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# Engel-like characterization of radicals in finite dimensional Lie algebras and finite groups

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**Abstract.** A classical theorem of R. Baer describes the nilpotent radical of a finite group $G$ as the set of all Engel elements, i.e. elements $y \in G$ such that for any $x \in G$ the $n$th commutator $[x, y, \ldots, y]$ equals 1 for $n$ big enough. We obtain a characterization of the solvable radical of a finite dimensional Lie algebra defined over a field of characteristic zero in similar terms. We suggest a conjectural description of the solvable radical of a finite group as the set of Engel-like elements and reduce this conjecture to the case of a finite simple group.

## Contents



## 1. Introduction

In the present paper we study certain Engel-like characterizations of the solvable radical of finite dimensional Lie algebras and finite groups. Such a characterization for the nilpotent radical of a finite group is given by R. Baer [Ba], see also [H], [Ro]:

**Theorem 1.1.** *The nilpotent radical of a finite group G coincides with the collection of all Engel elements of G.*

This theorem of Baer generalizes Zorn's theorem [Zo] which gives a characterization of finite nilpotent groups in terms of special two-variable Engel identities.

Our goal is to establish an analog of Baer's theorem for the solvable radical of a finite group. As a first step, we consider a similar problem for finite dimensional Lie

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algebras. Towards this end, we introduce some sequences of words which provide the solvability property. These sequences play the role of the Engel sequences that work in the nilpotent case.

All groups in the paper are assumed to be finite, all Lie algebras are finite dimensional. Observe, however, that Baer's theorem is valid for many classes of infinite groups satisfying some finiteness conditions. For example, Baer himself proved it for noetherian groups, a similar result is valid for radical groups [Plo1], [Plo2], linear groups [Pla], and PI-groups [Plo3] (see also [Pr], [To]).

The paper is organized as follows. In Section 2 all main definitions and statements of the problems are given. Section 3 deals with the case of Lie algebras. In Section 4 we prove a reduction theorem for the case of groups.

## 2. Definitions and main problems

We follow the terminology introduced in [Plo3].

Let $F_2 = F(x, y)$ be the free two generator group, $W_2 = W(x, y)$ the free two generator Lie algebra.

**Definition 2.1.** *We say that a sequence* $\overrightarrow{u} = u_1, u_2, u_3, \ldots, u_n, \ldots$ *of elements from* $F_2$ *is* **correct** *if the following conditions hold:*

*(i)* $u_n(a, 1) = 1$ *and* $u_n(1, g) = 1$ *for all sufficiently big n, every group G, and all elements* $a, g \in G$*;*

*(ii) if* $a, g$ *are elements of* $G$ *such that* $u_n(a, g) = 1$*, then for every* $m > n$ *we have* $u_m(a, g) = 1$.

Thus, if the identity $u_n(x, y) \equiv 1$ is satisfied in $G$, then for every $m > n$ the identity $u_m(x, y) \equiv 1$ also holds in $G$.

A similar definition can be given for Lie algebras.

**Definition 2.2.** *For every correct sequence* $\overrightarrow{u}$ *define the class of groups (resp. Lie algebras)* $\Theta = \Theta(\overrightarrow{u})$ *by the rule: a group (resp. Lie algebra) G belongs to* $\Theta$ *if and only if there is n such that the identity* $u_n(x, y) \equiv 1$ *(resp.* $u_n(x, y) \equiv 0$*) holds in G.*

**Definition 2.3.** *For every group (resp. Lie algebra) G denote by* $G(\overrightarrow{u})$ *the subset of G defined by the rule:* $g \in G(\overrightarrow{u})$ *if and only if for every* $a \in G$ *there exists* $n = n(a, g)$ *such that* $u_n(a, g) = 1$ *(resp. 0). Elements of* $G(\overrightarrow{u})$ *are viewed as Engel elements with respect to the given correct sequence* $\overrightarrow{u}$*. We call these elements* $\overrightarrow{u}$ **-Engel-like** *or, for brevity,* $\overrightarrow{u}$ **-Engel elements**.

**Examples 2.4.**

(1) If $\overrightarrow{u} = \overrightarrow{e} = e_1, e_2, e_3, \ldots$, where

$$e_1(x, y) = [x, y] = xyx^{-1}y^{-1}, \ldots, e_n(x, y) = [e_{n-1}(x, y), y], \ldots,$$

then $\Theta(\overrightarrow{e})$ is the class of all Engel groups. In the case of finite groups the class $\Theta(\overrightarrow{e})$ coincides with the class of finite nilpotent groups [Zo].

Clearly, $\overrightarrow{e}$-Engel elements of any group $G$ are none other than usual Engel elements. If $G$ is finite, the set $G(\overrightarrow{e})$ coincides with the nilpotent radical of $G$ [Ba].

(2) If $\overrightarrow{u}$ is determined by the following correct sequence of words:

$$u_1 = xy^{-1}, u_2 = u_1(xy, yx) = [x, y], \dots, u_n = u_{n-1}(xy, yx), \dots,$$

then for finite groups the class $\Theta(\overrightarrow{u})$ coincides with the class of all finite nilpotent-by-two groups [BP].

(3) The correct sequence $\overrightarrow{u}$, where

$$u_1(x, y) = x^{-2}y^{-1}x, \dots,$$
$$u_n(x, y) = [\, x\, u_{n-1}(x, y)\, x^{-1},\ y\, u_{n-1}(x, y)\, y^{-1}\,], \dots,$$

determines the class $\Theta(\overrightarrow{u})$ which coincides with the class of finite solvable groups [BGGKPP1], [BGGKPP2].

(4) The correct sequence $\overrightarrow{s}$, where

$$s_1(x, y) = x, \dots, s_n(x, y) = [s_{n-1}(x, y)^{-y}, s_{n-1}(x, y)], \dots$$

determines the class $\Theta(\overrightarrow{s})$ which also coincides with the class of finite solvable groups [BWW].

(5) The correct sequence $\overrightarrow{w}$, where

$$w_1(x, y) = [x, y], \dots, w_n(x, y) = [[w_{n-1}, x], [w_{n-1}, y]], \dots$$

and $[\ ,\ ]$ stands for the Lie bracket in a Lie algebra, determines the class $\Theta(\overrightarrow{w})$ of finite dimensional solvable Lie algebras over an infinite field $k$, $\mathrm{char}(k) \neq 2, 3, 5$ [GKNP].

It is easy to see that if $g$ is a $\overrightarrow{u}$-Engel element in $G$ then $g$ is a $\overrightarrow{u}$-Engel element in every subgroup $H$ containing $g$. If $H$ is a normal subgroup in $G$ and $g$ is a $\overrightarrow{u}$-Engel element in $G$, then $\bar{g} = gH$ is a $\overrightarrow{u}$-Engel element in $\bar{G} = G/H$.

The following natural problem arises:

**Problem 2.5.** Describe the class of sequences $\overrightarrow{u}$ such that $\Theta(\overrightarrow{u})$ is the class of finite solvable groups.

