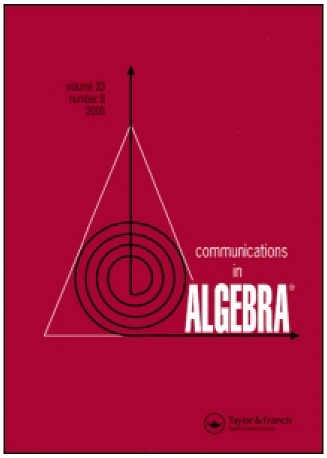

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GAUSS DECOMPOSITIONS OF KAC-MOODY GROUPS

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Dedicated to Eiichi Abe on his 70th birthday

Abstract. We present an axiomatic approach to both a Gauss decomposition of a Kac-Moody group and a Gauss decomposition of the associated Steinberg group. We study also a prescribed version in case of rank 2.

1. Axiomatic approach

Here we call $(G, U, T, V, \{\phi_1, \cdots, \phi_n\})$ a triangular system (or a Gauss system) if

(1) G is a group, and $U, T, V \leq G$ are subgroups,
(2) $\phi_i : \mathrm{SL}_2(K) \to G$ is a group homomorphism of SL_2 over a field K into G with

$$\text{(a)}\quad \phi_i\left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in K\right\} = U_i \leq U;$$

$$\text{(b)}\quad \phi_i\left\{\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \mid t \in K^\times\right\} = T_i \leq T;$$

$$\text{(c)}\quad \phi_i\left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in K\right\} = V_i \leq V,$$

(3) $G = \langle U, V_1, \cdots, V_n \rangle$, and $TU_i = U_iT$ for $1 \leq i \leq n$,
(4) there exist the subgroups of G called U_i' and V_i' for $1 \leq i \leq n$ such that

$$\begin{aligned} U &= U_i'U_i = U_iU_i', \\ V &= V_i'V_i = V_iV_i', \\ V_iU_i' &= U_i'V_i, \\ U_iV_i' &= V_i'U_i. \end{aligned}$$

Then we can establish

$$\begin{aligned} G &= UVTU \\ &= \textstyle\bigcup_{u \in U} u(VTU)u^{-1} \end{aligned}$$

(see below). We call this decomposition a Gauss decomposition of G.

First, we shall review the situation in the case when

$$G = \mathrm{SL}_2(K),\ n = 1,\ \phi_1 = identity$$

and

$$U = U_1,\ T = T_1,\ V = V_1.$$

Let us take an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K).$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

if $a \neq 0$. Otherwise, $c \neq 0$ and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix} \begin{pmatrix} 1 & \frac{b-c+d}{c} \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \mathrm{SL}_2(K) &= VTU \cup u(VTU)u^{-1} \\ &= UVTU \end{aligned}$$

with

$$u = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now, we shall return to a general case. We want to confirm $G = UVTU$. Let $X = UVTU \subset G$. Then $UX = X$ and

$$\begin{aligned} V_i X &= V_i U V T U \\ &= V_i\, U_i'\, U_i\, V_i'\, V_i\, T\, U_i\, U_i' \\ &= U_i'\, V_i\, V_i'\, U_i\, V_i\, T\, U_i\, U_i' \\ &= U_i'\, V_i'\, V_i\, U_i\, V_i\, U_i\, T\, U_i' \\ &\subset U_i'\, V_i'\, U_i\, V_i\, T_i\, U_i\, T\, U_i' \\ &= U_i'\, U_i\, V_i'\, V_i\, T_i\, T\, U_i\, U_i' \\ &= UVTU = X \end{aligned}$$

for $1 \leq i \leq n$. Therefore,

$$G = X = UVTU$$

and

$$G = \bigcup_{u \in U} u(VTU)u^{-1}.$$

Theorem 1. Every triangular system has a Gauss decomposition.

We can find such a system for a standard Kac-Moody group and for a Marcuson-type Kac-Moody group. One can also find a Gauss decomposition for the Steinberg groups associated with Kac-Moody groups. We will discuss them in the next section.

