SURJECTIVE STABILITY FOR THE K_1-FUNCTOR FOR SOME EXCEPTIONAL CHEVALLEY GROUPS

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Proofs are given for results on surjective stability for the K_1-functor for some exceptional Chevalley groups of normal types. Conditions are given (in terms of absolute stable rank) for stability for inclusions of Chevalley groups associated with maximal standard inclusions of root systems. The proofs are based on ideas of M. R. Stein.

In the present paper, proofs are given for the results on Chevalley groups of normal types announced in [17, 18]. We use the approach developed in a fundamental paper of M. Stein [35]. In fact, the purpose of our paper is to consider those cases of the inclusions of root systems that were not considered in [35] and, therefore, to obtain a complete list of conditions under which stability appears for all maximal inclusions of root systems. On the other hand, another "pedagogical" idea was kept in mind, namely, to use systematically the technique of base representations and weight diagrams developed in [33, 35], etc., which proved to play an important role in various questions on the structure of Chevalley groups, see [33, 35, 8, 6, 15, 34, 7], etc.

1. Notation and Statement of the Problem

Let R be a commutative ring with identity, Φ an irreducible root system, and let $G(\Phi,)$ be the Chevalley—Demazur group scheme. The value of this scheme at the ring R is denoted by $G(\Phi, R)$ and is said to be the Chevalley group of type Φ over R. Let $\Pi = \{\alpha_1, \ldots, \alpha_m\}$ and let Φ^+, Φ^- be the sets of positive and negative fundamental roots of the system Φ with respect to some order. The elementary unipotent root elements will be denoted by $x_\alpha(t)$ for $\alpha\in\Phi$, $t\in R$.

For a set X, we let $\langle X\rangle$ denote the object generated by X in the following sense: $\langle X\rangle$ denotes the subgroup generated by X if X is a subset of elements of a group G, $\langle X\rangle$ is the ideal generated by X if X is a subset of elements of a ring R, and $\langle X\rangle$ is the minimal closed root subsystem containing X if X is a subset of elements of a root system Φ.

We set, as usual,

$$E(\Phi,R)=\langle x_\alpha(t),\ \alpha\in\Phi,\ t\in R\rangle$$
$$V(\Phi,R)=\langle x_\alpha(t),\ \alpha\in\Phi^-,\ t\in R\rangle$$
$$U(\Phi,R)=\langle x_\alpha(t),\ \alpha\in\Phi^+,\ t\in R\rangle.$$

The subgroup $E(\Phi, R)$ is said to be the elementary subgroup of the Chevalley group.

It is well known that $G(\Phi, R) = E(\Phi, R)$ if $G(\Phi, R)$ is a simply connected Chevalley group and R is a field or, more generally, a semilocal ring.

Let $T(\Phi, R)$ be a split maximal torus in the group $G(\Phi, R)$. Then for a semilocal ring R, we have (see [24])

$$G(\Phi,R)=E(\Phi,R)T(\Phi,R)$$

Let

$$w_\alpha(t) = x_\alpha(t)x_\alpha(-t^{-1})x_\alpha(t),\ t\in R^*,$$
$$h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1},$$

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where R^* is the multiplicative group of the ring R. Let

$$N(\Phi,R) = \langle w_\alpha(t), \alpha\in\Phi, t\in R^*\rangle,$$
$$H(\Phi,R) = \langle h_\alpha(t), \alpha\in\Phi, t\in R^*\rangle.$$

For an arbitrary R, we have $H(\Phi,R)=E(\Phi,R)\cap T(\Phi,R)$. We will denote the Weyl group of the root system Φ by $W(\Phi)$. The group $W(\Phi)$ can be canonically identified with the factor group $N(\Phi, R)/H(\Phi, R)$.

Generally speaking, the Chevalley group $G(\Phi, R)$ depends on the representation of the corresponding complex simple Lie algebra of type Φ. We will consider only simply connected Chevalley groups $G(\Phi, R)$.