Denote by $F_2^{(n)}(x, y)$ the $n$-th term of the derived series of the free group $F_2(x, y)$.

**Proposition 2.6.** *Let $\overrightarrow{u}$ be a correct sequence. Then $\Theta(\overrightarrow{u})$ coincides with the class of finite solvable groups if and only if*

*(i) for every $n$ there exists $k = k(n)$ such that $u_k$ belongs to $F_2^{(n)}(x, y)$;*

*(ii) there is no $n$ such that $u_n(x, y) = 1$ is an identity in one of the following groups:* (1) $G = \mathrm{PSL}(2, \mathbb{F}_q)$ *where $q \geq 4$ is a prime power,* (2) $G = \mathrm{Sz}(2^m)$, $m \in \mathbb{N}$, $m \geq 3$ *and odd,* (3) $G$=$\mathrm{PSL}(3, \mathbb{F}_3)$.

*Here* $\mathrm{PSL}(m, \mathbb{F}_q)$ *denotes the projective special linear group of degree m over* $\mathbb{F}_q$. *For* $q = 2^m$ *we denote by* $\mathrm{Sz}(q)$ *the Suzuki group (the twisted form of* ${}^2B_2$, *see [HB, XI.3]).*

*Proof.* The proof repeats the proof of Theorem 2.1 from [BGGKPP1] and is based on the list of minimal simple non-solvable groups [Th]. □

*Remark 2.7.* Even for the case of the group $G = \mathrm{PSL}(2, \mathbb{F}_q)$ the basis of identities is known only for small fields $\mathbb{F}_q$ and not known in general [CMS], [So]. The bases look highly complicated and do not provide any hint to check explicitly if a particular identity follows from the basis.

**Problem 2.8.** Describe the class of sequences $\overrightarrow{u}$ such that $\Theta(\overrightarrow{u})$ is the class of finite dimensional solvable Lie algebras.

Denote by $W_2^{(n)}(x, y)$ the $n$-th term of the derived series of the free Lie algebra $W_2(x, y)$.

**Proposition 2.9.** *Let* $\overrightarrow{u}$ *be a correct sequence. Then* $\Theta(\overrightarrow{u})$ *coincides with the class of finite dimensional solvable Lie algebras over an infinite field k,* $\mathrm{char}(k) \neq 2, 3, 5$ *if and only if*

*(i) for every n there exists* $k = k(n)$ *such that* $u_k$ *belongs to* $W_2^{(n)}(x, y)$;
*(ii) there is no n such that* $u_n(x, y) = 0$ *is an identity in the simple Lie algebra* $\mathrm{sl}_2$.

*Proof.* The proof immediately follows from the fact that if the characteristic of $k$ differs from 2,3,5, then every simple Lie $k$-algebra contains a subalgebra which has the algebra $\mathrm{sl}_2$ as a quotient (see [GKNP] and references therein). □

*Remark 2.10.* The assumption $\mathrm{char}(k) \neq 3, 5$ is technical and can probably be dropped.

*Remark 2.11.* The situation with bases of identities for the minimal simple non-solvable Lie algebras is different from the group case. A basis of identities of the algebra $\mathrm{sl}_2$ over a field of characteristic zero is known and consists of two identities [Ra], [Bah] (see [MK] for the case where the ground field is finite, and [Ko] and references therein for the case where the ground field is infinite of characteristic $p$). However, it is difficult to verify whether a particular $u_n$ is an identity in $\mathrm{sl}_2$ using this basis.

**Conjecture 2.12.** *There is a sequence* $\overrightarrow{u} = \overrightarrow{u}(x, y)$ *such that for every finite group G the solvable radical of G coincides with* $G(\overrightarrow{u})$.

**Conjecture 2.13.** *There is a sequence* $\overrightarrow{u} = \overrightarrow{u}(x, y)$ *such that for every finite dimensional Lie algebra L the solvable radical of L coincides with* $L(\overrightarrow{u})$.

Of course, it is highly desirable not only to prove the existence of such a $\overrightarrow{u}$, but to exhibit an explicit sequence in each of the above cases (groups and Lie algebras). We give a partial positive answer to Conjecture 2.13 in Section 3 (see Theorems 3.7 and 3.12), and reduce Conjecture 2.12 to the case of simple groups in Section 4 (see Theorem 4.9).

## 3. Case of Lie algebras

Let $L$ be a finite dimensional Lie algebra over a field $k$. Denote by $[,]$ the Lie operation. For $t \in L$ the linear operator $\operatorname{ad} t\colon L \to L$ is defined by $(\operatorname{ad} t)x = [t, x]$.

By the *solvable radical* of $L$ we mean the largest solvable ideal $R$ of $L$ (Bourbaki [Bou] and Jacobson [J] call $R$ the radical of $L$). By the *nilpotent radical* of $L$ we mean the largest nilpotent ideal $N$ of $L$ (Jacobson [J] calls $N$ nil radical, and Bourbaki [Bou] calls it just the largest nilpotent ideal).

Define the sequence $\overrightarrow{e}$ by $e_1(x, y) = [x, y]$ and, by induction, $e_{n+1}(x, y) = [e_n(x, y), y]$. Then $e_n(x, y) = (-\operatorname{ad} y)^n x$.

**Definition 3.1.** *An element $y \in L$ is called an* Engel *element if it is $\overrightarrow{e}$-Engel, i.e. for every $x \in L$ there exists $n$ such that $e_n(x, y) = 0$ (i.e. $(\operatorname{ad} y)^n x = 0$).*

The following proposition is well known.

**Proposition 3.2.** *An element $y \in L$ is Engel if and only if* $\operatorname{ad} y$ *is nilpotent (i.e. there exists $n$ such that $(\operatorname{ad} y)^n = 0$).*

We need a lemma.

**Lemma 3.3.** *Let $V$ be a vector space of dimension $d$ over a field $k$, let $x \in V$, and let $A\colon V \to V$ be a linear map. If $A^m x = 0$ for some $m$, then $A^d x = 0$.*

*Proof of the lemma.* By Fitting's lemma (see [J, Sect. II.4]), $V = V_0 \oplus V_1$, where $V_0$ and $V_1$ are invariant subspaces of $A$, the restriction $A_0$ of $A$ to $V_0$ is nilpotent, and the restriction $A_1$ of $A$ to $V_1$ is invertible. Write $x = (x_0, x_1)$, where $x_i \in V_i$ $(i = 0, 1)$. We have $A_1^m x_1 = 0$, hence $x_1 = 0$ (because $A_1$ is invertible), i.e. $x \in V_0$. Since $A_0$ is nilpotent, we have $A_0^{d_0} = 0$ where $d_0 = \dim V_0$. We thus have $A^{d_0} x = 0$ (because $x \in V_0$). Since $d \geq d_0$, we conclude that $A^d x = 0$. □

*Proof of the proposition.* If $\operatorname{ad} y$ is nilpotent, then clearly $y$ is Engel. Conversely, if $y$ is Engel, then by Lemma 3.3 $(\operatorname{ad} y)^d x = 0$ for any $x \in L$, where $d = \dim L$. Hence $(\operatorname{ad} y)^d = 0$ and $\operatorname{ad} y$ is nilpotent. □

Define the sequence $\overrightarrow{v}$ by $v_1(x, y) = x$ and, by induction, $v_{n+1}(x, y) = [v_n(x, y), [x, y]]$. Then $v_{n+1}(x, y) = (-\operatorname{ad}[x, y])^n x = e_n(x, [x, y])$.

