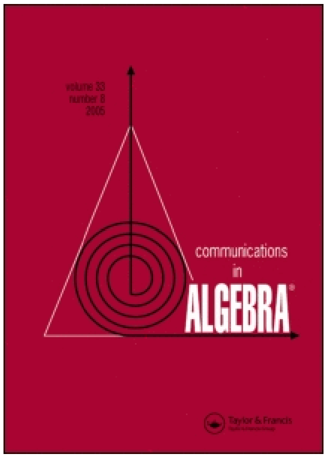

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GEOMETRICAL EQUIVALENCE OF GROUPS

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Abstract. The notion of geometrical equivalence of two algebras, which is basic for this paper, is introduced in [5], [6]. It is motivated in the framework of universal algebraic geometry, in which algebraic varieties are considered in arbitrary varieties of algebras. Universal algebraic geometry (as well as classic algebraic geometry) studies systems of equations and its geometric images, i.e., algebraic varieties, consisting of solutions of equations. Geometrical equivalence of algebras means, in some sense, equal possibilities for solving systems of equations.

In this paper we consider results about geometrical equivalence of algebras, and special attention is paied on groups (abelian and nilpotent).

Equivalence of universal algebras

1. Let Θ be a variety of algebras, $W = W(X)$ be the free algebra in Θ over the finite set X, and G an algebra in Θ . A congruence T of W is called G–closed if

$$T = \bigcap_{\mu \in A} \mathrm{Ker}\mu,$$

where A is a set of homomorphisms $\mu : W \to G$. The set of all G-closed congruences in given W is denoted by $Cl_G(W)$. This set can be considered as semilattice since the intersection of two G-closed congruences is also G-closed congruence.

Definition. *Two algebras G_1 and G_2 are called geometrically equivalent (or, shortly, equivalent) if*

$$Cl_{G_1}(W(X)) = Cl_{G_2}(W(X)),$$

for every finite set X.

Theorem 1 [6]. *Algebras G_1 and G_2 are geometrically equivalent if and only if every finitely generated subalgebra of G_1 can be approximated by subalgebras of G_2 and vice versa.*

Theorem 2. *If the algebras G_1 and G_2 are geometrically equivalent then they generate the same quasivariety.*

Proof.

Let the quasiidentity u of the form

$$w_1 \equiv w_1' \wedge \ldots \wedge w_n \equiv w_n' \to w \equiv w',$$

be fulfilled in G_1. Check that u is fulfilled in G_2. It is enough to check that u is fulfilled in every finitely generated subalgebra H of G_2. Since H is approximated by subalgebras of G_1, we have the injection

$$\mu : H \to \prod_{\alpha} G_{1\alpha},$$

where all $G_{1\alpha}$ are subalgebras in G_1. Since u is fulfilled in G_1, then u is fulfilled in each $G_{1\alpha}$, and, therefore, in $\prod_\alpha G_{1\alpha}$. Since μ is injection, u is fulfilled in H. Thus, u is fulfilled in G_2. Analogously, if u is fulfilled in G_2, then this quasiidentity is fulfilled in G_1.

2. A class of Θ-algebras $\mathfrak{X}$ is called a *prevariety* if it is closed under Cartesian products and subalgebras ([3], [7]). We call a prevariety $\mathfrak{X}$ *locally closed* if for a given $G \in \Theta$ the following property takes place: if every finitely generated subalgebra of G lies in $\mathfrak{X}$ then $G \in \mathfrak{X}$.

Every algebra $G \in \Theta$ generates the locally closed prevariety, and every quasivariety of algebras of Θ is a locally closed prevariety.

Theorem 3. *Algebras G_1 and G_2 are geometrically equivalent if and only if they generate the same locally closed prevariety.*

Proof.