2. Kac-Moody groups

Let $A = (a_{ij})$ be an $n \times n$ generalized Cartan matrix. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra over a field $\mathbf{C}$ defined by A with the so-called Cartan subalgebra $\mathfrak{h}$ (cf. [8], [9],[11], [16], [17]). Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ with the fundamental system $\Pi = \{ \alpha_1, \cdots, \alpha_n \}$. Let Δ_+ (resp. Δ_-) be the set of positive (resp. negative) roots defined by Π, and Δ^{re} the set of real roots. Put $\Delta_\pm^{re} = \Delta_\pm \cap \Delta^{re}$. Then we obtain

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha \in \Delta} \mathfrak{g}_\alpha \qquad \text{(root space decomposition)}$$

and

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+ \qquad \text{(triangular decomposition)},$$

where $\mathfrak{g}_\pm = \oplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$. Let M be an integrable $\mathfrak{g}$-module, which means that

$$M = \oplus_{\mu \in \mathfrak{h}^*} M_\mu,$$

where $M_\mu = \{ v \in M \mid hv = \mu(h)v\ (\forall h \in \mathfrak{h}) \}$, and that x is locally nilpotent on M for all $x \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta^{re}$. For the set of real roots, Δ^{re}, we can choose and fix a Chevalley basis

$$\{ e_\alpha \mid \alpha \in \Delta^{re} \}$$

(cf. [19]). We now suppose that M has a basis $\{ v_\gamma \mid \gamma \in \Gamma \}$ whose $\mathbf{Z}$-span, $M_{\mathbf{Z}}$, is invariant under the action of

$$\frac{e_\alpha^m}{m!}$$

for all $m \geq 0$. Such basis exists, for example, for the cases of adjoint representations, highest weight integrable representations, lowest weight integrable representations, and some others (see [9], [25] and references therein). Then, for any field K, we put $M(K) = K \otimes M_{\mathbf{Z}}$ and define $x_\alpha(t) \in \mathrm{GL}(M(K))$ by

$$x_\alpha(t)(s \otimes v) = \sum_{m=0}^{\infty} t^m s \otimes \frac{e_\alpha^m}{m!} v.$$

Let G be the subgroup of $\mathrm{GL}(M(K))$ generated by $x_\alpha(t)$ for all $\alpha \in \Delta^{re}$ and $t \in K$. We call G a standard (or elementary) Kac-Moody group (cf.[7], [18], [21], [22], [24], [25], [26]). Let

$$\begin{aligned} w_\alpha(t) &= x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t), \\ h_\alpha(t) &= w_\alpha(t) w_\alpha(-1) \end{aligned}$$

for $t \in K^\times$. Then, we put

$$U = \langle\, x_\alpha(a) \mid \alpha \in \Delta_+^{re},\ a \in K \,\rangle,$$

$$T = \langle\, h_\alpha(t) \mid \alpha \in \Delta^{re},\ t \in K^\times \,\rangle,$$

$$V = \langle\, x_\alpha(a) \mid \alpha \in \Delta_-^{re},\ a \in K \,\rangle,$$

Define the maps ϕ_i by

$$\phi_i : \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_{\alpha_i}(a),$$

$$\phi_i : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto h_{\alpha_i}(t),$$

$$\phi_i : \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mapsto x_{-\alpha_i}(a),$$

These maps are not necessarily injective. The subgroups U_i' and V_i' are defined as follows

$$\begin{aligned} U_i' &= \langle\, x_{\alpha_i}(s) x_\beta(t) x_{\alpha_i}(-s) \mid s,t \in K,\ \beta \in \Delta_+^{re} \setminus \{\alpha_i\} \,\rangle, \\ V_i' &= \langle\, x_{-\alpha_i}(s) x_\beta(t) x_{-\alpha_i}(-s) \mid s,t \in K,\ \beta \in \Delta_-^{re} \setminus \{-\alpha_i\} \,\rangle. \end{aligned}$$

Then $(G, U, T, V, \{\phi_1, \cdots, \phi_n\})$ is a triangular system. Hence,

$$\begin{aligned} G &= UVTU \\ &= \textstyle\bigcup_{u \in U} u(VTU)u^{-1}. \end{aligned}$$

Theorem 2. Every standard Kac-Moody group G over a field has a Gauss decomposition.

Remark 1. Gauss decomposition is tightly connected with the existence of Bruhat decomposition. Since for Kac-Moody groups the existence of Tits system is known (cf. [18], [12], [21]), it is quite natural to look for Gauss decomposition for these groups, as well as for groups, associated with them (cf. [1]).