**Theorem 3.4.** *Let $L$ be a finite dimensional Lie algebra over a field $k$ of characteristic zero. Then $L$ is solvable if and only if for some $n$ the identity $v_n(x, y) \equiv 0$ holds in $L$.*

*Proof.* Since $\operatorname{char}(k) = 0$, if $L$ is solvable, then $L' = [L, L]$ is nilpotent [J, Cor. II.7.1]. Hence every pair $z, t$ of elements of $L'$ satisfies the identity $(\operatorname{ad} t)^m z = 0$, where $m = \dim L'$. On putting $z = [x, [x, y]]$, $t = [x, y]$, we get $v_{m+2}(x, y) = 0$.

In the opposite direction, we mimic the proof of [GKNP, Thm. 3.1]. Namely, we first reduce to the case when $k$ is algebraically closed. This reduction immediately follows from the following easy lemma (cf. [GKNP]).

**Lemma 3.5.** *Let $k$ be an infinite field and let $P(x_1, \dots, x_n)$ be a polynomial in $n$ variables over $k$. If $P(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in k$, then for any field extension $K/k$ we have $P(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in K$.*

*Proof.* Since $k$ is infinite, by [L, Cor. IV.1.7] $P$ is the zero polynomial, and the lemma follows. □

*Remark 3.6.* A similar statement is valid for the elements of free associative and free Lie algebras, i.e. for non-commutative associative and Lie polynomials.

Thus we now assume $k$ algebraically closed and suppose that $L$ satisfies $v_n(x, y) \equiv 0$ and is not solvable. Denote by $R$ the solvable radical of $L$. The algebra $L/R$ is not zero and semisimple and also satisfies the identity $v_n(x, y) \equiv 0$. It contains a subalgebra $S$ isomorphic to $\mathrm{sl}_2$ which also must satisfy the same identity. Being isomorphic to $\mathrm{sl}_2$, the Lie algebra $S$ has a basis $\{e_+, e_-, h\}$ such that $[h, e_+] = 2e_+$, $[h, e_-] = -2e_-$, $[e_+, e_-] = h$. Take $x = e_+$, $y = e_-$. We have $[x, y] = h$, hence $v_2(x, y) = [e_+, h] = -2e_+$, and for any $n$ we get $v_n(x, y) = (-2)^{n-1}e_+ \neq 0$, contradiction. The theorem is proved. □

It turns out that the same sequence $v_n$ allows one to describe the solvable radical of $L$.

**Theorem 3.7.** *Let $L$ be a finite dimensional Lie algebra over a field $k$ of characteristic zero. Then its solvable radical $R$ coincides with the set of all $\overrightarrow{v}$-Engel elements of $L$.*

*Proof.* Let us first prove that every element $y$ of $R$ is $\overrightarrow{v}$-Engel. Since $\mathrm{char}(k) = 0$, by [J, Thm. II.7.13], $[L, R]$ is a nilpotent ideal. Hence every element $t$ of $[L, R]$ is Engel in $L$ [J, Thm. II.3.3], i.e. for any $z \in L$ there exists $n$ such that $(\mathrm{ad}\, t)^n z = 0$. Let $x$ be an arbitrary element of $L$. On setting $t = [x, y]$, $z = x$, we get $v_{n+1}(x, y) = 0$.

Conversely, let us prove that every $\overrightarrow{v}$-Engel element lies in the solvable radical. Let us first consider the case where $k$ is algebraically closed. It is enough to prove that there are no nonzero $\overrightarrow{v}$-Engel elements in a simple $k$-algebra $L$.

Let $L$ be a simple Lie algebra over $k$. Choose a Cartan subalgebra $H \subset L$ and a Borel subalgebra $B \supset H$. Let

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

be the root decomposition. Here $\Phi = \Phi(L, H)$ is a reduced irreducible root system, and all the spaces $L_\alpha$ are 1-dimensional. Let $\Phi^+$ be the set of positive roots determined by $B$. We have

$$L = L^- \oplus H \oplus L^+, \text{ where } L^+ = \bigoplus_{\alpha \in \Phi^+} L_\alpha,\ L^- = \bigoplus_{\alpha \in \Phi^+} L_{-\alpha}\ .$$

We write an element $y \in L$ in the form $y = y^- + h + y^+$, where $y^- \in L^-$, $h \in H$, $y^+ \in L^+$. For every $\alpha \in \Phi$ choose a nonzero element $e_\alpha \in L_\alpha$. The set $\{e_{-\alpha} : \alpha \in \Phi^+\}$ is a basis in $L^-$. Write $y^- = \sum_{\alpha \in \Phi^+} c_{-\alpha}(y) e_{-\alpha}$, where $c_{-\alpha}(y) \in k$.

Let $y \in L$, $y \neq 0$, $y = y^- + h + y^+$. Since $\Phi$ is irreducible, it has a unique maximal root; we denote it by $\gamma$. We may assume that $H$ and $B$ are chosen so that the coefficient $c_{-\gamma}(y)$ of $y^-$ at $e_{-\gamma}$ is nonzero.

Indeed, let Aut $L$ denote the algebraic group of automorphisms of $L$, and let $G$ be the identity component of Aut $L$. It suffices to prove that there exists $g \in G(k)$ such that $c_{-\gamma}(gy) \neq 0$. Assume the contrary, i.e. that $c_{-\gamma}(gy) = 0$ for any $g$. Set $V = \{z \in L \mid c_{-\gamma}(z) = 0\}$, it is a subspace of $L$ of codimension 1. Let $W(y)$ denote the vector space generated by the elements of the form $gy$ for $g \in G(k)$, it is a nonzero $G$-invariant subspace of $L$. Since by assumption $W(y) \subset V$, we conclude that $W(y) \neq L$. This leads to a contradiction, because the representation of $G$ in $L$ is irreducible for a simple Lie algebra $L$.

