, Polish, and German. His mathematical writings were brimming with quotes and epigraphs, and upon reflection, one couldn't help but marvel at how precisely they aligned with the corresponding mathematical phenomena.

"I am an intellectual," Kolya would say. "I am not an intelligentsia member; I am an intellectual." Of course, he wasn't entirely truthful. His natural-born refinement contradicted this claim, but when he embodied a certain image, Kolya would exaggerate with abandon, and his penchant for provocation became an integral part of his persona. It was effortless for him to declare that the classification of finite simple groups was the most significant event of the 20th century or to write that the study of linear groups had been ongoing for a thousand years. "So what if the concept of a group appeared only 200 years ago? They've been studied for a millennium!" In debates, Kolya was unbound by conventions or temporal constraints. He once claimed that all Russian youth drink only French wines of the Grand Cru category. During my first visit to Paris, he informed me that one must exclusively eat hippopotamus cutlets there. He relished the self-deprecating humor characteristic of Leningrad, quoting Kuryokhin in the 1990s and Shnurov in the 2010s — alongside, for instance, Camões or Rilke.

His factual memory was phenomenal. In particular, N. A. (here and henceforth, N. A. refers to the abbreviation for Nikolai Alexandrovich) possessed an extraordinary mathematical memory. It was something incredible, miraculous, and unparalleled. In the 1980s, Boris Borisovich Venkov was considered the benchmark of universal mathematical knowledge in St. Petersburg. Over time, I believe N. A. Vavilov surpassed even that. In his professional field, he undoubtedly had an encyclopedic repertoire of knowledge at his disposal. Moreover, this "computer" never malfunctioned...

Nothing human was alien to Nikolai Alexandrovich. He loved good food, especially Italian cuisine, and was an excellent cook himself. He mastered this craft, as he did all others, to perfection. I remember our first trip to Paris when he enthusiastically narrated stories about everything— from flamboyant Gothic architecture and the Champs-Élysées to the girls of Place Pigalle, the nuances of carpaccio and Parma ham, the differences between cheeses, wines, and sausages, and, naturally, about bibliophiles on the Seine.

Kolya Vavilov knew almost as much about wine as he did about the structure of Chevalley groups. He knew brands, vineyards, soils, regions, and accolades. He studied labels passionately, drank with pleasure, and was particularly proud that one of his mathematical protégés was a professional sommelier. Simplicity in communal living came hard to Kolya, and the same applied to wine. In general, he loved anything of high quality and distinction — from champagne on a plane or in a restaurant to pencils, clothes, or facsimile edition books. He was infinitely distant from sports. However, I recall the early 1980s in Jurmala, where N. A. encountered the traditional Baltic game of novus, a simplified version of billiards. Kolya quickly lost about ten games in this new, unfamiliar game and, as he put it, became "saddened". At five in the morning, I stepped out of the house to find Kolya feverishly practicing with the novus pieces — hitting, hitting, and hitting his targets. Paramosha turned out to be a gambler, after all...

But let us return to N. A. Vavilov's mathematical creativity.

2 Algebraic Creativity

The first pressing mathematical question that arose for N. A. Vavilov in the early 1980s was “Quo vadis?” — where are you going? And alongside it, “Where to lead others?” — “Quo ducere?” Since, as we know, we are responsible for those we tame. By this time, he already had students, followers, a seminar, and the first graduate students were on the horizon. Unconsciously, he stood at the origins of the unknown, and this role imposed a certain level of responsibility.

The first research direction suggested itself: we fix the general or special linear group and strive to move as far away as possible from the field case while preserving the ability to describe, over some ring, the structure of subgroups containing something reasonable, for example, the diagonal. This direction led to a series of papers with increasingly complex rings: local, semilocal, Gaussian, rings with stability conditions, Dedekind rings of arithmetic type, and so on (see, for example, [9], [8], [65], [68], [69]). At some point, N. A. Vavilov realized that it was possible to describe all subgroups of the general linear group over an arbitrary commutative ring containing the group of block-diagonal matrices with sufficiently large blocks ([10], [11]). This was a moment of truth, greatly influenced by the ideas of A. A. Suslin and his work [63]. It turned out that arbitrary commutative rings were entirely achievable.

The second research direction was no less natural. If we can vary the ring, then, of course, we can vary the group as well. Thus arose a series of works devoted to classical groups over various rings. In particular, symplectic, orthogonal, spinor, unitary, and other similar groups were studied (see [13], [66], [14], [15], [67], [70], [71]). It was during this time that Nikolai Alexandrovich mastered working virtuously with the corresponding matrices, mentally multiplying and summing weights and roots and transforming them into transvections.

Classical groups remained a lifelong testing ground for N. Vavilov for exploring ideas and honing intuition. Many profound works are devoted to them. He considered the symplectic group $Sp(4, R)$ the most challenging and exceptional case among all classical groups, while he referred to the general linear group only as “Your Majesty”.

All this led to the third research topic, which became the source of N. A. Vavilov’s lifelong scientific inspiration. Chevalley groups over rings appeared in his works in the early 1980s and very quickly became a world where events unfolded, creation occurred, and connections and laws were established over the next 40 years. Today, this is known as the “structural theory of Chevalley groups over rings”, but at that time, it was merely terra incognita – a mysterious land with unknown creatures and no clear approach.

From 1980 to the 1990s, N. A. Vavilov accumulated knowledge, formulated principles, tasks, and methods suitable for Chevalley groups over rings. This eventually coalesced into a well-defined circle of ideas. The world of Chevalley groups came to life and took shape. With form came a worldview. By 1990, N. A. Vavilov had become the world’s leading expert on Chevalley groups over rings and was ready to establish a school. The factual manifestation of this status was his survey “Structure of Chevalley Groups over Commutative Rings”, [72], published as part of the International Mathematical Congress in Kyoto in 1990, along with earlier or contemporaneous survey works [70], [73], [98].

The process of understanding occurred gradually. Initially, N. A. Vavilov realized that no canonical decompositions of elements worked in the case of general rings. What was left to do? The first thought was that for general rings, the only path was to work within suitable irreducible representations. But how? Large and complex matrices arise.

One day, while we were laboriously multiplying symplectic matrices, he suddenly said, “Everything is wrong. We need to work as proposed by M. Stein and H. Matsumoto”, and drew the first weight diagram. A preprint of Stein’s article appeared in Leningrad thanks to the efforts of C. Soule and A. Suslin. Andrei Alexandrovich immediately handed it to N. A. Vavilov, saying, “You know, I only work in the special linear group, at most in the orthogonal one”. This preprint played a special role in N. A. Vavilov’s work. It became clear that working in representations was possible. It was only necessary to properly organize computations and replace matrices with weight diagrams. Kolya then formulated it this way: “Harish-Chandra was right. Everything that can be proven for SL_n can be proven for any Chevalley group over any ring.”

Between 1980 and 1990, N. A. Vavilov published over 20 papers on Chevalley groups. Bruhat decompositions of various elements were studied, weight elements of Chevalley groups were introduced, net subgroups were described, subgroups containing the maximal split torus were examined, and so on. Stein’s article also catalyzed a series of works by N. A. Vavilov and his students, dedicated to stable computations, stability of the K_1-functor, descriptions of certain special subgroups, and much more. In 1989 and 1990, N. A. Vavilov delivered two extremely important lecture courses on Chevalley groups at the University of Crete and the University of Notre Dame in Indiana. Concurrently, conceptual lectures were given at various universities in Poland, Italy, America, Japan, England, and, of course, Russia. His name became widely known worldwide, and as soon as it became possible, N. A. Vavilov received visits from prominent scientists such as B. Cooperstein, G. Seitz, E. Abe, A. Hahn, and others. Life’s intensity was incredible. His doctoral dissertation (1988) was long behind him, and N. A. was entirely focused on science. He managed to absorb new results in related areas of algebra, refract them through his understanding of the subject, and immediately produce numerous ideas.

