PRESCRIBED GAUSS DECOMPOSITION FOR KAC-MOODY GROUPS OVER FIELDS.

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Abstract. We will show the Gauss decomposition with prescribed torus elements for a Kac-Moody group over a field in general case.

1. Introduction

Kac-Moody groups are equipped with canonical decompositions of different kinds. Let us note, for instance, the decompositions of Bruhat, Birkhoff and Gauss. As in finite dimensional case, they play an important part in calculations with these groups. However, for Kac-Moody situation all of them have their own peculiarities (look, for example, [19] where J.Tits gives a comparison of the Bruhat and the Birkhoff decompositions and some of their applications).

The aim of this paper is to establish the so-called prescribed Gauss decomposition for Kac-Moody groups.

The prescribed Gauss decomposition appeared in [17], [9] for the general linear group and then it was proved in full generality in [3],[4], [5] for Chevalley and twisted Chevalley groups. It happens to be the main tool for proving the substantial Ore and Thompson conjectures.

Namely, Ore [13] conjectured that every element of a finite simple group is a single commutator.

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The proof of this statement for all simple groups of Lie type is given in [6]. There are similar facts for infinite simple groups as well. Let us mention the paper of Ree [15], who proved that every element of a connected semisimple algebraic group over an algebraically closed field is a commutator. The survey of the results on Ore problem (before the results of Ellers-Gordeev and A.Lev) can be found in [21].

This paper is the continuation of the paper [12], in which the prescribed Gauss decomposition was established for Kac-Moody groups of the rank 2. The general result became possible due to the recent paper [2] where the elegant idea of V.Chernousov gives rise to a uniform proof of prescribed Gauss decomposition for all groups of Lie type. Moreover, we mostly follow the method of this paper, adjusting it to Kac-Moody case.

We do not consider in this paper Ore and Thompson type conjectures for Kac-Moody groups. These groups are perfect but not, generally speaking, simple, and their commutator structure can be very delicate.

2. Kac-Moody groups

Let $A = (a_{ij})$ be an $n \times n$ generalized Cartan matrix. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra over a field $\mathbf{C}$ defined by A with the so-called Cartan subalgebra $\mathfrak{h}$ (cf. [7], [10]). Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ with the fundamental system $\Pi = \{ \alpha_1, \cdots, \alpha_n \}$. Let Δ_+ (resp. Δ_-) be the set of positive (resp. negative) roots defined by Π, and Δ^{re} the set of real roots. Put $\Delta^{re}_\pm = \Delta_\pm \cap \Delta^{re}$. Then we obtain

$$\mathfrak{g} = \mathfrak{h} \oplus \bigsqcup_{\alpha \in \Delta} \mathfrak{g}_\alpha \qquad \text{(root space decomposition)}$$

and

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+ \qquad \text{(triangular decomposition)},$$

where $\mathfrak{g}_\pm = \oplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$. Let M be an integrable $\mathfrak{g}$-module, which means that

$$M = \oplus_{\mu \in \mathfrak{h}^*} M_\mu,$$

where $M_\mu = \{ v \in M \mid hv = \mu(h)v \ (\forall h \in \mathfrak{h}) \}$, and that x is locally nilpotent on M for all $x \in \mathfrak{g}_\alpha$ with $\alpha \in \Delta^{re}$. For the set of real roots, Δ^{re}, we can choose and fix a Chevalley basis $\{ e_\alpha \mid \alpha \in \Delta^{re} \}$ (cf. [11]). We now suppose that M has a basis $\{ v_\gamma \mid \gamma \in \Gamma \}$ whose $\mathbf{Z}$-span, $M_{\mathbf{Z}}$, is invariant under the action of $e_\alpha^m/m!$ for all $m \geq 0$. Such a basis exists, for example, for the cases of adjoint representations, highest weight integrable representations, lowest weight integrable representations, and some others (see [7], [19] and references therein). Then, for any field K, we put $M(K) = K \otimes M_{\mathbf{Z}}$ and define $x_\alpha(t) \in \mathrm{GL}(M(K))$ by

$$x_\alpha(t)(s \otimes v) = \sum_{m=0}^{\infty} t^m s \otimes \frac{e_\alpha^m}{m!} v.$$