So let $y \in L$, $y \neq 0$, and $c := c_{-\gamma}(y) \neq 0$. We shall find $x$ such that $v_n(x, y) \neq 0$ for any $n$. Take $x = e_\gamma$. Denote $h_\gamma = [e_\gamma, e_{-\gamma}]$, it is a nonzero element of $H$. We have $[h_\gamma, e_\gamma] = ae_\gamma$ with $a \neq 0$. Since $\gamma$ is the maximal root, we have $[x, y] = ch_\gamma + y_1^+$ where $y_1^+ \in L^+$. Hence we have $[x, [x, y]] = [e_\gamma, ch_\gamma + y_1^+] = -cae_\gamma$, and, by induction, $v_n(x, y) = (-ca)^{n-1}e_\gamma$. Since $a \neq 0$ and $c \neq 0$, we see that $v_n(x, y) \neq 0$ for all $n$. Thus there are no nonzero $\overrightarrow{v}$-Engel elements in $L$. This proves our theorem in the case when $k$ is algebraically closed.

To reduce the general case to the case of the algebraically closed ground field, we shall show that if $y$ is a $\overrightarrow{v}$-Engel element of a finite dimensional Lie algebra $L$, then one can choose $n$ in the condition $v_n(x, y) = 0$ to be independent of $x$. Actually we shall show that one can choose $n$ to be independent of $x$ and $y$.

**Lemma 3.8.** *Assume that $v_n(x, y) = 0$ for some $x, y \in L$ and for some natural $n$. Then $v_{d+1}(x, y) = 0$, where $d = \dim L$.*

*Proof.* We apply Lemma 3.3 to the linear operator $A = -\mathrm{ad}\,[x, y]\colon z \mapsto [z, [x, y]]$ in the linear space $L$. □

We can now complete the proof of Theorem 3.7. Let $y \in L$ be a $\overrightarrow{v}$-Engel element of $L$. This means that for any $x \in L$ there exists $n$ such that $v_n(x, y) = 0$. By Lemma 3.8, we then get $v_{d+1}(x, y) = 0$ for any $x \in L$. This is a polynomial identity in $x$. Since the field $k$ is infinite, it follows from Lemma 3.5 that $v_{d+1}(\bar{x}, y) = 0$ for any $\bar{x} \in \overline{L}$, where $\overline{L} = L \otimes \bar{k}$. In other words, $y$ is a $\overrightarrow{v}$-Engel element of $\overline{L}$. Since our theorem is already proved over an algebraically closed field, we see that $y \in \overline{R}$, where $\overline{R}$ is the solvable radical of $\overline{L}$. But $\overline{R} = R \otimes_k \bar{k}$ [Bou, Ch. I, §5, n°6], hence $y \in R$. □

*Remark 3.9.* The sequence $\overrightarrow{v}$ is adjusted to the case of Lie algebras over a field of characteristic zero. Indeed, the key point in the proof of Theorems 3.4 and 3.7 was the fact that if $L$ is solvable then $[L, L]$ is nilpotent. This is no longer true in positive characteristic. Here is an explicit counter-example to the corresponding statements in positive characteristic, based on the fact that $[L, R]$ is no longer contained in the nilpotent radical of $L$.

*Example 3.10.* We use an example given in [J, II.7, pp. 52–53]. Let $k$ be a field of characteristic $p > 0$, and $L$ be a vector $k$-space of dimension $p + 2$. Denote by

$\{e, f, e_1, \dots, e_p\}$ a basis of $L$ and define a structure of Lie algebra by the following multiplication table:

$$[e, f] = e,\ [e, e_i] = e_{i+1} \quad (1 \le i \le p-1),\ [e, e_p] = e_1,$$
$$[f, e_i] = (i-1)e_i \quad (1 \le i \le p),\ [e_i, e_j] = 0 \quad (1 \le i, j \le p). \tag{3.1}$$

From formulas (3.1) it follows immediately that the subalgebra $M = \langle e_1, \dots, e_p \rangle$ is an abelian ideal of $L$ and the quotient $S = L/M$ is a two-dimensional solvable algebra, hence $L$ is solvable (see [J, *loc. cit.*]). We shall show that $L$ does not satisfy any of identities $v_n(x, y) \equiv 0$. Indeed, take $x = f + e_1$, $y = e + e_2$. We have

$$t := [x, y] = -e,\ v_1(x, y) = x,\ v_2(x, y) = [x, t] = e + e_2,\ v_3(x, y) = e_3, \dots,$$

and, by induction, $v_p(x, y) = e_p$, $v_{p+1}(x, y) = e_1, \dots, v_{mp+r}(x, y) = e_r, \dots$. Thus for all $n$ we have $v_n(x, y) \neq 0$, and Theorem 3.4 fails in characteristic $p$.

The same algebra $L$ provides a counter-example in characteristic $p$ to the statement of Theorem 3.7, because it is solvable but not all of its elements are $\overrightarrow{v}$-Engel (for example, $y$ as above is not).

Let us now consider another sequence $\overrightarrow{w}$ which hopefully will be extendable to the case of positive characteristic. Define $w_1(x, y) = [x, y]$ and, by induction, $w_{n+1}(x, y) = [[w_n(x, y), x], [w_n(x, y), y]]$.

**Theorem 3.11.** [GKNP] *Let $L$ be a finite dimensional Lie algebra over a field $k$ of characteristic different from 2, 3, 5. Then $L$ is solvable if and only if for some $n$ the identity $w_n(x, y) \equiv 0$ holds in $L$.*

We hope that the same sequence $w_n$ allows one to describe the solvable radical of $L$.

**Theorem 3.12.** *Let $L$ be a finite dimensional Lie algebra over an algebraically closed field $k$ of characteristic zero. Then its solvable radical $R$ coincides with the set of all $\overrightarrow{w}$-Engel elements of $L$.*

*Proof.* Let us first prove that every element $y$ of $R$ is $\overrightarrow{w}$-Engel. Since $R$ is an ideal of $L$, we have $w_1(x, y) \in R$ for any $x \in L$. From the definition of $w_n$ it follows that $w_2(x, y) \in [R, R]$, and, by induction, $w_n(x, y)$ belongs to the $(n-1)$th term of the derived series of $R$. Since $R$ is solvable, for some $n$ we have $w_n(x, y) = 0$.

In the opposite direction, we have to prove that every $\overrightarrow{w}$-Engel element lies in the solvable radical. It is enough to prove that there are no nonzero $\overrightarrow{w}$-Engel elements in a simple $k$-algebra $L$.

Let $L$ be a simple Lie algebra over $k$. As in the proof of Theorem 3.7, we choose a Cartan subalgebra $H \subset L$ and a Borel subalgebra $B \supset H$, and we choose $e_\alpha \in L_\alpha$, $e_\alpha \neq 0$, for all $\alpha \in \Phi = R(L, H)$. We write an element $y \in L$ as $y = y^- + h + y^+$, where $y^- \in L^-$, $h \in H$, $y^+ \in L^+$. Write $y^- = \sum_{\alpha \in \Phi^+} c_{-\alpha}(y) e_{-\alpha}$. Let $\gamma$ denote the maximal root in $\Phi$.

Let $y \in L$, $y \neq 0$. As in the proof of Theorem 3.7, we may and shall assume that $c := c_{-\gamma}(y) \neq 0$. We shall find $x$ such that $w_n(x, y) \neq 0$ for any $n$. Take $x = e_\gamma$.