In his survey [72], he wrote: “Our main goal is to show why and how, rather than what exactly to do. We will focus more on methods of proof than on the results themselves. The entire exposition will be geared toward two problems: the normality of the elementary subgroup and the classification of normal divisors. Solutions to these problems are called the fundamental structural theorems”. In one paragraph, Vavilov explicitly stated his desire to move beyond immediate goals and deeply understand the structure of the polynomial equations defining the group. He quoted Zhang Zhou (whose existence, by the way, I highly doubt — this may be a mystification): “The view from beyond the horizon always sharpens vision, while the swamp always provides a limited picture of reality.” Or, in his characteristic linguistic style: “Wenn der Horizont verschieden ist sind es auch die Gedanken” — “When the horizon is different, the thoughts are different.” He always believed that essence was primary, and the results would follow naturally. Vavilov provided four different proofs of the normality of the elementary subgroup. Each had its own unique features and would later be used in specific circumstances. This was preceded by a characteristically brilliant Vavilov argument, which he attributed to “the main Daoist thinker Winnie-the-Pooh: different proofs should prove different things; otherwise, it would be the same proof!” Life proved him absolutely right.

3 Problems, Results, and the School

Of course, the survey by N. Vavilov, based on his talks in Kyoto and Hiroshima, was unique. In fact, it was a revelation, a glimpse into the future. This is especially evident now that this future has become the past and present. And it wasn’t just about the formulated problems. N. A. Vavilov had already published his first list of problems at the end of his previous survey [70]. The essence lay in the harmony of conceptualism and practice. Now we understand that he laid the foundation of a house that, starting at a certain point, began to build itself. This is evident today. Back then, it was not so clear. There were working days, fascinating Japan, an endless series of meetings, and communication with colleagues. Life. And as a result, a concrete philosophy emerged, which turned out to be the most practical of all practical sciences. The result of these reflections, the very "horizon" mentioned by Vavilov in his survey, culminated in the following principles:

- There is no substantial difference between classical groups and exceptional groups if the latter are considered in an appropriate representation. Thus, N. A. Vavilov consistently studied the structure and geometry of the minimal modules for Chevalley groups. Along the way, it turned out, as A. V. Mikhalev liked to say, that the key storyline was as follows.

- The combinatorics of basic representations, particularly micro-weight representations, and their visualization using weight diagrams. As soon as N. A. Vavilov understood the weight diagram for the minimal representation of a group of type E_6, he realized that he held an absolute weapon in his hands. The difference between matrix calculations in classical and exceptional groups was leveled and transformed from a problem into a challenge. This challenge required understanding the mechanism of stable and elementary computations, i.e., computations that replace arbitrary matrix multiplication with operations involving elementary matrices and control the action of elementary matrices on rows and columns through the visualization of Steinberg relations. By stable computations, we mean computations that deal with a single row or column.

- Simultaneously, N. A. Vavilov identified another main research direction: finding explicit equations, independent of characteristic, that define exceptional Chevalley groups in minimal representations, as well as studying realizations of exceptional groups as isometry groups of appropriate forms.

Returning to the problems where this philosophy is applicable, Vavilov concentrated on structural theory. The primary themes became the following:

- The normality of the group of elementary matrices in the full Chevalley group over a commutative ring and the nilpotency of the K_1-functor;

- The normal structure of Chevalley groups, specifically the description of subgroups of Chevalley groups normalized by the elementary subgroup. Description of the lattice of subgroups of Chevalley groups;

- The stability of the K_1 and K_2-functors, as well as the centrality of the K_2-functor.

The resolution of these and other related problems became the dominant focus for N. A. Vavilov and his students for many years. The spiral of knowledge kept unfolding, leading to the creation of hundreds of works. Whatever Nikolai Alexandrovich wrote about, his scientific maximalism and enjoyment of the process of discovery always shone through in the background. He had a habit of returning to core motifs over time, continuously refining and deepening his mastery of the subject. The titles of this mathematical "Bolero" speak for themselves: "Decomposition of Transvections: A Theme with Variations" [60], "A Third Look at Weight Diagrams" [74], "Structure of Chevalley Groups: A Proof from the Book" [16], "Commutator Yoga" [21], "Commutator Yoga: Further Applications" [22], "Commutators of Elementary Subgroups: Curiouser and Curiouser" [103], "Calculations in Exceptional Groups: Five Years Later" [38], "Unipotent Decompositions for E_6 and E_7: 25 Years Later" [77].

For an arbitrary Chevalley group, N. A. Vavilov identified five methods for solving the problem of the normality of the elementary subgroup. These include Suslin's direct factorization, the factorization and patching method (also by Suslin), the Quillen-Suslin-Vaserstein localization and patching method, Bak's localization and completion method, and the Stepanov-Vavilov method of unipotent decomposition, discovered by A. Stepanov in 1987. Alexei Stepanov remained a leading disciple and follower of Nikolai Alexandrovich throughout the years. The normality of the elementary group was proven by A. A. Suslin for the general linear group and V. Kopeiko for classical groups ([63], [26]). Later, D. Taddei proved this result for arbitrary Chevalley groups [64]. N. Vavilov and R. Hazrat [25] demonstrated that, under reasonable ring restrictions, the quotient group of the simply connected Chevalley group by its elementary subgroup (i.e., the K_1-functor) is a nilpotent group. Subsequently, N. Vavilov, along with A. Bak and R. Hazrat [3], showed that even in the relative case, the K_1-functor is an extension of a nilpotent group by an abelian group.

The description of Chevalley group subgroups normalized by the elementary matrix subgroup was obtained by Abe-Suzuki-Vaserstein using localization methods. In a series of works (1976-1995), they proved the standard nature of such descriptions, in the sense that each such subgroup is squeezed between an appropriate relative elementary subgroup of level A (where A is an ideal) and the corresponding congruence subgroup. N. A. Vavilov and his students found other proofs of this fact (1980-2015), see [16], [17], [76]. These conceptually distinct proofs allow for obtaining finer results, as well as visualizing the reasoning process and understanding "quo" all this "vadis"?. Perhaps this is what Nikolai Alexandrovich meant when, in his work [16], he called the result "a proof from the Book".

Throughout his scientific life, N. A. Vavilov held a special reverence for the classification theorem of finite simple groups. He followed the "Atlas", the new proofs, and the entire project as a whole, constantly emphasizing that science had never produced anything like it. Thus, when in 1984 M. Aschbacher [2] described the maximal subgroups of finite classical groups modulo the classification and introduced the Aschbacher classes $C_1 ... C_8$, and G. Seitz [48] and others extended the result to all Chevalley groups over algebraically closed fields, N. A. Vavilov was mentally prepared for this. He immediately realized that the Aschbacher subgroups, with slight modifications in their definitions, remain large for arbitrary Chevalley groups over commutative rings and admit standard descriptions of overgroup lattices. Thus, the lattice of subgroups of Chevalley groups over rings took shape. The topic of Aschbacher classes became the subject of numerous studies by N. A. Vavilov and his students and the focus of several dissertations ([11], [70], [52], [53], [58], [59], [1], [44], [45], [34], [35], [36], [32], [33], [18], [19]).