Consider operators on classes $\mathfrak{X}$ of algebras of Θ. As usual, $C\mathfrak{X}$ is the class of Cartesian products of algebras of $\mathfrak{X}$, $S\mathfrak{X}$ consists of subalgebras of algebras of $\mathfrak{X}$, and $L\mathfrak{X}$ is defined by the rule: $G \in L\mathfrak{X}$, if every finitely generated subalgebra H of G belongs to $\mathfrak{X}$. We show that for any $\mathfrak{X}$ the class $LSC(\mathfrak{X})$ is the locally closed prevariety generated by $\mathfrak{X}$. It is well known that $CS < SC$. Therefore, the class $SC\mathfrak{X}$ is a prevariety generated by $\mathfrak{X}$. Let us prove that $LSC(\mathfrak{X})$ is a locally closed prevariety.

First note that if $S\mathfrak{X} = \mathfrak{X}$ then $\mathfrak{X} \subset L\mathfrak{X}$. Indeed, if $G \in \mathfrak{X}$, and H is a finitely generated subalgebra in G, then $H \in \mathfrak{X}$ and, thus, $G \in L\mathfrak{X}$.

Check that $SL = L$. Take $G \in SL(\mathfrak{X})$. Then G is a subalgebra in $G_1 \in L\mathfrak{X}$. Every finitely generated subalgebra of G_1 is contained in the class $\mathfrak{X}$. Therefore, every finitely generated subalgebra of G belongs to $\mathfrak{X}$. Thus, $G \in L\mathfrak{X}$. On the other hand, if $G \in L\mathfrak{X}$, then $G \in SL(\mathfrak{X})$. Thus, every class of the type $L\mathfrak{X}$ is S-closed. Therefore, $L\mathfrak{X} \subset L(L\mathfrak{X}) = L^2\mathfrak{X}$. Let us check the inverse inclusion. Take $G \in L^2\mathfrak{X}$ and let H be a finitely generated subalgebra in G. Then $H \in L\mathfrak{X}$, and $H \in \mathfrak{X}$. This means that $G \in L\mathfrak{X}$ and $L^2\mathfrak{X} \subset L\mathfrak{X}$. Since $L\mathfrak{X} \subset L^2\mathfrak{X}$, we have $L^2\mathfrak{X} = L\mathfrak{X}$. Hence, every class $L\mathfrak{X}$ is L and S closed. In particular, this relates to the class $LSC(\mathfrak{X})$.

Check that $LSC(\mathfrak{X})$ is also C-closed. First, check that $CL < LSC$. Take an arbitrary $\mathfrak{X}$ and $G \in CL\mathfrak{X}$. Then $G = \prod_\alpha G_\alpha$, $G_\alpha \in L\mathfrak{X}$. Let H be a finitely generated subalgebra in G, with generators $g_1, \ldots, g_n$. For every α in G_α take subalgebra H_α, generated by the elements $g_1(\alpha), \ldots, g_n(\alpha)$. We have $H_\alpha \in \mathfrak{X}$ and $H \subset \prod_\alpha H_\alpha$, $H \in SC\mathfrak{X}$, $G \in LSC(\mathfrak{X})$. Then, $CLSC(\mathfrak{X}) \subset LSCSC(\mathfrak{X}) = LSC(\mathfrak{X})$. The inverse inclusion is also holds and the class $LSC(\mathfrak{X})$ is C-closed. Thus, this class is locally closed prevariety, which is the minimal locally closed prevariety, containing $\mathfrak{X}$. In particular, if G is an algebra in Θ then $LSC(G)$ is locally closed prevariety, generated by G.

Now let algebras G_1 and G_2 be equivalent. Every finitely generated subalgebra of G_1 is approximated by subalgebras of G_2 and, hence, belongs to prevariety $SC(G_2)$. Therefore, $G_1 \in LSC(G_2)$. Analogously, $G_2 \in LSC(G_1)$. Thus,

$$LSC(G_1) = LSC(G_2).$$

Let, conversely, the equality above holds. Then every finitely generated subalgebra H from G_1 is contained in $SC(G_2)$. Such H is approximated by subalgebras in G_2. Similarly, every finitely generated subalgebra in G_2 is approximated by subalgebras in G_1. Using Theorem 1 we conclude that G_1 and G_2 are equivalent.