Denote $h_\gamma = [e_\gamma, e_{-\gamma}]$, it is a nonzero element of $H$. We have $[h_\gamma, e_\gamma] = ae_\gamma$ with $a \neq 0$, then $[h_\gamma, e_{-\gamma}] = -ae_{-\gamma}$. Since $\gamma$ is the maximal root, we have

$$w_1(x, y) = [x, y] = ch_\gamma + y_1^+ \text{ where } y_1^+ \in L^+.$$

Hence we have

$$[w_1(x, y), x] = [ch_\gamma + y_1^+, e_\gamma] = c[h_\gamma, e_\gamma] = cae_\gamma$$

(we have $[y_1^+, e_\gamma] = 0$ because $\gamma$ is the maximal root). Furthermore,

$$c_{-\gamma}([w_1(x, y), y]) = c_{-\gamma}([ch_\gamma, y]) = c_{-\gamma}([ch_\gamma, ce_{-\gamma}]) = -c^2a,$$

(here once again we use the assumption that $\gamma$ is the maximal root). As at the first step, we obtain

$$w_2(x, y) = [[w_1(x, y), x], [w_1(x, y), y]] = -c^3a^2h_\gamma + y_2^+, \text{ where } y_2^+ \in L^+.$$

By induction we conclude that for all $n \geq 2$ we have

$$w_n(x, y) = -c^{2^n-1}a^{2^n-2}h_\gamma + y_n^+, \text{ where } y_n^+ \in L^+.$$

Since $c \neq 0$ and $a \neq 0$, we see that $w_n(x, y) \neq 0$ for all $n$. Thus there are no nonzero $\overrightarrow{w}$-Engel elements in $L$. □

*Remark 3.13.* To extend Theorem 3.12 to the case of any field $k$ of characteristic zero, it is enough to show that any $\overrightarrow{w}$-Engel element $y \in L$ remains $\overrightarrow{w}$-Engel in $\bar{L} = L \otimes_k \bar{k}$. To do that, it suffices to reverse the order of quantifiers in the definition of a $\overrightarrow{w}$-Engel element, i.e. to show that if $y \in L$ is a $\overrightarrow{w}$-Engel element, then there exists $n$ (depending only on $y$) such that for all $x \in L$ we have $w_n(x, y) = 0$. This would imply that $y$ remains an Engel element in $\bar{L}$. Indeed, with such a choice of $n$ the relation $w_n(x, y) = 0$ is a polynomial identity in $L$ (with respect to the variable $x$), and it remains true as an identity in $\bar{L}$ (cf. Lemma 3.5). In other words, it is enough to show that the sequence $A_n(y) = \{x \in L : w_n(x, y) = 0\}, n = 1, 2, \ldots,$ of subsets of $L$ stabilizes provided $y$ is a $\overrightarrow{w}$-Engel element. In light of Theorem 3.12, this is valid in the case when $k$ is an algebraically closed field. We can also prove this fact for any uncountable field $k$, see Proposition 3.14 below.

Let $k$ be a field. We fix an algebraically closed field $\Omega$ containing $k$. Let $S$ be a set of polynomials in the polynomial ring $k[X_1, \ldots, X_m]$ in $m$ variables. By a $k$-closed set (or algebraic $k$-set) in the affine space $\mathbb{A}^m$ we mean the set $B$ of common zeros in $\Omega^m$ of such a set of polynomials $S$ (cf. [L, Ch. IX, §§1,2]). In particular, we say that $B = \mathbb{A}^m$ if $B = \Omega^m$. By $B(k)$ we denote the set of zeros of $S$ in $k^m$, i.e. $B(k) = B \cap k^m$.

**Proposition 3.14.** *Let $k$ be an uncountable field, and $\mathbb{A}^m$ be an affine space over $k$. Consider a sequence of $k$-closed sets $B_n \subset \mathbb{A}^m$, such that $B_1 \subseteq B_2 \subseteq \ldots B_n \subseteq \ldots$ and $\bigcup_1^\infty B_i(k) = \mathbb{A}^m(k)$. Then there is $n_0$ such that $B_{n_0} = \mathbb{A}^m$.*

*Proof.* We use induction on $m$.

*Step* 1. If $m = 1$ then any $k$-closed set is either finite or equal to $\mathbb{A}^1$. Thus,

— either $B_i$ are all finite, and $\bigcup_1^\infty B_i(k)$ contains at most countable set of elements, which contradicts to the assumption $\bigcup_1^\infty B_i(k) = \mathbb{A}^m(k)$;

— or there is $n_0$ such that $B_{n_0} = \mathbb{A}^1$.

*Step* $m$. Assume that the claim is valid for all $m' < m$. Let $x_1, \dots, x_m$ be the coordinates in $\mathbb{A}^m$, and let $H_0$ denote the hyperplane defined by the equation $x_1 = 0$.

Fix a point $\alpha = (0, a_2, \dots, a_m) \in H_0(k)$ and a line

$$L_\alpha = \{(t, a_2, \dots, a_m) \mid t \in \Omega\}.$$

Since $\cup(B_i \cap L_\alpha)(k) = (\cup B_i(k)) \cap L_\alpha(k) = L_\alpha(k)$, there is $n(\alpha)$ such that $B_{n(\alpha)} \cap L_\alpha = L_\alpha$ (see Step 1).

For a natural number $l$ define a set

$$D_l = \{\alpha \in H_0(k) : n(\alpha) \le l\} \subset H_0(k).$$

In other words, $\alpha \in D_l$ means that $L_\alpha \subset B_l$. Let us show that $D_l = V_l(k)$ for some $k$-closed subset $V_l$ of $H_0$.

Let $B_l$ be defined in $\mathbb{A}^m$ by a set of polynomial equations

$$\begin{cases} F_{1,l}(x_1, \dots, x_m) = \sum x_1^i p_{i,1,l}(x_2, \dots, x_m) = 0; \\ F_{2,l}(x_1, \dots, x_m) = \sum x_1^i p_{i,2,l}(x_2, \dots, x_m) = 0; \\ \dots \\ F_{s_l,l}(x_1, \dots, x_m) = \sum x_1^i p_{i,s_l,l}(x_2, \dots, x_m) = 0. \end{cases}$$

The condition $L_\alpha \subset B_l$ implies that

$$F_{j,l}(t, a_2, \dots, a_m) = 0$$

for any $t \in \Omega$, i.e. $p_{i,j,l}(a_2, \dots, a_m) = 0$ for all possible $i$, $j$. Thus

$$D_l = \{\alpha \in H_0(k) : \ p_{i,j,l}(\alpha) = 0\},$$

and clearly $D_l = V_l(k)$ for the $k$-closed subset $V_l$ of $H_0$ defined by these equations.