Stable computations appeared in N. A. Vavilov's toolkit in 1977 and became a powerful universal instrument that he applied throughout his scientific career. They unified a wide range of studies. It all began with M. Stein's article on low-dimensional K-theory [57]. N. A. Vavilov polished ideas developed for stabilizing lower K-functors. Along the way, he developed the critically important technique of stable matrix computations in exceptional groups, intricately linked with triangular factorizations of the Bruhat or

Gaussian type and parabolic factorizations of the Dennis-Vaserstein type. The latter is genetically associated with the injective stabilization of the K_1-functor (see [102], [4] and references therein). This topic from the early 1980s captivated N. A. Vavilov throughout his career, reappearing in various incarnations, see [95], [38]. It received new impetus in recent works by N. A. Vavilov on the bounded generation of Chevalley groups. For example, the unresolved challenging problem of injective stabilization of the K_2-functor for root systems with multiple connections was discussed in connection with the bounded generation of Steinberg groups. Another important object of N. A. Vavilov's studies, related to the stabilization of lower K-functors, is the unipotent decomposition method, which, together with the Chevalley-Matsumoto decomposition, enables induction on the group rank. Finally, N. A. Vavilov's students, E. Voronetsky, A. Lavrenov, and S. Sinchuk, completed the resolution of the fundamental problem of the centrality of the K_2-functor over a commutative ring [31], [30], [54], [104].

One of N. A. Vavilov's innovations was what he called numerology. The idea is as follows. Consider a Chevalley group in a certain representation, typically the minimal or adjoint representation. We draw a weight diagram. How can one, by looking at the diagram—the number and arrangement of squares — extract complete information about the quadratic equations defining the orbit of the highest weight, including signs, or, equivalently, the signs of the structural constants arising from the action of unipotents? This topic, which originated from a remark by C. Ringel at a seminar in Bielefeld, was further developed in works [47], [46], [74], [78], [79], and others. The idea was to depart from the geometric theory of standard monomials and extract all the relevant information directly from the diagram.

In the works of V. Petrov, A. Luzgarev, N. Geldhauser, A. Stavrova, I. Pevzner, P. Gvozdevsky, A. Smolensky, and other followers of N. A. Vavilov, one can trace elegant numerological motifs. In this context, let us recall the memories of Nikita Geldhauser.

Recollections by Nikita Geldhauser

Nikolai Alexandrovich actively promoted the idea of numerology in algebraic groups. Many theorems (in his lectures) were proven this way: if the numerology aligned, then everything was correct. And if we didn't understand the numerology, we needed to figure out why it was like that (not just formally, but in essence), and everything would become clear. This idea, also familiar to specialists in the classification of finite simple groups, spread within the St. Petersburg algebraic school thanks to N. A. and his ability to transfer general principles from one field to another, as well as his broad perspective, which allowed him to view problems from a distance.

In my works and in the works of Viktor Petrov, much has been accomplished based on numerology. Moreover, I now extend this idea further in my lectures in Munich and in my presentations, and students greatly enjoy this approach to mathematics.

In his approach to science (apart from numerology and his broad erudition in algebra as a whole), what struck me most was his optimism. N.A. was never afraid to propose highly optimistic hypotheses, adjusting and refining them as the proofs unfolded. This speaks to his strength of character — most mathematicians are cautious by nature and tend to avoid formulating bold hypotheses.

Among his personal qualities, I believe it is essential to highlight his genuine joy at the successes of others — a rare trait that was an integral part of his personality.

Of course, I will always remember our numerous conversations on very broad *topics: linguistics, philosophy in the widest sense (his Taoist approach to life), ancient literature, art, and so on. During these discussions, I mostly listened and asked questions.*

The scientific legacy of N.A. Vavilov is immense, with Nikolai Alexandrovich being the author of over 200 scientific works. So far, we have only touched on some directions of his algebraic creativity. Before moving on to other research areas of N.A. Vavilov, it is fitting to pause and focus on his greatest creation — his scientific school.

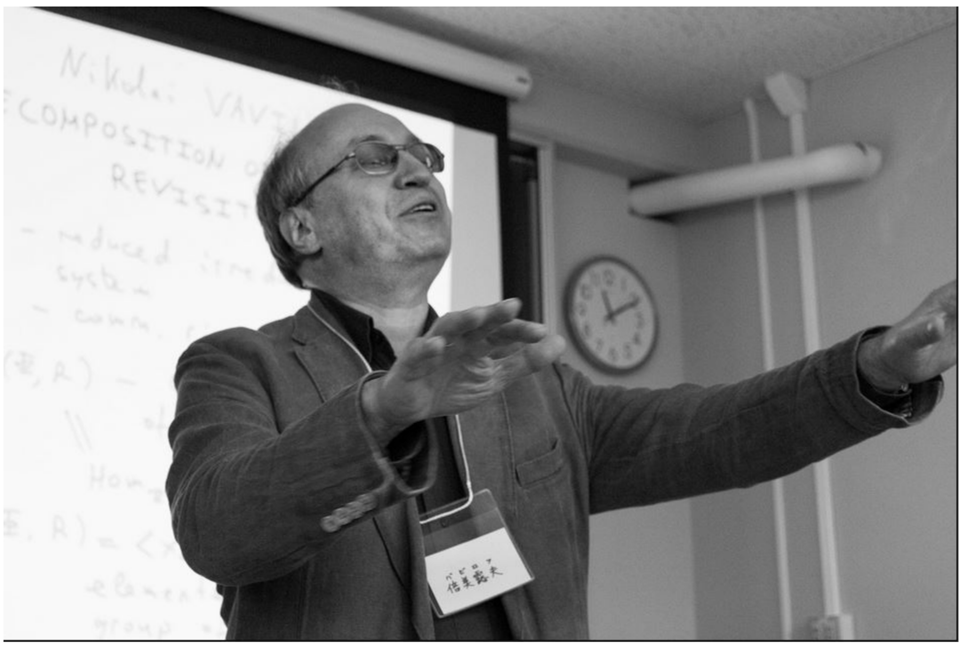

Figure 2: Lecture, Tsukuba University, 2014

"If you could know what gibberish empowers, the talent grows, from all abashment freed..." — indeed, what minute details, nuances, and interjections weave together the talent of a teacher and create the aura of shared action, which we call a lecture. N.A. Vavilov had a natural gift for captivating his audience with the illusion of understanding; his charisma radiated positivity down to the last comma, and his overwhelming mathematical optimism charged the audience with an energy they didn't even know they possessed. Often, Kolya didn't just present facts or explain theorems — he conducted the lecture like a maestro, gesticulating with his hands, rising and falling in tone. He distributed material across the board with precision, observed audience reactions, and performed everything with professionalism and confidence. Yet the key element was his ability to convey to his listeners his joy that "we are here, all together, doing something meaningful — not begrudgingly, not by force, but with confidence in ultimate success". N.A. Vavilov's lectures were imbued with mathematical hedonism, and it's no surprise that he loved discussing projects over a glass of wine or Belgian beer. "They cleanse the chakras and open harmonies," Kolya might say, or, quoting Hercule Poirot, "it's all about those little grey cells."