Let G be the subgroup of $\mathrm{GL}(M(K))$ generated by $x_\alpha(t)$ for all $\alpha \in \Delta^{re}$ and $t \in K$. We call G a (standard or elementary) Kac-Moody group (cf. [14], [16], [20]). Sometimes G is called of type A. Let $\mathcal{G} = \mathcal{G}(A, K)$ be the family of all Kac-Moody groups over a field K of type A. Then, there is a unique, up to isomorphism, element of $\mathcal{G}$ which dominates all other elements. We fix it, and also we call it G. Let

$$\begin{aligned} w_\alpha(t) &= x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t), \\ h_\alpha(t) &= w_\alpha(t) w_\alpha(-1) \end{aligned}$$

for $t \in K^\times$. Then, we put

$$U = \langle\ x_\alpha(a) \ \mid\ \alpha \in \Delta_+^{re},\ a \in K\ \rangle,$$

$$T = \langle\ h_\alpha(t) \ \mid\ \alpha \in \Delta^{re},\ t \in K^\times\ \rangle,$$

$$V = \langle\ x_\alpha(a) \ \mid\ \alpha \in \Delta_-^{re},\ a \in K\ \rangle,$$

Define the maps ϕ_i by

$$\phi_i \ :\ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_{\alpha_i}(a),$$

$$\phi_i \ :\ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto h_{\alpha_i}(t),$$

$$\phi_i \ :\ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mapsto x_{-\alpha_i}(a),$$

These maps are not necessarily injective. The subgroups U_i' and V_i' are defined as follows

$$\begin{aligned} U_i' &= \langle\ x_{\alpha_i}(s) x_\beta(t) x_{\alpha_i}(-s) \ \mid\ s, t \in K,\ \beta \in \Delta_+^{re} \setminus \{\alpha_i\}\ \rangle, \\ V_i' &= \langle\ x_{-\alpha_i}(s) x_\beta(t) x_{-\alpha_i}(-s) \ \mid\ s, t \in K,\ \beta \in \Delta_-^{re} \setminus \{-\alpha_i\}\ \rangle. \end{aligned}$$

Then $(G, U, T, V, \{\phi_1, \cdots, \phi_n\})$ is a triangular system. Hence,

$$\begin{aligned} G &= UVTU \\ &= \textstyle\bigcup_{u \in U} u(VTU)u^{-1}, \end{aligned}$$

and every Kac-Moody group G over a field has a Gauss decomposition (cf. [12]).

Put

$$N = \langle\ w_\alpha(t) \ \mid\ \alpha \in \Delta^{re},\ t \in K^\times\ \rangle,$$

then $T \lhd N$, and N/T is isomorphic to the Weyl group W (cf. [7], [10]). We sometimes identify W with N/T. For each $w \in W$, we can write

$$wUw^{-1} = \bar{w}U\bar{w}^{-1}$$

if $\bar{w} \in N$ and $w = \bar{w}$ mod T. Then

$$\bigcap_{w \in W} wUw^{-1} = 1$$

and

$$\bigcap_{w \in W} wVw^{-1} = 1$$

(cf. [8], [14]).

Therefore, if $1 \neq u \in U$ (resp. $1 \neq v \in V$), then there exists $\bar{w} \in N$ such that

$$\begin{array}{c} \bar{w}u\bar{w}^{-1} = u_i u_i', \quad u_i \neq 1 \\ (\text{resp.} \quad \bar{w}v\bar{w}^{-1} = v_i' v_i, \quad v_i \neq 1), \end{array}$$

for some $i = 1, 2, \cdots, n$, where $u_i \in U_i$, $u_i' \in U_i'$ (resp. $v_i \in V_i$, $v_i' \in V_i'$). This property is important for us, and later we will use it. And, we need one more thing about substructures of Kac-Moody groups. For a subset X of Π, we denote by Δ_X the sub-root system of Δ generated by X, and we define the following subgroups:

$$\begin{array}{rcl} G_X &=& \langle\, x_{\pm\alpha}(t) \mid \alpha \in X,\ t \in K \,\rangle, \\ T_X &=& \langle\, h_\alpha(t) \mid \alpha \in X,\ t \in K^\times \,\rangle, \\ T_X' &=& \langle\, h_\alpha(t) \mid \alpha \in \Pi \setminus X,\ t \in K^\times \,\rangle, \\ U_X &=& \langle\, x_\alpha(t) \mid \alpha \in \Delta_+^{re} \cap \Delta_X,\ t \in K \,\rangle, \\ U_X' &=& \langle\, y x_\beta(t) y^{-1} \mid \beta \in \Delta_+^{re} \setminus \Delta_X,\ t \in K,\ y \in G_X \,\rangle, \\ V_X &=& \langle\, x_\alpha(t) \mid \alpha \in \Delta_-^{re} \cap \Delta_X,\ t \in K \,\rangle, \\ V_X' &=& \langle\, y x_\beta(t) y^{-1} \mid \beta \in \Delta_-^{re} \setminus \Delta_X,\ t \in K,\ y \in G_X \,\rangle. \end{array}$$