On the other hand, $\bigcup V_l(k) = \bigcup D_l = H_0(k)$. Thus, by the induction hypothesis there exists $l_0$ such that $V_{l_0} = H_0$, and, consequently, $B_{l_0} = \mathbb{A}^m$. □

**Corollary 3.15.** *Let $L$ be a finite dimensional Lie algebra over an uncountable field $k$ of characteristic zero. Then its solvable radical $R$ coincides with the set of all $\overrightarrow{w}$-Engel elements of $L$.*

*Remark 3.16.* Proposition 3.14 is not valid for countable fields.

Here is an example. Let $x_1, \dots, x_n, \dots$ be the countable set of all the $k$-points of the affine space $\mathbb{A}^m$ over a countable field $k$ for $m > 1$. Denote by $L_n$ the straight line containing $x_n$ and the origin, and set $B_n = \bigcup_1^n L_i$. Then $\bigcup_1^\infty B_n(k) = \mathbb{A}^m(k)$.

Nevertheless, our conjecture is that Theorem 3.12 is valid for any field of characteristic zero.

*Remark 3.17.* Theorem 3.12 does not hold in positive characteristic because simple Lie algebras may then contain nonzero $\overrightarrow{w}$-Engel elements (see Example 3.18 below). However, we hope that the theorem remains true for "classical" Lie algebras (i.e. for those coming from algebraic groups).

*Example 3.18.* Let $L = W(1; 1)$ be the Witt algebra defined over a field $k$ of characteristic $p$. Recall (see, for example, [SF, 4.2, p. 148]) that $L$ is of dimension $p$ with multiplication table defined on a basis $\{e_{-1}, e_0, e_1, \ldots, e_{p-2}\}$ as follows:

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & \text{if } -1 \le i+j \le p-2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

If $p > 2$, the algebra $L$ is simple [SF, Thm. 2.4(1) on p. 149]. However, if $p > 3$, it contains nonzero $\overrightarrow{w}$-Engel elements. Indeed, let $y = e_{p-2}$, and let $x = \alpha_{-1}e_{-1} + \cdots + \alpha_{p-2}e_{p-2}$ be an arbitrary element of $L$. From formulas (3.2) it follows that

$$w_1(x, y) = \alpha_{-1}e_{p-3} + \alpha_0 e_{p-2}.$$

For $p > 3$ this implies $[w_1(x, y), y] = 0$ and hence $w_2(x, y) = 0$. Thus $y$ is a $\overrightarrow{w}$-Engel element, and the statement of Theorem 3.12 does not hold for $L$.

In order to extend the characterization of the solvable radical obtained in Theorem 3.12 to a broader class of Lie algebras, we suggest a little change in our original approach.

**Definition 3.19.** *We say that an element $y \in L$ is* strictly Engel*, if it is Engel and for any $x \in L$ the element $[x, y]$ is Engel (see Definition 3.1 for the notion of an Engel element).*

For finite dimensional Lie algebras the following counterpart of Baer's theorem holds.

**Theorem 3.20.** *Let $L$ be a finite dimensional Lie algebra a field $k$,* $\mathrm{char}(k) = 0$. *The nilpotent radical $N$ of $L$ coincides with the set of all strictly Engel elements of $L$.*

*Proof.* By [J, Thm. II.3.3] every element $y$ of $N$ is an Engel element. Since $N$ is an ideal, $[x, y] \in N$ for every $x \in L$, $y \in N$. Thus every element of $N$ is strictly Engel.

Let us show that there are no nonzero strictly Engel elements outside $N$. Let $y$ be a strictly Engel element of $L$. Denote by $R$ the solvable radical of $L$. First we prove that $y \in R$ . Indeed, if $y \notin R$, then by Theorem 3.7 there exists $x \in L$ such that $(\mathrm{ad}\,[x, y])^n x \neq 0$ for any $n$. Hence $[x, y]$ is not Engel and therefore $y$ is not strictly Engel.

Thus $y \in R$. The set of Engel elements of $L$ which are contained in $R$ coincides with $N$, see [Bou, Ch. I, §5, n°5, Cor. 7 of Thm. 1] (see also [AS, Ch. 16, §4)]. Thus $y \in N$. □

*Remark 3.21.* There is another approach allowing one to characterize the nilpotent radical in a slightly different manner.

Let $L$ be a finite dimensional Lie algebra, and $L^+$ be the corresponding vector space. Denote by $(\operatorname{ad} L)^*$ the subalgebra of the associative algebra End $(L^+)$ generated by the linear operators of the form $\operatorname{ad} x$ where $x \in L$. Let $y \in L$. Observe that the principal ideal $\langle y \rangle$ in the Lie algebra $L$ generated by $y$ consists of the elements $u(y)$, where $u \in (\operatorname{ad} L)^*$.

We say that an Engel element $y$ is *totally Engel* if $u(y)$ is an Engel element for every $u \in (\operatorname{ad} L)^*$.

The next theorem follows from the definitions.

**Theorem 3.22.** *Let $L$ be a finite dimensional Lie algebra over a field $k$. The nilpotent radical $N$ of $L$ coincides with the set of all totally Engel elements of $L$.*

*Proof.* An argument similar to that in the proof of Theorem 3.20, using [J, Theorem II.3.3], shows that all elements of $N$ are totally Engel.

Conversely, if $y$ is totally Engel, then the ideal $\langle y \rangle$ consists of Engel elements. By Engel's theorem, this ideal is nilpotent and therefore is contained in $N$. In particular $y \in N$. □

*Remark 3.23.* Observations from the previous remark are also relevant to the case of the solvable radical and $\overrightarrow{w}$-Engel elements. We could define strictly (resp. totally) $\overrightarrow{w}$-Engel elements in the same way as it was done for strictly (resp. totally) Engel elements. Moreover, to treat the case of positive characteristic, one may be led to an even more restrictive definition of totally $\overrightarrow{w}$-Engel element requiring that it remains $\overrightarrow{w}$-Engel after applying any (not necessary inner) derivation of $L$. In this setting one can hope to get a characterization of the solvable radical as the set of all totally $\overrightarrow{w}$-Engel elements.

*Remark 3.24.* Using yet another approach, one can redefine the notion of strictly $\overrightarrow{w}$-Engel element using sequences of three (or more) variables. For example, define $r_n(x, y, z) = [z, [x, y], \ldots, [x, y]]$. Then an element $y \in L$ is strictly Engel if for every $x, z$ there exists $n = n(x, y, z)$ such that $r_n(x, y, z) = 0$ and $e_n(x, y) = 0$.

## 4. Group case. Reduction theorem

Let $G$ be a finite group. We have $[x, y] = xyx^{-1}y^{-1} = x\sigma_y(x^{-1})$ where $\sigma_y \in \operatorname{Aut} G$ takes $x$ to $yxy^{-1}$.

Let $\overrightarrow{u} = u_1(x, y), \ldots, u_n(x, y), \ldots$ be a sequence in $F_2 = F_2(x, y)$. We want to define $\overrightarrow{u}$-Engel-like automorphisms of $G$.