Here is a non-exhaustive list of some of Nikolai Alexandrovich's formal and informal students: E. Voronetsky, M. Gavrilovich, P. Gvozdevsky, N. Geldhauser, V. Golubovsky, E. Denisova, V. Kazakevich, A. Lavrenov, R. Lubkov, A. Luzgarev, M. Mitrofanov, V. Nesterov, I. Pevzner, E. Perelman, V. Petrov, E. Plotkin, A. Semenov, S. Sinchuk,

A. Smolensky, E. Sopkina, A. Stavrova, A. Stepanov, I. Khamdam, and A. Shchegolev.

Figure 3: Vavilov's Seminar. R. Lubkov, M. Buryakov, S. Sinchuk, A. Stavrova, V. Petrov, E. Voronetsky, D. Mamaev, K. Tsvetkov, A. Smolensky, A. Shchegolev, N. Vavilov, A. Stepanov. 2017

For one of his anniversaries, students gifted Nikolai Alexandrovich a poster depicting him as a mother hen surrounded by chicks in his coop. Behind this simple joke lay profound meaning. N.A. Vavilov meticulously crafted problems for those who wanted and could work with him, weighed combinations of ideas, guided and assisted them, coauthored articles, and constantly kept everyone informed about the latest mathematical developments. He shared his ideas, knowledge, and thoughts freely. He was immensely proud of the successes of his students and the mathematicians of the St. Petersburg school. He spoke of them in superlatives, generously using epithets: "What? You haven't read Viktor's, or Katya's, or Nastya's, or Lesha's latest result? It's absolutely brilliant! Now everything falls into place; everyone should know about it — it sheds light, connects, explains..." He organized online seminars of various levels, tirelessly working, commenting, posing questions, and discussing. It was a true "cosa nostra" — a shared endeavor, in his beloved Italian. "Carissimi — dear ones", he would address seminar participants, and indeed, they were "dear." Most importantly, he managed to forge everyone into a cohesive organism, which worked harmoniously under the roof of the structural theory of Chevalley groups. Of course—and thankfully!—many had long become independent scholars with their own perspectives, interests, and topics, often far removed from Chevalley groups. Others remained within the realm of the original scientific preferences, while some left active algebra altogether. But their regard for N.A. as a mentor remained forever. Thus, starting from the late 1990s and early 2000s, when discussing results and theorems, they could confidently be attributed to the achievements of Vavilov's school.

Returning to the results, it is worth noting several other topics of interest to N.A.

Vavilov that were, in various ways, closely related to those mentioned above. In the early 1990s, N.A. Vavilov, together with Lino Di Martino, obtained a number of interesting results on the (2,3)-generation of classical groups [12]. Warm, friendly relations with Lino persisted throughout his life. They frequently called each other and spoke extensively in Italian, often discussing not necessarily mathematical topics, but simply sharing various news. Research on the geometry of root subgroups and tori initiated with Di Martino, later was continued with V. Nesterov and I. Pevzner [39], [40], [97], [80], [81]. This direction aligns with a series of works (in collaboration with A. Semenov) on weight elements and semisimple root elements ([99], [100], [101]) and small Bruhat cells (with M. Mitrofanov) [96]. Finally, the combinatorics of root systems and Weyl groups, as well as closed sets of roots, were studied in collaboration with A. Kharebov and N. Kharchev [20], [94].

Commutation formulas and related problems of generating absolute and relative subgroups were one of the leitmotifs of Nikolai Alexandrovich's mathematical creativity. He revisited this topic constantly, and not without reason, as commutation relations and conjugacy classes emerged under various guises for at least the last twenty years. With his characteristic flair, N.A. dubbed the entire project "Yoga of Commutators". An article under this title was published in 2011 (jointly with Roozbeh Hazrat, A. Stepanov, and Zhang Zhuhong) [21]. Subsequently, in collaboration with the same team of co-authors, N.A. published around 20-25 works related to this "yoga". The last of these, co-authored with Zhang Zhuhong, appeared in 2023 [103]. It bore the same title as the first, with the addition "Curioser and Curioser"... Unfortunately, this work remains unfinished. Essentially, the project concerns a set of sophisticated commutation formulas that generalize classical commutation formulas and provide a universal approach for proving various structural theorems — both those described above and new ones. In particular, they allowed A. Stepanov and N. Vavilov to prove several results on the finiteness of commutator width in elementary generators for groups over commutative rings [61], [23], [62]. The primary tools used to prove the formulas in the "yoga" are two types of localization: Quillen–Suslin's "localization and patching" and A. Bak's "localization-completion", which, unlike unipotent decomposition, enable dimension reduction of the base ring.

Collaboration with Tony Bak marked a milestone in N.A. Vavilov's career, see [24]. Their acquaintance began during the Kyoto Congress in 1990. Tony delivered a major lecture on a topic related to the nilpotency of the K_1-functor. Following the lecture, a lively and engaging discussion unexpectedly began about K-theory in general and the K_1-functor in particular. Later, they went out for drinks but found nothing, so the discussion continued. Tony was somewhat surprised and openly delighted by the animated response. Thus began their collaboration.

In addition to purely scientific results related to structural theorems for Bak's unitary groups, the algebraic theory of quadratic forms, and fundamental ideals of group rings, this collaboration proved to be highly significant for many students from St. Petersburg University. These students visited Bielefeld and had opportunities to participate in numerous conferences and scientific schools. N.A. Vavilov envisioned this collaboration on a grand scale. After winning the Humboldt Prize, N.A. spent several years at Bielefeld, filled with scientific contacts and intense work. In fact, through the SFB 343 program, Bielefeld became a second home for many Russian mathematicians, serving as a kind of catalyst for international cooperation on German soil. Strengthening his reputation, N.A., in collaboration with Bak, managed to promote a bilateral agreement between the universities of Bielefeld and St. Petersburg. As a result of this work, talented students from St. Petersburg appeared in Bielefeld, and some even defended their dissertations there.

In addition to Chevalley groups, N.A. Vavilov was interested in all their close relatives, such as the Steinberg group, Bak's unitary group, and isotropic reductive groups. The latter became the focus of A. Stavrova's dissertation (2009). In fact, V. Petrov and A. Stavrova initiated active research on isotropic but not necessarily split reductive groups. Together with A. Stepanov and A. Luzgarev, they demonstrated how and what could be transferred from split groups to isotropic reductive groups [82], [49], [50], [43], [51], [37]. It can be said that the resulting theory combines the structural theory of Chevalley groups over rings with the deep theory of algebraic groups by Borel-Tits and Grothendieck-Demazure. In particular, an analogue of the elementary subgroup was defined, and its normality within the entire reductive group was proven. The theory of relative roots emerged, along with associated commutation formulas. This developed theory led to a beautiful result of the standardness of the normal structure of reductive groups over commutative rings of isotropic rank at least 2, provided the structural constants are invertible. Even more importantly, this theory has bright prospects for further development. It is worth noting that, as a side product of this activity, significant progress was made on the Serre–Grothendieck problem for isotropic groups (in collaboration with I. Panin and A. Stavrova) [41].

N.A. Vavilov showed broad interest in the properties of the unitary group introduced by A. Bak [5]. This group became the subject of research by V. Petrov, who generalized its definition and established its fundamental structural properties [42]. Later, these studies were continued and expanded by E. Voronetsky [105]. The collaboration between N.A. Vavilov's students and A. Bak's students, including R. Hazrat, G. Tang, R. Preusser, and others, has remained active.

In recent years, Nikolai Alexandrovich, together with B. Kunyavskii and E. Plotkin, devoted considerable attention to the problem of bounded generation of Chevalley, Kac-Moody, and Steinberg groups [28], [27], [29]. He sensed that the time had come for this topic, which is closely related to the congruence subgroup problem, representation theory, and model theory. Ideas originating from the famous work of Bass–Milnor–Serre [6] came to full fruition, leading to a new stage of understanding. N.A. Vavilov's extraordinary calculations with groups of type C_2 and the work of A. Trost helped fill the final gaps for the functional case of global fields, bringing the entire picture together. It was shown that Chevalley groups over Dedekind rings of arithmetic type admit uniformly bounded generation with a constant depending only on the root system [29].