3. An algebra $G \in \Theta$ is called subdirectly indecomposable if there is a non-zero congruence T in G which is contained in all non-zero congruences in G. Simple algebra is an algebra which has only zero congruence and unity congruence. Simple algebra is subdirectly indecomposable and the corresponding congruence is unity congruence.

Proposition 1. *If two subdirectly indecomposable finite algebras G_1 and G_2 are equivalent then they are isomorphic.*

Proof.

Equivalency of G_1 and G_2 implies injections $G_2 \to G_1 \to G_2$. Therefore, $|G_1| = |G_2|$ and the injection $G_1 \to G_2$ is an isomorphism.

Corollary. *Finite simple algebras are equivalent if and only if they are isomorphic.*

In particular, finite simple groups are equivalent if and only if they are isomorphic. The same reasoning can be applied to finite dimensional simple associative and Lie algebras.

Equivalence of groups

1. From Theorem 2 immediately follows

Proposition 2. *If the groups G_1 and G_2 are geometrically equivalent and one of them is torsion free, then the other one is also torsion free.*

Now, let G_1 and G_2 be Abelian groups.

Theorem 4. *Abelian groups G_1 and G_2 are geometrically equivalent if and only if they have the same quasiidentities.*

Proof.

It suffices to prove that if G_1 and G_2 have the same quasiidentities then they are equivalent.

A.Berzinsh has proved (see [6], [1]) that two Abelian groups G_1 and G_2 are equivalent if and only if.

1. Groups G_1 and G_2 have the same exponents.

2. For every prime number p the exponents of the corresponding Sylov subgroups G_1p and G_2p coincide.

Let us consider the special quasiidentity u:

$$x^{p^{n+1}} = 1 \Rightarrow x^{p^n} = 1.$$

This quasiidentity is fulfilled in a group G if and only if every p-element in G has the order, which divides p^n. Indeed, let the order of each p-element divides p^n. Then $g^{p^n} = 1$ for every p-element g. Let now $g^{p^{n+1}} = 1$. Then g is a p-element and $g^{p^n} = 1$.

Let g be a p-element, and the quasiidentity u be fulfilled in G. Then for some m we have $g^{p^m} = 1$. Suppose p^m is the order of g and p^m does not divide

p^n. Then $m > n$, $m = m_0 + n + 1$, $m_0 \geq 0$. Therefore, $g^{p^m} = g_1^{p^{n+1}} = 1$, $g_1 = g^{p^{m_0}}$. By the condition $g_1^{p^n} = 1$ and $g^{p^{m_0+n}} = 1$. Since $m_0 + n < m$ and p^m is the order of g, we get contradiction. Thus, the orders of all p-elements of G divide p^n.

Let Abelian groups G_1 and G_2 have the same quasiidentities. Then they have the same identities, and, therefore, their exponents coincide. Let G_{1p} and G_{2p} be Sylow p-subgroups. Suppose the exponent of G_{1p} is p^n. If G_{1p} is trivial, then $n = 0$. All p-elements of G_1 belong to G_{1p} and their orders divide p^n. The quasiidentity $x^{p^{n+1}} = 1 \Rightarrow x^{p^n} = 1$ holds in G_1 and G_2. Therefore, G_{2p} has the exponent, which divides p^n. If this exponent less than p^n, then acting backwards we get contradiction. Hence, exponents of Sylow subgroups G_{1p} and G_{2p} coincide.

Now let the exponent of G_{1p} be infinite. If the exponent of G_{2p} is finite, then there is a quasiidentity in G_2 which does not exist in G_1. Thus, the exponent of G_{2p} should be infinite too. Conditions of the criterion are fulfilled and the theorem is proved.