Remark 2. If V is an integrable highest weight module generated by a maximal vector with a highest weight, then we can construct a bigger group $\tilde{G}$, called a Marcuson-type Kac-Moody group, than G (cf. [12], [24]). Namely, $\tilde{U}$, which is bigger than U, is corresponding to all $\alpha \in \Delta_+$. On the other hand, $\tilde{T}$, $\tilde{V}$, $\tilde{\phi}_i$ are the same as T, V, ϕ_i. Then we see that $(\tilde{G}, \tilde{U}, \tilde{T}, \tilde{V}, \{\tilde{\phi}_1, \cdots, \tilde{\phi}_n\})$ is a triangular system. Hence,

$$\begin{aligned}\tilde{G} &= \tilde{U}\tilde{V}\tilde{T}\tilde{U} \\ &= \bigcup_{u\in\tilde{U}} u(\tilde{V}\tilde{T}\tilde{U})u^{-1}.\end{aligned}$$

Therefore, every Marcuson-type Kac-Moody group has a Gauss decomposition.

Remark 3. Let $\mathrm{St}_2(K)$ be the Steinberg group of rank one over a field K (cf. [15], [22]). Then, it is also easily seen that there is a Gauss decomposition for $\mathrm{St}_2(K)$ naturally induced from $\mathrm{SL}_2(K)$. Therefore, we can replace $\mathrm{SL}_2(K)$ by $\mathrm{St}_2(K)$ in the condition (2) of a triangular system. Hence, there is a Gauss decomposition of the Steinberg group, $\hat{G} = \mathrm{St}(A, K)$, associated with a Kac-Moody group G (cf. [26]). Namely,

$$\begin{aligned}\hat{G} &= \hat{U}\hat{V}\hat{T}\hat{U} \\ &= \bigcup_{u\in\hat{U}} u(\hat{V}\hat{T}\hat{U})u^{-1}.\end{aligned}$$

where $\hat{U}, \hat{T}, \hat{V}$ are the corresponding subgroups of $\hat{G}$. Therefore, every Steinberg group associated with a standard Kac-Moody group over a field has a Gauss decomposition.

Remark 4. If we replace the condition (3) of a triangular system by

(3') $G = \langle U, T, V_1, \cdots, V_n \rangle$, $TU = UT$, $VT = TV$, and $TU_i = U_iT$ for $1 \leq i \leq n$,

then we can apply our method to a Tits-type Kac-Moody group (cf. [26]), $\bar{G}$, whose torus $\bar{T}$ is bigger than the group T above. Hence,

$$\begin{aligned}\bar{G} &= \bar{U}\bar{V}\bar{T}\bar{U} \\ &= \bigcup_{u\in\bar{U}} u(\bar{V}\bar{T}\bar{U})u^{-1},\end{aligned}$$

where $\bar{U}$ and $\bar{V}$ are the same as U and V above. Therefore $\bar{G}$ has a Gauss decomposition.

Remark 5. The same method can be applied for other groups associated with Kac-Moody data. In particular, we think that it can be used in the framework of the scheme theoretical approach to Kac-Moody groups, introduced by Mathieu [13], [14].

Remark 6. Furthermore, axiomatically we can replace the group $\mathrm{SL}_2(K)$ in the condition (2) of a triangular system by a certain group G^* with subgroups U^*, T^*, V^* satisfying

$$G^* = U^* V^* T^* U^*.$$

In this case, we must set

$$\phi_i(U^*) = U_i, \quad \phi_i(T^*) = T_i, \quad \phi_i(V^*) = V_i.$$

Then the same method works.

3. Prescribed version

Let $(G, U, T, V, \{\phi_1, \cdots, \phi_n\})$ be a triangular system. Then, as in Section 1, we obtain

$$G = \bigcup_{u \in U} u(VTU)u^{-1}.$$

We now take an element $h^* \in T$. Put

$$G(h^*) = Z(G) \cup \bigcup_{g \in G} g(Vh^*U)g^{-1},$$

where $Z(G)$ is the center of G. Then we want to consider whether $G = G(h^*)$ or not. If $G = G(h^*)$ for all $h^* \in T$, then we say that G has a Gauss decomposition with prescribed elements in T. This is equivalent to the fact, that for every non central element $g \in G$ there exists an element $g_1 \in G$ satisfying $g_1 g g_1{}^{-1} = vhu$, where $v \in V$, $u \in U$, and h is a prescribed element from T.