The question whether the subgroup $E(\Phi, R)$ is normal in $G(\Phi, R)$ was open for a long time. This is not the case for the groups of rank one (see [29, 38, 21]). Moreover, as for arbitrary associative rings, there exists a ring A such that even the subgroup $E_n(A)$ of elementary matrices is not normal in $GL_n(A)$ (see [13]). However, for all Chevalley groups of rank ≥ 2 and for any commutative ring R, we have

$$E(\Phi,R) \lhd G(\Phi,R)$$

[39, 40, 25, 8, 20, 22, 16]. We can therefore consider the factor group

$$K_1(\Phi,R)=G(\Phi,R)/E(\Phi,R)$$

This is nothing but the K_1-functor for Chevalley groups. If $\Phi = A_n$, we obtain the usual SK_1-functor for the special linear group [26]. An inclusion of root systems $\Delta\subseteq\Phi$ induces homomorphisms $\nu: G(\Delta,R)\to G(\Phi,R)$, $\nu: E(\Delta,R)\to E(\Phi,R)$, and, therefore, $\bar{\nu}: K_1(\Delta,R)\to K_1(\Phi,R)$.

By a problem of surjective (injective) stability, we mean the following: Find conditions on the ring R depending on $\Delta \to \Phi$ under which the homomorphism $\bar{\nu}$ is surjective (injective). Surjective stability is equivalent to the decomposition $G(\Phi,R)=E(\Phi,R)G(\Delta,R)$, and injective stability is equivalent to the following formula:

$$E(\Delta,R)=G(\Delta,R)\cap E(\Phi,R)$$

Therefore, when speaking about stability for K_1-functor for an inclusion $\Delta\subseteq\Phi$, we can simply mean the existence of such a decomposition.

The stability problem for the K_1-functor for Chevalley groups over rings is a natural expansion of the analogous problem for classical groups. This problem is mentioned, among others, by H. Bass in [1]. After the papers of Bass, L. Vaserstein, H. Bass—J. Milnor—J.-P. Serre, and others on classical groups (commutativity of the ring was not assumed), a considerable advance for the general case of Chevalley groups was achieved in [33, 30] and in Stein's series of papers [35-37]. The definition below follows [35].

A commutative ring R is said to satisfy the absolute stable rank condition ASR_n if for any row $(r_1, \ldots r_n)\in R^n$, there exist $t_1, \ldots, t_{n-1}\in R$ such that any maximal ideal in R containing $\langle r_1+t_1r_n, r_2+t_2r_n, \ldots, r_{n-1}+t_{n-1}r_n\rangle$ also contains the ideal $\langle r_1, \ldots r_n\rangle$ (see [31, 35]).

If the row $(r_1, \ldots, r_n)\in R^n$ is unimodular, i.e., $\langle r_1, \ldots, r_n\rangle = R$, the ASR_n condition could be replaced by the stable rank condition SR_n. Both of the conditions ASR_n and SR_n are known to follow from the condition $\dim Max(R) = n-2$.

The condition ASR_n has the usual properties, i.e., it is inherited by a factor ring and, moreover, ASR_m implies ASR_n if $n > m$. In the next section, we will again discuss the role of the conditions SR_n and ASR_n.

2. Base Representations and the Chevalley—Matsumoto Theorem

With every irreducible representation π of finite dimention of a complex simple Lie algebra g of type Φ, we can associate a representation of the Chevalley group $G(\Phi, R)$ in a free R-module V (see [3, 19, 33]). By irreducibility of an induced representation we mean that it is irreducible for $G(\Phi,\mathbb{C})$. For the study of Chevalley groups over rings, it is extremely important to choose the representation π in an appropriate way.

Let $\Lambda(\pi)$ denote the set of all weights of the representation π, let $\Lambda^*(\pi)$ be the set of nonzero weights, and let $\Delta(\pi)$ be the set of fundamental roots of the system Φ that belong to the set of weights of the representation π.

A natural way to transfer arguments for classical groups to the general case is to choose a representation π so that all weight submodules V^λ associated with the weights $\lambda\in\Lambda^*(\pi)$ are one-dimensional.