Let $G_1 = G \leftthreetimes \operatorname{Aut} G$ be the semidirect product of $G$ and $\operatorname{Aut} G$. Recall that it can be viewed as the set of all pairs of the form $(g, \sigma)$, where $g \in G$, $\sigma \in \operatorname{Aut} G$, with multiplication $(g_1, \sigma_1) \cdot (g_2, \sigma_2) = (g_1\sigma_1(g_2), \sigma_1\sigma_2)$. The natural embeddings $G \to G_1$ and $\operatorname{Aut} G \to G_1$, together with the above formula for the group law in $G_1$, imply that the equality $\sigma g \sigma^{-1} = \sigma(g)$ holds inside $G_1$, for any $g \in G$ and $\sigma \in \operatorname{Aut} G$ (informally, an arbitrary automorphism of $G$ becomes an inner automorphism inside a bigger group $G_1$). In particular, this implies that $\sigma g \sigma^{-1}$ belongs to $G$.

A $\overrightarrow{u}$-Engel-like automorphism $\sigma$ of $G$ should be defined in such a way that in the group $G_1 = G \leftthreetimes \operatorname{Aut} G$ it will be presented by a usual $\overrightarrow{u}$-Engel-like element.

Let us take a correct sequence $\overrightarrow{u}(x, y)$ in $F_2(x, y)$. Given a group $G$, consider a homomorphism $\mu\colon F_2(x, y) \to G_1$. Denote $\mu(x) = g$, $\mu(y) = \sigma$, where $g \in G$, $\sigma \in \operatorname{Aut} G$, and define $u_n(g, \sigma) = \mu(u_n(x, y)) \in G_1$ (informally, we "substitute" $g$ instead of $x$ and $\sigma$ instead of $y$).

**Lemma 4.1.** *Suppose that a correct sequence $\overrightarrow{u} = \overrightarrow{u}(x, y)$ in $F_2(x, y)$ satisfies the property:*

*(iii) For every $x, y \in F_2$ the word $u_n(x, y)$ belongs to the minimal subgroup containing the $\sigma_y$-orbit of the element $x$.*

*Then for every group $G \leftthreetimes \operatorname{Aut} G$, every $g \in G$, and every $\sigma \in \operatorname{Aut} G$, the element $u_n(g, \sigma)$ belongs to $G$.*

*Proof.* According to condition (iii), every element $u_n(x, y)$ can be represented as a product of elements $yxy^{-1}$, $y^k x^{-1} y^{-k}$. Then $u_n(g, \sigma)$ is a product of elements $\sigma^k g^{\pm 1} \sigma^{-k} = \sigma^k(g^{\pm 1})$ and thus belongs to $G$. □

**Definition 4.2.** *We call a sequence $\overrightarrow{u}$ satisfying the hypotheses of Lemma* 4.1 *autocorrect.*

We are now able to define $\overrightarrow{u}$-Engel automorphisms:

**Definition 4.3.** *Let $\overrightarrow{u}$ be an autocorrect sequence in $F_2$ and let $G$ be a group. We say that an automorphism $\sigma \in \operatorname{Aut} G$ is $\overrightarrow{u}$-Engel if for any $g \in G$ there exists $n$ such that $u_n(g, \sigma) = 1$.*

*Example 4.4.* For $\sigma \in \operatorname{Aut}(G)$ define $[x, \sigma] = x\sigma(x^{-1})$.

Consider the Engel sequence $\overrightarrow{e}$ defined by

$$e_1(x, \sigma) = [x, \sigma], \dots, e_n(x, \sigma) = [e_{n-1}(x, \sigma), \sigma].$$

It satisfies the hypotheses of Lemma 4.1. We say that an automorphism $\sigma$ of $G$ is Engel if for any $x \in G$ there exists $n$ such that $e_n(x, \sigma) = 1$.

Another example of a sequence satisfying the hypotheses of Lemma 4.1 is $\overrightarrow{w} = \overrightarrow{w}(x, \sigma)$, where

$$w_1 = [x, \sigma], \dots, w_n = [[w_{n-1}, x], [w_{n-1}, \sigma]].$$

Note that $\sigma = \sigma_y$ is a $\overrightarrow{u}$-Engel automorphism if and only if $y$ is a $\overrightarrow{u}$-Engel element. Thus if $G$ has no non-trivial $\overrightarrow{u}$-Engel automorphisms, then $G$ has no non-trivial $\overrightarrow{u}$-Engel elements. Let $A$ be the cyclic subgroup of $\operatorname{Aut}(G)$ generated by $\sigma$. Denote $\widetilde{G} = G \leftthreetimes A$. If $\sigma$ is a $\overrightarrow{u}$-Engel automorphism, then $\sigma$ is a $\overrightarrow{u}$-Engel element in $\widetilde{G}$.

Let $G$ be a finite semisimple group, i.e. $R(G) = 1$, and let $\overrightarrow{u}$ be an autocorrect sequence.

Consider two classes of groups:

- the class of semisimple groups with non-trivial $\overrightarrow{u}$-Engel elements;
- the class of semisimple groups with non-trivial $\overrightarrow{u}$-Engel automorphisms.

Our aim is to show that for some $\overrightarrow{u}$ these classes are empty.

Any finite semisimple group $G$ contains a unique maximal normal centreless completely reducible (CR) subgroup (by definition, CR means a direct product of finite non-abelian simple groups) which is called the CR-radical of $G$ (see [Ro, 3.3.16]). We call a product of the isomorphic factors in the decomposition of the $CR$-radical *an isotypic component* of $G$.

Denote the $CR$-radical of $G$ by $V = V(G)$. This is a characteristic subgroup of $G$.

**Theorem 4.5.** *Let $\overrightarrow{u}$ be a correct sequence, and let $G$ be a semisimple group of smallest order having non-trivial $\overrightarrow{u}$-Engel elements. Then $G$ has the following structure:*

*(i) all non-trivial quotients of $G$ are solvable;*
*(ii) the CR-radical of $G$ contains only one isotypic component.*

*Proof.* (i) Let $H$ be an arbitrary normal subgroup in $G$. Denote $\bar{G} = G/H$. Let $R(\bar{G}) = H_1/H$ be the solvable radical of $\bar{G}$. Then $\bar{\bar{G}} = \bar{G}/R(\bar{G})$ is semisimple and of order strictly smaller than the order of $G$. Hence, for a $\overrightarrow{u}$-Engel element $g \in G$ we have $\bar{\bar{g}} = 1$, and therefore $\bar{g} \in H_1/H$. Thus $g \in H_1$. Since $H_1$ is normal in $G$, it is semisimple. The order of $H_1$ is smaller than the order of $G$ and $g \in H_1$, therefore $g = 1$.

Thus $G = H_1$, and $G/H = H_1/H$ is solvable.