Then, using the commutator formula (cf. [11]), we see $U = U_X U_X' \rhd U_X'$ and $V = V_X V_X' \rhd V_X'$. Also clearly we obtain $T = T_X T_X'$. At the end of this section, we will deal with the center $Z(G)$ of G. Actually, $Z(G) \subset T$ and we can explicitly describe as folows:

$$Z(G) = \{\ \prod_{i=1}^{n} h_{\alpha_i}(t_i) \mid \prod_{i=1}^{n} t_i^{\beta(h_{\alpha_i})} = 1 \text{ for all } \beta \in \Pi\ \}.$$

3. Theorems

Here first, we review the result on the prescribed version of Gauss decompositions for the rank two Kac-Moody groups. For this result and for the definition of Gauss decompositions with prescribed torus elements associated with triangular systems, see [12].

Theorem ([12]). Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ be a generalized Cartan matrix with $ab \geq 4$. Put $m = \max \{ a, b \}$. Let K be a field with $\mid K \mid > m+3$. Then every Kac-Moody group, $G \in \mathcal{G}(A, K)$, over K of type A has a Gauss decomposition with prescribed elements in T. This means that given arbitrary

semisimple element element h^*,

$$G = Z(G) \cup \bigcup_{g \in G} g(Vh^*U)g^{-1},$$

where $Z(G)$ is the center of G.

The similar result holds for $0 \leq ab \leq 3$ in rank two case, in which case there is no restriction to the cardinality of K. Moreover, in the case of Chevalley groups, there is a general result about Gauss decomposition with prescribed semisimple elements.

Theorem ([2]). Let A be a Cartan matrix, and let K be a field. Then every Chevalley group, $G \in \mathcal{G}(A, K)$, over K of type A has a Gauss decomposition with prescribed elements in T.

In the remaining of this paper, we will present the following result on Kac-Moody groups G for all generalized Cartan matrices. For a generalized Cartan matrix $A = (a_{ij})$, we put

$$m = \max \{ \mid a_{ij} \mid \ (1 \leq i \neq j \leq n) \}.$$

Theorem 1. Suppose $\mid K \mid > m+3$. Then, every Kac-Moody group G over K has the Gauss decomposition with prescribed elements in T.

Corollary. Every element of a Kac-Moody group G can be expressed as a product of two unipotent elements in G.

4. Some inductive method

Here we will show the following proposition. We put $I = \{ 1, 2, \cdots, n \}$.

Proposition 1. Let Γ be the group generated by an abstract symbol σ and our Kac-Moody group G satisfying that σ acts on G by conjugation as an diagonal automorphism. Let $Z(\Gamma)$ be the center of Γ. Suppose $\mid K \mid > \ m+3$. Then, for every element $\sigma g \in \Gamma$ with $g \in G$ and $\sigma g \notin Z(\Gamma)$, and for every element $h^* = \prod_{i=1}^{n} \ h_{\alpha_i}(t^*_{\alpha_i}) \in T$, there exists an element $z \in G$ such that

$$z(\sigma g)z^{-1} = \sigma(vh^*u)$$

for some $v \in V$ and $u \in U$.

The proof of this proposition can be given exactly in the same way as in [2]. We proceed by induction on n. We have already known that Proposition 1 holds for $n = 1, 2$ (cf. [2], [12]). We suppose $n \geq 3$.

We take $\sigma g \in \Gamma$ with $g \in G$ in general. Since $\sigma U = U\sigma$ and $G = UVTU$, we see $\sigma G = U\sigma VTU$. Hence, for our purpose, we can assume that σg is just of the form $\sigma g = \sigma vhu$ with $v \in V,\ h \in T,\ u \in U$. Then, using the conjugate action of $W = N/T$, we fall into one of the following three cases.

(Case 1) There exists $w \in W$ satisfying

$$\gamma = w(\sigma g)w^{-1} = \sigma(v_i' v_i h' u_i u_i')$$

with $v_i' \in V_i',\ v_i \in V_i,\ h' \in T,\ u_i \in U_i,\ u_i' \in U_i'$ and $v_i u_i \neq 1$ for some $i \in I$.