Definition [33]. An irreducible representation π is said to be a base representation if the Weyl group $W(\Phi)$ acts transitively on the set $\Lambda^*(\pi)$.

This definition is equivalent to the following: if $\lambda_1-\lambda_2 = \alpha$, with $\lambda_1,\lambda_2\in\Lambda^*(\pi)$, $\alpha\in\Pi$, then we have $w_\alpha\lambda_1=\lambda_2$ where w_α is a fundamental reflection.

Such representations satisfy the above condition. Moreover, the dimension of the weight submodule V^0 associated with the zero weight of the representation π is equal to the number of those fundamental roots of Φ that are the weights of the representation π, i.e., $\dim V^0$ = card $\Delta(\pi)$. In the sequel, it will be convenient to consider the zero weights $\hat{\alpha}_i$, $\alpha_i\in\Delta(\pi)$. We can choose a basis of the module $V=\sum V^\lambda\oplus V^0$, $\lambda\in\Lambda^*(\pi)$ in a special way: $v^\lambda\in V^\lambda, \lambda\in\Lambda^*(\pi)$, $v^0_\alpha\in V^0, \alpha\in\Delta(\pi)$, so the action of the elementary root unipotents $x_\alpha(t), \alpha\in\Phi, t\in R$, in this basis has a simple structure. A description of this action is given by Lemma 2.3 of [33], which we will use widely, with some additions (see [7]).

We denote the action of elementary root unipotents of the representation π on the vectors v^λ by simply $x_\alpha(t)v^\lambda$. Recall that an element v of an R-module V is said to be unimodular if there exists an $f\in V^* = \mathrm{Hom}(V, R)$ such that $f(v)$ is invertible in R. For a free R-module V, this means that v may be included in a basis of V.

LEMMA 1 ([33]). The elementary root unipotent elements $x_\alpha(t)$ act on the above basis of the R-module V according to the rule

1. if $\lambda\in\Lambda^*(\pi)$, $\lambda+\alpha\notin\Lambda(\pi)$, then $x_\alpha(t)v^\lambda=v^\lambda$;
2. if $\lambda,\lambda+\alpha\in\Lambda^*(\pi)$ then $x_\alpha(t)v^\lambda=v^\lambda\mp tv^{\lambda+\alpha}$;
3. if $\alpha\notin\Lambda(\pi)$ then $x_\alpha(t)v^0=v^0$ for every $v^0\in V^0$;
4. if $\alpha\in\Lambda(\pi)$ then $x_\alpha(t)v^{-\alpha}=v^{-\alpha}\mp tv^0(\alpha)\mp t^2v^\alpha$, $x_\alpha(t)v^0=v^0\mp t\alpha_*(v^0)v^\alpha$,

where α_* is a unimodular element of $(V^0)^* = \mathrm{Hom}_R(V^0, R)$, and $v^0(\alpha)$ is a unimodular element of V^0. More precisely, $x_\alpha(t)v^0_\beta = v^0_\beta \mp \delta_{\alpha,\beta}tv^\beta$, where α, $\beta\in\Lambda(\pi)$.

Furthermore, the elements α_*, $\alpha\in\Delta(\pi)$, form a basis of $(V^0)^*$. The choice of signs in the lemma is a topic for separate investigation (see [32]) and is not important for us.

Let μ be the highest weight of the representation π. We denote by $(G(\Phi, R),\mu)$ the group $G(\Phi, R)$ considered in this representation. Any element g of the group $(G(\Phi, R), \mu)$ is represented in the above basis by a matrix $g = (g_{\lambda,\nu})$; λ, $\nu\in\Lambda^*(\pi)\cup\Delta(\pi)$, whose rows and columns are indexed by weights of the representation π. Furthermore, we denote by $\lambda(g) = g_{\lambda,\nu}$ the λ-th coordinate of the first column of g.