2. Now let G be a nilpotent group and $P(G)$ be its periodic part.

Theorem 5. *If groups G_1 and G_2 are geometrically equivalent then $P(G_1)$ and $P(G_2)$ are equivalent too.*

Proof.

Let H be a finitely generated and, hence, finite subgroup in $P(G_1)$. H is approximated by subgroups of G_2, which are also finite and, therefore, belong to $P(G_2)$. Analogously, every finitely generated subgroup of $P(G_2)$ is approximated by subgroups of $P(G_1)$. Thus, $P(G_1)$ and $P(G_2)$ are equivalent.

Similarly, one can prove that if G_1 and G_2 are equivalent then their Sylow p-subgroups G_{1p} and G_{2p} are equivalent.

Theorem 6. *Let G_1 and G_2 be periodic nilpotent groups. They are geometrically equivalent if and only if for every prime p Sylow subgroups G_{1p} and G_{2p} are equivalent.*

Proof.

For every prime p consider Sylov decomposition of G_1 and G_2:

$$G_1 = \prod_{p \in \Pi_1} G_{1p}, \qquad G_2 = \prod_{p \in \Pi_2} G_{2p}.$$

Since unity group is equivalent only to unity group, it follows that if G_1 and G_2 are equivalent then $\Pi_1 = \Pi_2 = \Pi$ and for every $p \in \Pi$ groups G_{1p} and G_{2p} are equivalent.

According to [5] it follows that G_1 and G_2 are equivalent.

Now consider torsion free nilpotent groups.

Such a group G is called subdirectly indecomposable in the class of torsion free groups, if there is a nontrivial normal subgroup H in G, such that G/H is torsion free and H lies in every normal subgroup with this property.

Proposition 3. *If G_1 and G_2 are equivalent finitely generated torsion free groups, which are subdirectly indecomposable in the class of torsion free groups, then there exist injections $G_1 \to G_2$ and $G_2 \to G_1$.*

Proof.

By the condition G_1 can be approximated by torsion free subgroups of G_2. Since G_1 is subdirectly indecomposable in the class of torsion free groups, there is an injection $G_1 \to G_2$. Similarly, there is an injection $G_2 \to G_1$.

Recall that a nilpotent torsion free group has finite rank $r = r(G)$, if in G there is normal series of the length r whose factors are isomorphic to subgroups of the additive group of rational numbers. Every finitely generated nilpotent group without torsion has a finite rank. If H is a subgroup of a group of rank r, then $r(H) \leq r$. A subgroup H is called isolated, if $x^n \in H \Rightarrow x \in H$. If H is a proper isolated subgroup, then $r(H) < r$. A group G is called divisible, if for every $a \in G$ and $n \in \mathbb{Z}$ there exists $b \in G$ such that $b^n = a$. Every torsion free nilpotent group G can be embedded into its completion, which is the minimal divisible nilpotent torsion free group, containing G [4].

Theorem 7. *Let G_1 and G_2 be torsion free finitely generated nilpotent groups, which are subdirectly indecomposable in the class of torsion free groups. If they are geometrically equivalent then $r(G_1) = r(G_2)$, and their completions are isomorphic.*

Proof.

Using Proposition 3 we have injections $G_2 \to G_1 \to G_2$. Then $r(G_2) \leq r(G_1) \leq r(G_2)$. Therefore, $r(G_1) = r(G_2)$. We also have injections for the completions

$$\tilde{G}_2 \to \tilde{G}_1 \to \tilde{G}_2.$$

Since $r(G) = r(\tilde{G})$, we have $r(\tilde{G}_1) = r(\tilde{G}_2)$.

Complete subgroup in a nilpotent torsion free group is an isolated subgroup. Therefore, if the injection $\tilde{G}_1 \to \tilde{G}_2$ is not a surjection, then $r\tilde{G}_1 < r\tilde{G}_2$. Now $r(\tilde{G}_1) = r(\tilde{G}_2)$ implies that the injection $r(\tilde{G}_1) \to r(\tilde{G}_2)$ is an isomorphism.