To summarize, the school of N.A. Vavilov has resolved many fundamental problems and created new directions in mathematics. Notably, the school provided various proofs of the normality of the elementary subgroup and the normal structure of Chevalley, unitary, and isotropic reductive groups; classified intermediate subgroups of various Aschbacher classes; derived commutation formulas for congruence subgroups; proved the nilpotency and homotopy invariance of unstable K_1-functors; demonstrated the centrality of unstable K_2-functors; and established bounded generation of Chevalley groups over Dedekind domains, among other achievements.

4 Philosophy and History of Mathematics

Nikolai Alexandrovich was by no means a philosopher of mathematics — he was a thinker by nature! This lofty phrase captures the essence of how he spoke, the angles from which he approached problems, and, above all, the sheer breadth of his knowledge. The image of a muscular Rodin-esque hero hardly suits him. Instead, one should picture him at his computer, surrounded by books and papers, with a glass of wine in one hand, a folio in the other, and a pen in the third. The quintessence of his approach to philosophical problems in mathematics is laid out in his foundational 2019 work, "Reshaping the metaphor of proof" [83]. This work is a true cascade of names, facts, and ideas. Immersing oneself in it once is not enough; it warrants rereading, again and again, until all this wealth of information finds its rightful place in one's mind. And yes, it would be helpful to have shelves for all that knowledge — if only one knew where to find them.

The essay delves into intricate issues related to the concept of mathematical proof. It begins with a fascinating historical journey through mathematics, where the names of d'Alembert, Gauss, Cauchy, Riemann, Abel, Hilbert, and many others come alive, leaving one amazed at how all these historical facts and personas could be retained in such active memory. Following this, the concept of knowledge is analyzed through the lens of mathematical proofs. Here, N.A. Vavilov's penchant for paradox shines brilliantly. He never missed an opportunity to highlight the duality of concepts, the incompleteness of knowledge, the imperfection of formal theories, the ambiguity of conclusions, and all else that characterizes the process of understanding. He always had examples, counterexamples, and counter-counterexamples at his disposal, along with a fitting quote from Confucius for any assertion. The volume of information he used grew exponentially, incorporating achievements and errors of the present to provide new arguments.

As the saying goes, style is the man himself, and here is one quote from the article that captures his essence: "The pathos, ethos, spirit, and values of mathematical research have changed little over the past 2,500 years. Purely intellectually, Archimedes, de Fermat, fon Leibniz, Euler, Lagrange, Dirichlet, Jacobi, Hamilton, and Riemann are still closer to us than most of our contemporaries." And further: "On a more frivolous note, my late friend Oleg Izhboldin used to specify: "Who proved what and to whom?" Essentially, at the time this could translate, for instance, into 'Voevodsky has proven the Milnor's conjecture to Suslin."

Quoting Yuri Manin, N.A. Vavilov wrote: "A good proof is one that makes us wiser". Yet, as always, he added his unique twist: what is "good", for whom is it "good", and what does "good" even mean? He then introduced Voevodsky's reflections on K-theory, prefacing them with his characteristic positive maximalism: "This is an absolutely stunning human document, of immense intellectual honesty, and perhaps it has roughly the same historical significance as Fermat's famous letter to Descartes or Galois' last letter. It is absolutely essential reading for anyone who wants to understand what creative Mathematics is."

When discussing the aesthetics of mathematical proofs, N.A. would venture into the cozy, familiar world of the Renaissance. He was well-versed in the history of art and perceived subtle artistic analogies as naturally as an ordinary person enjoys ice cream on a hot day. Great theorems, for him, were reminiscent of quattrocento frescoes, the works of Piero della Francesca, Benozzo Gozzoli, or Filippo Lippi. For shorter proofs, he envisioned the art of Rogier van der Weyden, Robert Campin, or Hans Memling. "Everyone writes as they breathe", taught Bulat Okudzhava. And Nikolai Alexandrovich breathed exactly that way! He expressed this himself: "Mathematics is not so different from other sublime manifestations of free creative spirit, such as Language, Painting, or Music. The only reason I became a professional mathematician was that, for me, as for any cognisant individual, mathematical constructions and concepts have supreme intellectual and emotional appeal, or, switching from koine to katharevousa, combine the highest possible level of explicitness with the highest possible level of suggestivity. Seen as a supreme human activity, mathematical constructions and concepts possess the highest intellectual and emotional appeal.”

Sergei Pilyugin continues: Vavilov as a Philosopher and Historian of Mathematics

Figure 4: S. Pilyugin and N. Vavilov. ICM 2014, Seoul

From this humanistic credo in the article [83], purely scientific conclusions are drawn. N.A. begins by formulating several (commonly accepted) postulates about the structure of mathematical proof. Simplifying, here are examples of some of them:

- *A proof is a formal text where the result is derived from a set of axioms and previously obtained results according to strictly defined rules;*
- *Sometimes it is very difficult to present a proof, but its verification is a purely technical process (in particular, accessible to a computer);*
- *There are universally accepted criteria for rigor in proofs across all fields of mathematics;*
- *All statements presented in sufficiently advanced courses are accompanied by complete and precise proofs.*

Using very compelling examples and arguments, N.A. refutes most of the above theses. According to his concept, the proof of any substantial mathematical claim is not merely a formal derivation of a result from axioms and prior theorems but rather a "roadmap" that allows a professional mathematician (by applying efforts sometimes equal to those made by the author in constructing the proof) to verify the validity of the claimed result.

It is precisely at this stage of verifying new knowledge that a reasonable combination of logic and intuition is required, interacting on the foundation of fundamental mathematical training.

In 2020–2022, N.A. Vavilov published a series of articles [84]–[89], united under the somewhat dry title "The Computer as a New Reality of Mathematics". The author himself downplays the significance of these texts (or at least some parts of them), stating that "they are not scientific or historical but methodological and pedagogical in nature" [85]. In reality, the articles [84]–[89] represent a serious, uniquely profound historical study of certain classical problems in number theory, written by a first-class mathematician. These works discuss Waring's problem in its various forms, Goldbach's conjecture, questions related to the behavior of Mersenne and Fermat numbers and their analogs. The articles include numerous references to results achieved in this field by both professional mathematicians and amateurs (including schoolteachers, rural priests, and even generals).

Much attention is given to the formulation of classical problems; as N.A. demonstrates, in most historical reviews and books, these formulations are inaccurately represented. The precise statements are verified through a meticulous examination of the original texts "under a magnifying glass". For instance, in [87], on page 10, N.A. writes about Goldbach's marginal note in his letter to Euler, in which he formulated his hypothesis: "In higher resolution, it is clear that the words 'die grosser ist als 1' are added below the line without any spaces, then the '1' is replaced with a '2,' and then again with a '1'." Well, it's that simple — anyone could wake up in the morning, examine Goldbach and Euler's correspondence, and check the peculiarities of their handwriting in German. Fortunately, N.A. Vavilov does this for us, and one can trust him, suppressing any sense of amazement — because one quickly grows accustomed to excellence.

As suggested by the series title, N.A. meticulously traces the incredible progress in number theory brought about by the use of modern computers. For example, in [87], the complete resolution of the odd Goldbach conjecture (stating that every odd natural number $n > 5$ *can be expressed as the sum of three odd primes) by H. Helfgott, published in 2013–2014, is thoroughly analyzed. This solution combined achievements based on classical approaches with substantial use of computers.*

A significant portion of the material is devoted to methods in number theory; the author provides numerous polynomial identities used by various mathematicians, starting from Euler and Liouville. Additionally, N.A. includes many problems for readers to solve using computers.