In such a form this decomposition first appeared in the paper [23] for the case of general linear group and then it was studied in detail by Ellers and Gordeev for all split semisimple algebraic groups(=Chevalley groups= finite dimensional Kac-Moody groups), see [3], [4], and for twisted Chevalley groups [5]. It turns out that the prescribed Gauss decomposition has various applications and is the main tool for solving remarkable Ore and Thompson conjectures (see [20], [2], [6] and references therein). Here, we will check this in the case when $G = \mathrm{SL}_2(K)$, which is the easiest but important in our discussion later. We choose and fix

$$h^* = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \in T.$$

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in VTU \subset \mathrm{SL}_2(K).$$

Then $a \neq 0$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

If $b \neq 0$, then

$$\begin{pmatrix} 1 & 0 \\ \frac{a-t}{b} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{t-a}{b} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c'}{t} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{t} \\ 0 & 1 \end{pmatrix},$$

where

$$c' = \frac{1}{b}\{ta + td - t^2 - ad + bc\}.$$

If $c \neq 0$, then

$$\begin{pmatrix} 1 & \frac{t-a}{c} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \frac{a-t}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{t} & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}\begin{pmatrix} 1 & \frac{b'}{t} \\ 0 & 1 \end{pmatrix},$$

where

$$b' = \frac{1}{c}\{ta + td - t^2 - ad + bc\}.$$

If $b = c = 0,\ a \neq \pm 1$, then

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a - \frac{1}{a} \\ 0 & \frac{1}{a} \end{pmatrix},$$

which arrives at the case of $b \neq 0$ above. If $b = c = 0,\ a = \pm 1$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \in Z(G).$$

Therefore, we obtain

$$G = G(h^*)$$

for all $h^* \in T$ when $G = \mathrm{SL}_2(K)$. Hence, $\mathrm{SL}_2(K)$ has a Gauss decomposition with prescribed elements in T.

4. In case of rank 2

Let

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

be a generalized Cartan matrix with $ab \geq 4$. Then, the corresponding Kac-Moody groups are called of type A. Here, we define

$$m = \max\ \{\ a, b\ \}.$$

Let

$$\mathcal{G} = \mathcal{G}(\mathcal{A}, \mathcal{K})$$

be the family of all standard Kac-Moody groups over a field K of type A. Then, there is a unique, up to isomorphism, element of $\mathcal{G}$ which dominates all other elements. We fix it, and also we call it G. Put

$$N = \langle\ w_\alpha(t)\ \mid\ \alpha \in \Delta^{re},\ t \in K^\times\ \rangle,$$

then $T \triangleleft N$, and $W = N/T$ is an infinite dihedral group. For each $w \in W$, we can write

$$wUw^{-1} = \bar{w}U\bar{w}^{-1}$$

if $\bar{w} \in N$ and $w = \bar{w} \bmod T$. Then

$$\bigcap_{w \in W} wUw^{-1} = 1$$

and

$$\bigcap_{w \in W} wVw^{-1} = 1$$

(cf. [10], [21]). Therefore, if $1 \neq u \in U$ (resp. $1 \neq v \in V$), then there exists $\bar{w} \in N$ such that

$$\bar{w}u\bar{w}^{-1} = u_1u_1', \quad u_1 \neq 1$$
$$(\text{resp. } \bar{w}v\bar{w}^{-1} = v_1'v_1, \quad v_1 \neq 1)$$

or

$$\bar{w}u\bar{w}^{-1} = u_2u_2', \quad u_2 \neq 1$$
$$(\text{resp. } \bar{w}v\bar{w}^{-1} = v_2'v_2, \quad v_2 \neq 1).$$

where $u_i \in U_i$, $u_i' \in U_i'$ (resp. $v_i \in V_i$, $v_i' \in V_i'$) for $i = 1, 2$.

To consider a prescribed version, we will assume, from here on, $|K| > m+3$. Let $h^* = h_{\alpha_1}(t_1^*)h_{\alpha_2}(t_2^*) \in T$ and fix it. Let

$$g = vhu \in VTU$$

with $u \in U$, $h \in T$, $v \in V$. Then we fall into one of the following four cases.