Problem 1. Is it true that every torsion free nilpotent group is geometrically equivalent to its completion?[1] If not, consider the conditions when geometrical equivalence takes place.

3. Examples.

Consider the group $G = UT_n(\mathbb{Z})$ of n-unitriangular matrices over integers. This group is torsion free nilpotent group of nilpotency class $(n-1)$ with finite number of generators. It has isolated cyclic center which is contained in every isolated normal subgroup of G. Thus, G is subdirectly indecomposable in the class of torsion free groups. Denote by $\tilde{G}$ the similar group $UT_n(\mathbb{Q})$ over the field of rational numbers, which is, in fact, completion of G. Let us show (A.Tsurkov) that

Proposition 4. *The groups $G = UT_n(\mathbb{Z})$ and $\tilde{G} = UT_n(\mathbb{Q})$ are equivalent.*

Proof.

The set $\sigma = \{\sigma_{ij}\}$, where $\sigma_{ij} = \frac{\mathbb{Z}}{r^k}$, $r \in \mathbb{N}$, $k \leq j - i$, if $j \leq i$ and $\sigma_{ij} = 0$ otherwise, is the net of additive subgroups (see, for example [2], [8]). Consider the net subgroup G_σ in $GL_n(\mathbb{Q})$ and take the subgroup $U_\sigma = G_\sigma \cap UT_n(\mathbb{Q})$ generated by transvections $x_{ij}(u)$, $u \in \sigma_{ij}$. It coincides with the subgroup G_r, $r > 0$, $r \in \mathbb{Z}$, of $\tilde{G}$, which consists of matrices $g = (g_{ij})$ of the form

$$\begin{pmatrix} 1 & \frac{a_{12}}{r} & \dots & \frac{a_{1n}}{r^{n-1}} \\ 0 & 1 & \dots & \frac{a_{2n}}{r^{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

where $a_{ij} \in \mathbb{Z}$. It is easy to see that every finitely generated subgroup of $\tilde{G}$ lies in some G_r, where r is the common denominator of all entries of all matrices generated this subgroup. Besides that, the homomorphism, assigning to every g the matrix

$$\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

is the injection $G_r \to G$. Hence, G and $\tilde{G}$ are equivalent.

Using similar arguments and the notion of net subgroup for Chevalley group ([10], [11]), it can be shown, that the unipotent subgroup $U(\Phi, \mathbb{Z})$ of the Chevalley group $G(\Phi, \mathbb{Z})$, where Φ is a root system, is equivalent to its completion.

Consider other examples of group equivalence.

[1] This is true for the torsion free nilpotent groups of class 2 (A.Tsurkov). For torsion free nilpotent groups of class n, $n \geq 3$ there are counterexamples (V.Bludov).

Let P be a field and $R = P[x]$ be a polynomial ring. Let G_1 and G_2 be the groups of n-unitriangular matrices over P and R respectively. If $charP = 0$ then G_1 and G_2 are torsion free nilpotent groups. It is easy to check that if the field P is infinite then the groups G_1 and G_2 are equivalent.

Now let P be a finite field of odd characteristic p. Consider the group G of matrices over P of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

This is a nilpotent finite group of class 2 with exponent p. Let L be an extension of P of degree 2 and $\tilde{G}$ be the group of matrices g over $L = P(\alpha)$ of the form

$$\begin{pmatrix} 1 & x & z_1 + \alpha z_2 \\ 0 & 1 & y_1 + \alpha y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

This is also a nilpotent finite group of class 2 with exponent p. Then the homomorphism $g \to (g_1, g_2)$, where

$$g_1 = \begin{pmatrix} 1 & x & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & x & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

gives the injection $\tilde{G} \to G \times G$. Hence (A.Tsurkov), the groups G and $\tilde{G}$ are equivalent. Of course, they are non-isomorphic and non-decomposable in direct product.