(ii) Take $H = V(G)$, where $V(G)$ is the CR-radical of $G$. Suppose $V(G) = G_1 \times G_2$, where $G_1$ and $G_2$ are isotypic components of $V(G)$, i.e. products of isomorphic non-abelian simple groups. Then they are normal subgroups of $G$. By (i), $G/G_1$ and $G/G_2$ are solvable, hence so is $G/(G_1 \cap G_2)$. Since the intersection is 1, $G$ is solvable. The contradiction shows that $V(G)$ consists of one isotypic component. □

Observe that under hypotheses of Theorem 4.5, the CR-radical of $G$ coincides with the intersection of all its proper normal subgroups.

Let us now consider a group $G$ which is minimal with respect to the second property (to possess non-trivial $\overrightarrow{u}$-Engel automorphisms). We need the following auxiliary result.

**Proposition 4.6.** *Let $G$ be the group generated by $G_0$ and $g_0$, and let $G_0 \lhd G$. Let $G_0$ be a finite semisimple group, and $V_0$ its CR-radical. If $g_0$ acts trivially on $V_0$ then $g_0$ acts trivially on $G_0$.*

*Proof.* Denote $R(G) = R$, $\bar{G} = G/R$. We have $R \cap G_0 = 1$ and $[R, G_0] = 1$, $\bar{G}_0 = G_0R/R \cong G_0$, $\bar{V}_0 = V_0R/R \cong V_0$.

Consider the action of $g_0$ on $G_0$ and on $\bar{G}_0$. We have $\overline{g_0 g g_0^{-1}} = \bar{g}_0 \bar{g} \bar{g}_0^{-1} = g_0 \bar{g} g_0^{-1}$, $g \in G_0$. Thus, the actions of $g_0$ in $G_0$ and $\bar{G}_0$ are isomorphic. $\bar{V}_0$ is the CR-radical of each of the groups $\bar{G}_0$ and $\bar{G}$. Suppose that $\bar{g}_0$ belongs to the centralizer of $\bar{V}_0$. Then $\bar{g}_0 = \bar{1}$, and $g_0 \in R(G)$, $g_0$ and $G_0$ commute. Hence $g_0$ acts trivially on $G_0$. □

Proposition 4.6 implies that if $\sigma$ is an automorphism of a semisimple group $G$ acting trivially on the CR-radical of $G$, then $\sigma$ is the identity automorphism of $G$.

**Theorem 4.7.** *Let $\overrightarrow{u}$ be an autocorrect sequence, and let $G$ be a semisimple group of smallest order having a non-trivial $\overrightarrow{u}$-Engel automorphism. Then $G$ is a direct product of isomorphic non-abelian simple groups.*

*Proof.* Observe, first of all, that if $\sigma$ is a $\overrightarrow{u}$-Engel automorphism of $G$ and $H$ is a $\sigma$-invariant subgroup of $G$, then the corresponding restricted automorphism $\sigma_1$ is a $\overrightarrow{u}$-Engel automorphism of $H$. If now $H$ is a normal subgroup of $G$ invariant under a $\overrightarrow{u}$-Engel automorphism $\sigma$, then $\bar{\sigma}$, the automorphism of the group $G/H$ induced by $\sigma$, is a $\overrightarrow{u}$-Engel automorphism of $G/H$.

Let $G$ be a minimal semisimple group which has a non-trivial $\overrightarrow{u}$-Engel automorphism $\sigma$. Let $V$ be the CR-radical of $G$. If $V < G$ then the restriction of $\sigma$ to $V$ induces a $\overrightarrow{u}$-Engel automorphism $\sigma_1$ of $V$ which must be trivial. Then according to Proposition 4.6, $\sigma$ is trivial on $G$. Contradiction. Thus $V = G$.

Suppose $V = G$ consists of several isotypic components $V_1, V_2, \ldots, V_k$. By the minimality hypothesis on $G$, $\sigma$ acts trivially on each component, and $\sigma$ once again must be trivial. The contradiction shows that $V$ consists of one isotypic component. □

**Conjecture 4.8.** *There exists an autocorrect sequence $\overrightarrow{u}$ such that no group $G$ of the form $G = \prod G_i$, where all $G_i$'s are isomorphic simple non-abelian groups, has a non-trivial $\overrightarrow{u}$-Engel automorphism.*

**Theorem 4.9.** *Conjecture* 2.12 *is equivalent to Conjecture* 4.8.

*Proof.* First note that for any $G$ and any correct sequence $\overrightarrow{u}$ all elements of $R = R(G)$ are $\overrightarrow{u}$-Engel. Hence the assertion of Conjecture 2.12 is equivalent to the following one: if $G$ is semisimple, then it contains no non-trivial $\overrightarrow{u}$-Engel elements.

1. Suppose that $\overrightarrow{u}$ is a sequence such that for any group $G$ we have $R(G) = G(\overrightarrow{u})$. Let $G$ be any finite semisimple group, and let $\sigma$ be a non-trivial $\overrightarrow{u}$-Engel automorphism of $G$. Let us show that $\sigma = 1$. Let $\widetilde{G} = G \leftthreetimes \langle\sigma\rangle$, and consider $\bar{G} = \widetilde{G}/R(\widetilde{G})$. This group is semisimple, and $\bar{\sigma}$ is a $\overrightarrow{u}$-Engel element of the semisimple group $\bar{G}$. Then by hypothesis $\bar{\sigma} = 1$, and $\sigma$ is trivial. Thus the statement of Conjecture 4.8 is fulfilled.
2. Suppose Conjecture 4.8 is true, and take $\overrightarrow{u}$ as in its statement. We want to prove that there are no non-trivial $\overrightarrow{u}$-Engel elements in any semisimple group. Assume the contrary, and let $G$ be a semisimple group of smallest order containing a non-trivial $\overrightarrow{u}$-Engel element $g$. Then $\sigma_g$ is a non-trivial $\overrightarrow{u}$-Engel automorphism of $G$. According to Theorem 4.7, $G = \prod G_i$, where all $G_i$'s are isomorphic non-abelian simple groups. Conjecture 4.8 gives $\sigma_g = 1$. Hence $g = 1$. □

*Remark 4.10.* We believe that Conjecture 4.8 can be replaced with the following, *a priori* weaker

**Conjecture 4.11.** *There exists an autocorrect sequence $\overrightarrow{u}$ such that no finite simple non-abelian group $G$ has a non-trivial $\overrightarrow{u}$-Engel automorphism.*

Indeed, Theorem 3.3.20 of [Ro] says that if $G = \prod_i G_i$, where all $G_i$'s are isomorphic simple non-abelian groups, then Aut $G$ is isomorphic to the wreath product Aut $G_i$ $wr$ $S_n$, where $S_n$ is the symmetric group of degree $n$. This gives some hope to deduce Conjecture 4.8 from Conjecture 4.11.

*Remark 4.12.* Conjectures 4.11 and 4.8 require to specify a sequence $\overrightarrow{u}$. One of possible candidates is the sequence $\overrightarrow{w}$ from Example 4.4.

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