Finally, he discusses in detail the ideas and approaches that have transformed modern number theory into one of the most important branches of mathematics.

These articles were written by a person of remarkable erudition (and are undoubtedly intended for a highly educated and thoughtful reader). It is quite possible that the analogy between N.A. Vavilov and Rodin's muscular thinker is fitting—except that his muscles were intellectual. The texts include quotations (untranslated into Russian) in Latin, English, German, and French, as well as in the peculiar mixture of German and Latin used in the correspondence between Euler and Goldbach. They also feature excerpts in Italian, Polish, Greek, and more.

For N.A., the progress of mathematics was an integral part of the development of human culture as a whole, making it entirely natural for him to reference figures such as Saint Augustine and Leonardo da Vinci, O. Spengler, the Brothers Grimm, O. Wilde, L. Carroll, and J. L. Borges, alongside J.S. Bach, G.F. Handel, D. Scarlatti, L. Giustini, J.-P. Rameau, Marco Polo (and, of course, the artist Vasya Lozhkin).

To mark the 300th anniversary of St. Petersburg University, the journal "Vestnik SPbGU" began publishing a series of historical reviews on the achievements of Petersburg and Leningrad mathematicians associated with the university. As part of this series, N.A.

Vavilov started writing a cycle of articles dedicated to the contributions of Petersburg mathematicians to the theory of linear, classical, and algebraic groups. Two articles [90], [91] from the intended four-article cycle were published; unfortunately, we will never know what was planned for the continuation...

5 N.A. Vavilov: Teaching and Applied Research

In his youth, Nikolai Alexandrovich wrote a short novell. By misfortune—or perhaps fortune—the text was lost, and literature never became his profession. Nevertheless, he wrote his academic works brilliantly. His pen was light and swift, and his texts radiated clarity and harmony. He employed a rich and generous language, unafraid of verbosity or adding extra touches of color. He spared no effort in providing explanations, eschewing brevity, yet often achieved extraordinary precision in his formulations. Over the years, his style became increasingly refined, and certain phrases seemed to leap onto the page on their own, guiding the reader toward the core ideas of each specific text. It is particularly painful now that the monograph N.A. Vavilov had planned and partially written, "The Theory of Chevalley Groups", never came to fruition. With his encyclopedic knowledge and literary mastery, it would undoubtedly have stood alongside the classic works of Steinberg, Carter, and Springer. It simply didn't happen. However, his series of textbooks for university undergraduates remains, presenting modern algebra in a unique and colorful manner: "Not Quite Naive Set Theory", "Not Quite Naive Linear Algebra", and "Concrete Group Theory".

For N.A., clarity of exposition was not only a mathematical criterion but also an ethical measure of the correctness and significance of results.

Recollections by Alexei Stepanov

"After Sasha Sivatski and I proved the boundedness of the lengths of commutators $[a, b]$*, where* $a \in GL_n(R)$ *and* $b \in E_n(R)$ *[55], Kolya decided to extend this result to all Chevalley groups instead of just the general linear group.*

However, in our work with Sivatskiy, in addition to localization, we used the method of transvection decomposition, which is not applicable to all Chevalley groups. So, Kolya said, 'If we can't use transvection decomposition, we'll use double localization.' After several discussions, I wrote a draft of the text. Kolya returned it to me, deeply disappointed. He said it couldn't be written like that – it was as if there was no result at all because, even though everything was formally correct, it was completely unclear why it worked. At that point, we parted ways: Kolya focused on finding precise estimates for zero-dimensional rings, while I tried to simplify the proof.

A couple of years(!) later, we returned to my text, which I still hadn't managed to simplify, though I could now explain the ideas to Kolya without the technical details. He was still dissatisfied, and we postponed writing the paper for another couple of years. Finally, I managed to write down the ideas I had been explaining to him. This time, he said: 'Yes, now I see that it can't be done any simpler.' He rewrote my entire text, inserting his own part, and the paper was ready.

Afterward, Kolya wrote several more papers, refining the technique of double localization to perfection and clearly explaining to all readers how it works and what kinds of problems it can solve."

Reminiscences by Vladimir Khalin

“The productivity of Nikolai Alexandrovich was astounding: when inspired, he could write a fully-fledged, thirty-page article in just a few days, and in impeccable English. This is exactly how 'The Skies are Falling: Mathematics for Non-Mathematicians' [92] was born. His preparatory materials for the course 'Mathematics and Computers,' only a fraction of which made it into the concise book 'Mathematica for Non-Mathematicians' [93], reportedly amounted to over a thousand pages, according to Vavilov himself!

To him, mathematics was '...the highest manifestation of human spirit and culture, valuable regardless of any applications.' This belief underscored the unique role of mathematics education in society, which he divided into three fundamentally different levels: pre-university, mathematics for mathematicians, and mathematics for non-mathematicians. Nikolai Alexandrovich believed that '...the most important aspect of teaching mathematics at the elementary level is fostering intellectual honesty: the ability to distinguish between what you understand and what you don't; between what has a precise meaning and what doesn't; between what is stated and what is implied; between the possible and the impossible; the true and the false; the proven and the conjectured.' Another equally important aspect was 'mental gymnastics,' training the brain to tackle any challenging problems.

At the university level, different goals took precedence — foremost among them, cultivating the mathematical way of thinking: the ability to start from first principles, consider the simplest case, use analogies and metaphors, generalize and specialize, and so forth. Of course, this also included developing a genuine understanding of mathematics and practicing fundamental modes of reasoning.

Professor N.A. Vavilov was the author of a unique approach to teaching mathematics to non-mathematicians at the university level, leveraging symbolic computation systems and computer algebra. Nikolai Alexandrovich believed that 'teaching mathematics should intrigue, captivate, and enchant' and advocated a new approach to mathematics education: delegating most routine calculations to computer algebra systems and focusing entirely on the conceptual side of mathematics, emphasizing the most important, useful, intriguing, and exciting layers of mathematics—concepts, ideas, analogies, constructions, and metaphors. He insisted that non-mathematicians should be 'taught mathematics as we, mathematicians, understand it — by prioritizing UNDERSTANDING' [92].

Starting in 2005, N.A. Vavilov began teaching his signature course, 'Mathematics and Computers,' at the Faculty of Economics at Saint Petersburg State University. The course focused on core mathematical ideas rather than specific applications. Here is how Nikolai Alexandrovich himself described this work: 'Initially, we introduced new mathematical concepts and ideas, as well as formulated several key statements, sometimes with sketches of proofs. Full proofs were presented only when they were particularly concise and illustrative or contained powerful general ideas useful in many contexts. We then transitioned to algorithms, computer demonstrations, computations, graphics, etc. With active participation and interest from the students, we managed to cover significantly more mathematics — more diverse, interesting, and ultimately more useful mathematics — than would have been possible with a more traditional approach' [93].

Students greatly appreciated N.A. Vavilov's approach, and even years later, they fondly remember the course. One of them wrote: 'Mathematics and Computers was one of the subjects in the “Applied Informatics in Economics” program at SPbU, and the corresponding book left a pleasant aftertaste. The concepts and algorithms described in the book remain highly relevant for solving any tasks requiring mathematics, which is indispensable for analysts or data professionals.' Another added: 'Mathematica for Non-Mathematicians is the only university textbook I occasionally revisit even ten years after graduation.”'