Now we point out the problems related to equivalency of nilpotent groups.

Problem 2. Is it true that two torsion free (periodic) nilpotent groups with the same quasiidentities are equivalent?

Problem 3. Let G_1 and G_2 be two equivalent nilpotent groups. Is it true that torsion free groups $G_1/P(G_1)$ and $G_2/P(G_2)$ are also equivalent?

Problem 4. Let $F = F(X)$ be a free nilpotent group of class n. Describe all groups G which are equivalent to F.

Remarks on algebraic varieties over a finite group

1. Let us return to general definitions, given in the beginning of the paper. Closed congruences in free algebras, (in particular, closed norml subgroups in free groups) are dual to (affine) algebraic varieties.

Let Θ be a variety of algebras, $W(X) = W$, X-finite, be the free algebra in Θ. Fix an algebra $G \in \Theta$. The set $Hom(W, G)$ is considered to be an affine space, whose points are homomorphisms. Define the Galois correspondence between sets of points A in $Hom(W, G)$ and binary relations T in W by the rule

$$T = A' = \cap_{\mu \in A} Ker\mu,$$

$$A = T' = \{\mu, T \subset Ker\mu\}.$$

Recall that the congruence $T = A'$ is called G-closed congruence in W. The set of points $A = T'$ is called *an affine algebraic variety* over the algebra G in affine space $Hom(W, G)$. For every set $A \subset Hom(W, G)$ one can take its closure $A'' = (A')'$, which is the minimal affine space, containing A. There is , also, the closure $T'' = (T')'$.

Consider examples. Let $\mathfrak{X}$ be a class in Θ. Denote the set of all points $\mu : W \to G$, such that $Im\ \mu \in \mathfrak{X}$, by $A(\mathfrak{X})$. If $\mathfrak{X}$ is a variety, then

Proposition 5. *The set $A = A(\mathfrak{X})$ is an algebraic variety over the algebra G.*

Proof.

Let $T = A' = \cap_{\mu \in A} Ker\mu$. All $W/Ker\mu$ are contained in the variety $\mathfrak{X}$. Hence, $W/T \in \mathfrak{X}$. If now $\mu \in A''$, then $T \subset Ker\mu$. Therefore, $W/Ker\mu$ is the homomorphic image of W/T. Since $W/T \in \mathfrak{X}$, the algebra $W/Ker\mu$ belongs to $\mathfrak{X}$. Then $Im\ \mu \in \mathfrak{X}$, $\mu \in A = A(\mathfrak{X})$. Thus, $A = A''$.

Let us generalize this proposition. Let $\mathfrak{X}$ be a class of algebras, which is hereditary under subalgebras.

Theorem 8. *1.Let $\mathfrak{X}(G)$ be the set of all subalgebras in G, belonging to $\mathfrak{X}$, $A = A(X)$. Then, $T = A'$ is the verbal congruence in W of all identities of the class $\mathfrak{X}(G)$.*

2. The closure $A'' = T'$ consists of all points μ from $Hom(W, G)$, for which the identities of the set T are fulfilled in $Im\ \mu$.

Proof.

Let $\bar{T}$ be the congruence of identities of the class $\mathfrak{X}(G)$. For every $H \in \mathfrak{X}(G)$ denote by $\bar{T}(H)$ the congruence of identities in H. Then,

$$\bar{T} = \cap_{H \in \mathfrak{X}(G)} \bar{T}(H).$$

$$\bar{T}(H) = \cap_{\mu : W \to H} Ker\mu.$$

Since $\mathfrak{X}$ is hereditary, $Im\ \mu \in \mathfrak{X}$ for every $\mu : W \to H$. It follows that $T \subset \bar{T}$. Let $\mu : W \to G$ be an arbitrary homomorphism such that $Im\ \mu \in \mathfrak{X}$. Then $H = Im\ \mu \in \mathfrak{X}(G)$. This implies that $\bar{T} \subset T$ and $\bar{T} = T$.