(Case 2) The element σg is of the form

$$\gamma = \sigma g = \sigma h'$$

with $h' \in T$ and $\gamma \notin Z(\Gamma)$.

(Case 3) The element σg is of the form

$$\gamma = \sigma g = \sigma h'$$

with $h' \in T$ and $\gamma \in Z(\Gamma)$.

In (Case 2), we can find an element $x_{\alpha_j}(t)$ with $t \neq 0$ for some $j \in I$ such that

$$x_{\alpha_j}(t)(\sigma h')x_{\alpha_j}(-t) = \sigma(h' u_j)$$

with $u_j \neq 1$. Therefore, we can reach (Case 1) in this case. Since we need not consider the situation of (Case 3) by our assumption, we can assume that (Case 1) holds. And now we fix such $i \in I$ described in (Case 1).

Put $X = \{\ \alpha_1, \alpha_2, \cdots, \alpha_{n-1}\ \}$ and $Y = \{\ \alpha_2, \alpha_3, \cdots, \alpha_n\ \}$. We write

$$\begin{array}{rcl} h' & = & h_{\alpha_1}(t_1) \cdots h_{\alpha_n}(t_n), \\ h_X & = & h_{\alpha_1}(t_1) \cdots h_{\alpha_{n-1}}(t_{n-1}), \\ h_Y & = & h_{\alpha_2}(t_2) \cdots h_{\alpha_n}(t_n) \end{array}$$

with $t_1, \cdots, t_n \in K^\times$.

If $i > 1$, then we put $\sigma_Y = \sigma h_{\alpha_1}(t_1)$ and $\Gamma_Y = \langle\ \sigma_Y,\ G_Y\ \rangle$. Then,

$$\gamma = v_Y'(\sigma_Y v_Y h_Y u_Y)u_Y'$$

for some $v_Y' \in V_Y',\ v_Y \in V_Y,\ u_Y \in U_Y,\ u_Y' \in U_Y'$ with $v_Y u_Y \neq 1$. Then, we choose $t_2' \in K^\times$ such that $\sigma h^* h_{\alpha_2}(t_2')$ is noncentral in $\Gamma_X = \langle\ \sigma_X,\ G_X\ \rangle$, where

$\sigma_X = \sigma h_{\alpha_n}(t^*_{\alpha_n})$. This is actually possible since we have enough elements in K. Applying our induction to

$$\gamma_Y = \sigma_Y(v_Y h_Y u_Y) \in \Gamma_Y,$$

we can find an element $z_Y \in G_Y$ such that $z_Y \gamma_Y z_Y^{-1} = \sigma_Y v''_Y h''_Y u''_Y$ with $v''_Y \in V_Y,\ u''_Y \in U_Y$ and $\sigma h^* h_{\alpha_2}(t'_2) = \sigma_Y h''_Y$. Then, we have

$$\gamma' = z_Y \gamma z_Y^{-1} = \sigma v'' h'' u''$$

with $v'' \in V,\ u'' \in U$ and $h'' = h^* h_{\alpha_2}(t'_2)$. Next, we write

$$\gamma' = v_X(\sigma_X v_X h_X u_X) u'_X$$

with $v'_X \in V'_X,\ v_X \in V_X,\ u_X \in U_X,\ u'_x \in U'_X$ and

$$h_X = h_{\alpha_1}(t_1) h_{\alpha_2}(t^*_{\alpha_2} t'_2) h_{\alpha_3}(t^*_{\alpha_3}) \cdots h_{\alpha_{n-1}}(t^*_{\alpha_{n-1}}).$$

Then, we apply our induction to

$$\gamma_X = \sigma_X(v_X h_X u_X) \in \Gamma_X.$$

Since γ_X is noncentral in Γ_X, we can find an element $z_X \in G_X$ such that $z_X \gamma_X z_X^{-1} = \sigma_X v''_X h''_X u''_X$ with $v''_X \in V_X,\ u''_X \in U_X$ and $\sigma h^* = \sigma_X h''_X$. Therefore, we have

$$\gamma'' = z_X \gamma' z_X^{-1} = \sigma v''' h^* u'''$$

with $v''' \in V,\ u''' \in U$. This is what we wanted.

If i = 1, then we can take X first and then Y next. Then the same process as above works. We should choose $\alpha_n,\ \alpha_{n-1}$ and α_1 instead of $\alpha_1,\ \alpha_2$ and α_n respectively. Hence we have completed the proof of Proposition 1. Then, Proposition 1 implies Theorem 1.

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