When it comes to mathematics teaching, N.A. Vavilov delivered an incredible variety of courses throughout his career. Here are just a few of them: "Algebraic Geometry", "Lie Algebras and Groups", "Algebraic Groups", "Hopf Algebras and Galois Theory", "Kac-Moody Algebras and Groups", "Category Theory", "Central Simple Algebras", "Computer Algebra", "Non-commutative Rings", "Representation Theory of Finite Groups", "Finite Groups of Lie Type", "Modular Representations of Finite Groups", "Unipotent Decompositions", "Exceptional Objects in Algebra and Geometry", "Cartan-Type Algebras", and "The Jacobian Problem". This list could go on, as it would be easier to name areas of algebra he did not lecture on than to compile a complete list of his courses. Without delving into the specifics of his teaching, one thing stands out: he was a MASTER. He not only delivered material but also had the rare ability to identify those select students who developed a true sense of the subject. These students would later write their theses under his supervision, defend their dissertations, and become researchers—all of which began with his undergraduate algebra courses.

But it did not end there. Every scholar whose name is associated with a school of thought has their own working secrets. For instance, Niels Bohr claimed his success lay in "never being afraid to tell his students he was a fool". Conversely, Lev Landau "never hesitated to tell his students that they were fools". N.A. Vavilov's signature approach was the ability to say, "We". We can do it, we will prove it, we will undoubtedly achieve the best result, we will write this theorem for all Chevalley groups, and so on. In most cases, this was only *partially* true. But the secret was that he always spoke with conviction. There were no "halves" in his statements. Later, something specific might emerge, but by then, it hardly mattered. He had a remarkable intuition for knowing what to say, when to say it, and to whom.

Vavilov's student, V. Nesterov, writes: "One of N.A.'s traits during discussions was his ability to strongly motivate others to tackle the given problem. He would captivate you with the problem, highlight its potential, and often point to overarching goals that could be achieved in the future. With his profound knowledge of the history of mathematics, N.A. often cited fascinating and inspiring examples." And here are observations from A. Luzgarev:

Recollections by Alexander Luzgarev

"Nikolay Alexandrovich started teaching us algebra in the second semester, at the beginning of 2000. I was astonished: he delivered an extraordinary amount of information, covering completely different areas of mathematics. It seems he began with the definition of a group, and by the second lecture, Lie algebras had already appeared. We probably understood only a fraction, but we felt his erudition and the broad scope of his material; and I think it was thanks to him that I first began to understand the unity of mathematics. Later, I realized that this was an important part of his method. I remember him saying: 'Some believe you should only say things that a person can understand, using words they already know. By this logic, you shouldn't talk to a baby at all.'

A couple of years later, I started attending his specialized courses and seminars, and he always encouraged even junior students to attend talks where they had no chance of understanding anything. According to Nikolay Alexandrovich, learning mathematics, like learning a foreign language, should happen through "immersion in the environment". He said: 'You listen, and some words, concepts are repeated many times and settle into your subconscious; after a while, you are no longer afraid of them, and a bit later, you suddenly start to understand everything."

N.A. loved working outside of classrooms. During walks, in cafés, or over friendly conversations, he could endlessly veer into his favorite topics, and everyone who worked with him knew this well. But when the blend of cultural and culinary musings was exhausted, the time for mathematics would arrive, and all the prior prelude suddenly seemed perfectly fitting. Alexander Luzgarev writes:

"By the third or fourth year of university, some of the classes on the Faculty of Mathematics and Mechanics were held on Kamskaya Street, near the Smolensk Cemetery. It just so happened that my classes often ended at the same time as the lectures Nikolay Alexandrovich gave to the younger students. We would walk together from Kamskaya Street to the Vasileostrovskaya metro station and then take the metro — he would go to Sennaya Ploshchad, while I continued further to Baltiyskaya.

This happened once or twice a week for several months. During the walks, he would constantly talk: about mathematics, history, or details of his meetings with foreign mathematicians. I had no idea what to say; it was practically a monologue. Soon, Igor Pevzner and I asked him for topics for term papers, then for our theses, and later we entered graduate school and defended our dissertations — all while attending almost every specialized course and seminar Nikolay Alexandrovich taught.

But what had the greatest influence on me were those very first algebra lessons with Nikolay Alexandrovich in my first year, and those frequent walks with him around Vasilievsky Island."

In time, numerous recollections of N.A. by his friends, colleagues, and students will undoubtedly be compiled. Let us include here an excerpt from the vivid memories of N.A. Vavilov's student, Viktor Petrov. These recollections are also dedicated to the distinctive style of teaching mathematics, in which N.A. Vavilov was a virtuoso.

Viktor Petrov recalls

"The classes were conducted in an absolutely inimitable style. Algebra practice, of course, implies solving computational problems, but perhaps only about one-tenth of the time was dedicated to that. The main focus was on enlightenment in the broadest sense of the word. And not just in mathematics, but also in linguistics (the actual number of cases in the Russian language, Indo-European languages, the Nostratic theory, the structure of Chinese characters...), and in philosophy (Nikolai Alexandrovich's favorite work was the treatise "Zhuangzi", and his favorite quote from it was: 'A white horse is not a horse, an abelian group is not a group'). The structure of his exposition most resembled Borges' "The Garden of Forking Paths": having mentioned a concept or answered a question, Nikolai Alexandrovich would immediately start discussing a new topic. For instance, the definition of a maximal ideal would come up — and immediately we'd hear about Stone's theorem on Boolean algebras, ultrafilters, hyperreal numbers, model theory... His delivery was emotional; Nikolai Alexandrovich gesticulated, used intonation, and employed tautological phrases for emphasis ('Exactly, precisely this way'). It was evident that he was overflowing with knowledge and impressions he wanted to share, because, as the saying goes, 'Out of the abundance of the heart, the mouth speaks.' "

The remarks of N.A. Vavilov's closest student, Alexei Stepanov, resonate in harmony.

Alexei Stepanov recalls

Figure 5: N. Vavilov and V. Petrov. Luminy, Marseille, 2015

"In the late 1990s and early 2000s, Kolya and I were in Bielefeld. I was going through one of my creative slumps, which Kolya decided to end by proving a result with me that, as he assured me, would definitely work. We met in his office, and Kolya confidently declared that we were going to prove that the arrangement of subgroups between $E_n(R)$ and $GL_n(F)$ is standard, where F is the field of fractions of the ring R. 'Well, at least for factorial rings, we'll prove this without any problems,' he assured me.

We began with the case of a polynomial ring $R = K[x, y]$ over a field K. Using the usual method of transvection decomposition, we quickly derived some transvections from an arbitrary matrix in $GL_n(F)$. Kolya said, 'See, just a little more, and we'll prove everything.' However, I was slightly unnerved by the fact that in all the transvections we extracted, the numerators contained more variables than the denominators. That is, if the matrix consisted of rational functions of degree 0 (i.e., the degree of the numerator equaled the degree of the denominator), we could only extract rational functions of positive degree. That's where we left it.

After thinking it over at home, I realized that if we took the ring $A = K[x, y, y/x] \subseteq F$ and factored it by the ideal generated by x and y, we would obtain a ring isomorphic to $K[z]$, while R would transform into K.

Disheartened, I shared this with Kolya. He said, 'Even better — proving the result for the pair $R \subseteq F$ will automatically yield the result for the pair $K \subseteq K[z]$. When I asked whether he believed in the latter result, he replied, 'Not really, but it's definitely true for the pair $R \subseteq F$!