Case 1: g is conjugate to

$$g' = v_1'v_1h_1h_2u_1u_1'$$

with $v_1' \in V_1'$, $v_1 \in V_1$, $h_1 \in T_1$, $h_2 \in T_2$, $u_1 \in U_1$, $u_1' \in U_1'$ and

$$v_1u_1 \neq 1.$$

Case 2: g is conjugate to

$$g' = v_2'v_2h_1h_2u_2u_2'$$

with $v_2' \in V_2'$, $v_2 \in V_2$, $h_1 \in T_1$, $h_2 \in T_2$, $u_2 \in U_2$, $u_2' \in U_2'$ and

$$v_2u_2 \neq 1.$$

Case 3: g is conjugate to

$$g' = h_1h_2$$

with $h_1h_2 \notin Z(G)$.

Case 4: g is just an element of $Z(G)$.

In Case 3, we obtain

$$x_{\alpha_i}(1)g'x_{\alpha_i}(-1) = h_1h_2u_i$$

for some $i = 1, 2$ with $1 \neq u_i \in U_i$, which arrives at Case 1 or Case 2. In Case 2, we will change the numbering of 1 and 2 (and then a and b are exchanged), which arrives at Case 1. Therefore, for our purpose, we can assume that Case 1 holds. We choose and fix an element

$$t^* \in K^\times$$

such that

$$t^{*a} \neq t_2^2, \quad t^{*2} \neq t_2^{*b},$$

where $h_2 = h_{\alpha_2}(t_2)$. Then, as in Section 3, we obtain that g' is conjugate to

$$g'' = v' h_{\alpha_1}(t^*) h_2 u'$$

with $v' \in V$, $h_2 \in T_2$, $u' \in U$. We can rewrite

$$g'' = v_2' v_2 h_{\alpha_1}(t^*) h_2 u_2 u_2'$$

with $v_2' \in V_2'$, $v_2 \in V_2$, $u_2 \in U_2$, $u_2' \in U_2'$. If $v_2 u_2 = 1$, then g'' is conjugate to

$$x_{\alpha_2}(1) g'' x_{\alpha_2}(-1) = v_2'' h_{\alpha_1}(t^*) h_{\alpha_2}(t_2) u_2^* u_2''$$

with $v_2'' \in V_2'$, $1 \neq u_2^* = x_{\alpha_2}(t^{*a} t_2{}^{-2} - 1) \in U_2$, $u_2'' \in U_2'$. Therefore, we can assume

$$v_2 u_2 \neq 1.$$

Then, also as in Section 3, we see that g'' is conjugate to

$$g''' = v'' h_{\alpha_1}(t^*) h_{\alpha_2}(t_2^*) u''$$

with $v'' \in V$, $u'' \in U$. We can rewrite again

$$g''' = v_1' v_1 h_{\alpha_1}(t^*) h_{\alpha_2}(t_2^*) u_1 u_1'$$

with $v_1' \in V_1'$, $v_1 \in V_1$, $u_1 \in U_1$, $u_1' \in U_1'$. If $v_1 u_1 = 1$, then g''' is conjugate to

$$x_{\alpha_1}(1) g''' x_{\alpha_1}(-1) = v_1'' h_{\alpha_1}(t^*) h_{\alpha_2}(t_2^*) u_1^* u_1''$$

with $v_1'' \in V_1'$, $1 \neq u_1^* = x_{\alpha_1}(t^{*-2} t_2^{*b} - 1)$, $u_1'' \in U_1'$. Hence, we can assume

$$u_1 v_1 \neq 1.$$

Then, again as in Section 3, we obtain that g''' is conjugate to

$$g^* = v''' h_{\alpha_1}(t_1^*) h_{\alpha_2}(t_2^*) u''' = v''' h^* u'''$$

with $v''' \in V$, $u''' \in U$. Hence, combining this and Case 4, we see that g is conjugate to some element $g^* \in V h^* U$ if $g \notin Z(G)$. Thus,

$$G(h^*) = G$$

for all $h^* \in T$. Therefore, we obtain the following result.

Theorem 3. Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ be a generalized Cartan matrix with $ab \geq 4$. Put $m = \max\{a, b\}$. Let K be a field with $|K| > m + 3$. Then every standard Kac-Moody group, $G \in \mathcal{G}(\mathcal{A}, \mathcal{K})$, over K of type A has a Gauss decomposition with prescribed elements in T.

It remains to consider the same problem for (infinite dimensional) standard Kac-Moody groups of rank ≥ 3.

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