Let now $\mu \in T' = A''$. This means that $T \subset Ker\mu$. Then identities of T are fulfilled in $Im\ \mu$. Now let the identity of T be fulfilled in $W/Ker\mu$. This means that $T \subset Ker\mu$ and $\mu \subset T'$.

These two propositions illustrate the definition of the closure of a set A.

In the next two items we give the notions, which look very natural from the point of view of algebraic geometry of finite groups.

2. Let Θ be the variety of all groups and G be a finite group, $|G| = m$. Let $|X| = n$. Then $|Hom(W, G)| = |G|^{|X|} = m^n$. If $A = T'$ is an affine algebraic variety over G, then

$$v_T(G) = \frac{|A|}{m^n}$$

is called the *volume* of the variety A. Define

$$v(T) = sup_G \; v_T(G),$$

where supremum is taken over all finite G. The function $v(T)$ naturally characterizes the system of equations T in the class of finite groups. The similar characteristic can be considered also for the class of finite simple groups.

3. Consider the situation, when the set X consists of two variables $X = \{x, y\}$, and the finite group G is fixed. For $a, b \in G$, one can consider the homomorphism $\mu = \mu_{a,b} : F \to G$, given by $\mu(x) = a, \mu(y) = b$, where F is the free group with two generators. To each set of elements T in $F = F(x, y)$ corresponds binary relation $\rho = \rho(T)$ on the group G

$$a\rho b \Leftrightarrow \mu = \mu_{a,b} \in T' = A.$$

This means that $T \subset Ker\mu_{a,b}$ or $f(a, b) = 1$ for all $f \in T$.

Suppose that the set T is symmetric, i.e.

$$f(x, y) \in T \Leftrightarrow f(y, x) \in T,$$

and reflexive $f(x, y) \in T \Rightarrow f(x, x) \in T$. If T is a symmetric and reflexive set, then the algebraic variety $A = T'$ and the relation ρ are symmetric and reflexive. Such a relation ρ determines the graph on G, which is the union of its connected components.

Examples. For the set T take the commutator $[x, y]$. Since $[x, y] = 1 \Rightarrow [y, x] = 1$, this is symmetric relation. This T determines graph on the group G, which is called commuting graph of the group (see [9] for further information).

Consider generalizations of this graph.

Denote

$$u_1 = u_1(x, y) = [x, y], \quad u_2 = [u_1, y], \ldots, u_n = [u_{n-1}, y], \ldots$$

and $T = \{u_n(x,y), u_n(y,x)\}$. The corresponding relation ρ_n is called the relation of n-step nilpotency (Engel relation). Elements a and b from a group satisfy nilpotency relation, if there is n such that $a\rho_n b$. Recall that a finite group G is nilpotent if and only if there exists n such that the identity $u_n \equiv 1$ holds in G.

Let us introduce now the relation of solvability. Take $v_1 = v_1(x,y) = [x,y]$, and let $v_1' = [v_1, x]$, $v_1'' = [v_1, y]$. Set $v_2 = [v_1', v_1'']$. Having v_{n-1}, define $v_{n-1}' = [v_{n-1}, x]$, $v_{n-1}'' = [v_{n-1}, y]$, and $v_n = [v_{n-1}', v_{n-1}'']$. It is clear that if the group G is solvable of class n, then we have the identity $v_n \equiv 1$.

Here arises the natural conjecture:

A finite group G is solvable if and only if in G holds the identity $v_n \equiv 1$ for some n.

Let us take now $T_n = \{v_n(x,y), v_n(y,x)\}$, and $\rho_n = \rho(T_n)$. This relation ρ_n is the relation of n-step solvability between elements of the group. Elements a and b from a group satisfy relation of solvability, if there is n such that $a\rho_n b$.

These relations give rise to study of the corresponding graphs, and for such T it is natural to consider the function $v(T)$.

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