The contradiction was resolved only a couple of years later, when I proved that for the pair $K \subseteq K[z]$, there is no standard arrangement of subgroups, and hence none for the pair $R \subseteq F$ either. I must say, the ideas used in this work ended my slump, so Kolya's original goal was achieved. In this way, Nikolai Alexandrovich's scientific optimism often served as a serious stimulus for my work."

All of Nikolai Alexandrovich's students have their own stories about his scientific generosity.

Sometime around 2005–2007, I found myself in a mild depression. It was a slightly *delayed midlife crisis, compounded by the need to write up results on subring subgroups. I lacked the necessary knowledge and found the task uninteresting because, ideologically, everything was already clear and (in my opinion) beautiful, but technically challenging.*

And, as usual, I was broke. Actually, worse than usual because the invitations to Bielefeld had stopped (their SFB funding had ended, or something like that).

Then, in 2007, Kolya came up to me and said: "I've written our joint article. Take a look — maybe you'll add something or suggest corrections".

I was stunned and asked, "What joint article?"

He replied, "Well, don't you remember? We discussed the generalized commutator formula in your office in Bielefeld. So, I've written down the results of our discussion. Everything worked out just as you said."

I vaguely remembered the discussion, but I had no memory of contributing anything substantive.

*However, no amount of persuasion that this article should have a single author had any effect. So, I fixed a couple of typos and agreed. The article was published in the *Vestnik of St. Petersburg State University* under the title "Standard Commutator Formulas".*

A couple of years later, another article, "Once More on the Standard Commutator Formula", was published with roughly the same level of my involvement.

In addition to his generosity, Kolya had a healthy dose of scientific pragmatism. Many people probably remember this period — it was a time when Russian universities almost stopped paying salaries, and we survived on grants, primarily foreign ones.

To secure these grants, it wasn't enough to achieve good results; you needed a lot of publications. Regarding this, Kolya used to say, "Every day — an observation, three observations — a lemma, three lemmas — a theorem, three theorems — an article. That's how you work!"

Of course, it was a joke. But Kolya wrote many articles, which secured grants that enabled many of his students and colleagues to pursue science without going hungry.

Vladimir Nesterov Remembers

"In 1991 (in my fourth year), I approached N.A. Vavilov to ask for a diploma project. N.A. offered me a choice of three problems. I intuitively chose the task of describing subgroups generated by pairs of short root unipotent subgroups in a Chevalley group. As far as I know, at that time, N.A. had two graduate students: the Bulgarian Alan Kharebov (they co-authored one paper) and another woman whose name I cannot recall.

In the spring of 1992, I defended my diploma, and in the summer of 1992, Vavilov left for Europe, and I didn't see him again until 2001. We corresponded by mail. In 1994, I sent him the text of my dissertation and a paper for a preprint, which was later published in Bielefeld. I must say that N.A. did an enormous amount of work correcting my paper, not only adding deep remarks to the introduction but also refining the presentation and improving the English.

*Thanks to N.A., the journal version of the paper was published in *Doklady RAN* in 1995. N.A. wrote all sorts of glowing reviews (clearly exaggerated), which allowed me to secure grants for several consecutive years. N.A. also made great efforts to ensure I could attend a school-conference on Crete in 1995, where my trip was fully funded. In the* *early 2000s, I finally published the results from my dissertation, and N.A. submitted a grant proposal for further research on toral generation. We were awarded a substantial grant.*

*Here's an interesting story. In 2016 or 2017, Nikolai Alexandrovich and I met at the cafe *Sladkoezhka* on Marata Street. After discussing scientific matters, N.A. told me about the dialects of the Spanish language, and we headed home. But in the metro, I realized that my wallet had been stolen at the cafe. N.A. immediately called home and said he would be delayed. We went together to the police station, where we waited in line for over half an hour. N.A. stayed until I finished writing a statement, and only then did we part ways.*

It so happened that much of my learning from N.A. occurred remotely. Yet, I began to truly understand many of my works only thanks to certain phrases N.A. included in his feedback or suggested adding to the introduction. It's amazing how just a few words can deepen your understanding and allow you to see things in a broader context.

Under N.A.'s guidance, I initially worked on generation by short root unipotent subgroups in Chevalley groups. At his suggestion, I described subsystem subgroups in the group of type F_4, generated by short root subgroups. We then shifted to generation by microweight tori in Chevalley groups. Together, we developed a reduction theorem and described subgroups generated by pairs of microweight tori in $GL(5)$ and $GL(6)$, which do not embed into $GL(4)$. Currently, my student from China is working on the general and most challenging case of $GL(4)$."

6 About the Man and His Time

Nikolai Alexandrovich loved life in all its manifestations. A passionate traveler, he visited numerous countries on scientific visits: Japan, India, China, Iran, Turkey, Thailand, Cambodia, almost all European countries, Israel, Canada, and the United States. Though a lover of comfort, he also went on expeditions along the Volga River, to Solovki, and through Northwest Russia.

He greatly valued friendly and scientific communication. After his trip to Crete, he referred to "my friend Deriziotis", and following his visit to Israel, he spoke of "my astral twin Aner Shalev". He was a welcome guest in Japan, maintained close relationships with many Italian mathematicians, and regularly lectured in Perugia. Bielefeld and, of course, Tony Bak played an important role in his life, and several fundamental works were written in England. At that time, it was a window to Europe that remained open. Simultaneously, from the early 1980s, close friendly and professional relationships developed between N.A. and many mathematicians from Minsk, Novosibirsk, Moscow, and Kyiv.

However, N.A.'s soul, of course, belonged to St. Petersburg. He dedicated his entire scientific life to the University and the Chebyshev Laboratory. He always held an active, even uncompromising, social position. The future of mathematics in St. Petersburg, the paths and directions for the development of mathematical life and education at the university touched the core of his being. A child of the Department of Mathematics and Mechanics, he stood at the cradle of the Chebyshev Laboratory's establishment, heavily involved in launching its undergraduate program: creating curricula, syllabi for algebraic disciplines, and more. With the establishment of the new Faculty of Mathematics and Computer Science (MCS) in 2019, N.A. transitioned to work there. Vasilievsky Island, with its courtyards, passages, cafes, and restaurants, was his environment. He proudly

Figure 6: Kolya..., 2015

showed the memorial plaque dedicated to Georg Cantor, chatted with familiar waitresses at the Italian café, and dreamed of an international mathematical congress in St. Petersburg.

A part of N.A.'s life unfolded on the Petrograd Side, not far from Vyazemsky Garden, where the Euler International Mathematical Institute is now located. Another part was spent at the corner of Sadovaya and Gorokhovaya Streets, where every path to the Steklov Institute (POMI) was well-trodden, where familiar doors of shops and bakeries opened, where the Fontanka River flowed calmly, along with life itself. In recent years, the center of N.A.'s existence shifted to his dacha in Toksovo. Here, naturally, it was Olga Sergeyevna who ruled, their grandson Yarik was raised, and it was here, in a house surrounded by a large garden, that many significant works were created, and countless articles were written.

Blessed is the memory of Nikolai Alexandrovich. The life lived nearby rises before us like a vast cloud. We trample through the daily grind without thinking of the happiness of being close to one another, of picking up the phone or turning on the camera and saying: "Hi... How are you?" Now it is time to speak invisibly, feeling with every cell the gaping void left behind in this world.

Blessed is the memory of you, Kolya. You created so much that you changed the destinies of those who crossed paths with you; you created a science that will never disappear, remaining a part of nature intertwined with time; you created a light that, as we believe, will shine across generations.

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