We recall the list of base representations [7]. It is well known (see [33]) that $G(\Phi, R)$ has a unique base representation with zero weight. This is the representation on the short roots of the root system Φ whose highest weight is the dominant short root, i.e., it is simply the maximal root for those root systems in which all roots are of equal length. Thus, in the notation of [4], we have

A_l	$\mu = \varepsilon_1 - \varepsilon_{l+1}$	π is the adjoint representation
B_l	$\mu = \omega_1$	π is the representation of minimal dimension
C_l	$\mu = \omega_2$	
D_l	$\mu = \omega_2$	π is the adjoint representation

E_6	$\mu = \omega_2$	π is the adjoint representation
E_7	$\mu = \omega_1$	π is the adjoint representation
E_6	$\mu = \omega_1, \omega_6$	π is the two minimal dimensional representations
F_4	$\mu = \omega_4$	
G_2	$\mu = \omega_1$	

Now let $\Lambda(\pi) = \Lambda^*(\pi)$, i.e., $\Lambda(\pi) = W(\Phi)\mu$. Then the base representation with the highest weight μ is a microweight representation. We have

A_l, $\mu = \omega_k$, $k = 1,\ldots,l$	π is the k-th exterior power of the minimal representation
B_l, $\mu = \omega_2$,	π is the spinor representation
C_l, $\mu = \omega_1$	π is the minimal representation
D_l, $\mu = \omega_1$	π is the minimal representation
$\mu = \omega_{l-1}, \omega_l$	π is the semispinor representation
E_7, $\mu = \omega_7$	

There are no such representations for E_8, F_4, and G_2. All base representations are easily seen to be fundamental, except the adjoint representation for $\Phi = A_\ell$.

Note that the list of representations having the fundamental (for us) property $\dim V^\lambda = 1$ for $\lambda \in \Lambda^*(\pi)$ is not exhausted by the base representations. In particular, the adjoint representations for $\Phi = C_\ell$, F_4, and G_2 are not base representations.

It is an important fact that we can associate a weight diagram [35] with every base representation (see also [6, 7, 34, 15]), the diagram being constructed in the following manner [35, 7].

The diagram consists of vertices and edges.

1. With every weight $\lambda \in \Lambda^*(\pi)$ we associate a vertex indicated by the symbol λ, and we agree to read the diagram from the left to the right, i.e., a bigger weight stands on the left of a smaller one.

2. If $\lambda_1, \lambda_2 \in \Lambda(\pi)$ and $\lambda_1 + \lambda_2 = \alpha_i$ for some $\alpha_i \in \Pi$, then the corresponding vertices are connected by an edge indicated by α_i or simply by i.

3. With a zero weight of multiplicity K we associate K vertices K_i, the following sequence of length three being associated with every "zero weight" $\hat{\alpha}_i$:

$$\overset{\alpha_i}{\circ}\text{———}\overset{\hat{\alpha}_i}{\circ}\text{———}\overset{-\alpha_i}{\circ}$$

Fig. 1

where all $\hat{\alpha}_i$ are not connected with other weights.

The labels of vertices may be uniquely restored from the labels of edges and the highest weight, so, they are usually omitted. Note that irreducibility of a representation is equivalent to connectedness of its diagram. We now proceed to the Chevalley—Matsumoto theorem. Fix a root $\alpha_k \in \Pi$ and let Δ be the minimal root subsystem of Φ generated by $\Pi - \{\alpha_k\}$.

We set $\Sigma = \Phi - \Delta, \Sigma^{\pm} = \Phi^{\pm} \cap \Sigma$,

$$U(\Sigma,R) = \langle x_\alpha(t),\ \alpha\in\Sigma^+, t\in R\rangle$$
$$V(\Sigma,R) = \langle x_\alpha(t),\ \alpha\in\Sigma^-, t\in R\rangle.$$

Recall that if π is a base representation, then for all $G(\Phi, R)$ there exists a unique root $\alpha_k \in \Lambda(\pi)$ such that $\mu - \alpha_k \in \Sigma$. We take it for the definition of the root subsystem Δ. If $\Phi = A_\ell$, we set $\alpha_k = \alpha_\ell$ or α_1.

The Chevalley—Matsumoto theorem [28, 33] claims that if for an element $g\in(G(\Phi, R), \mu)$ we have $g_{\mu\mu}\in R^*$, then we have a decompositon $g = vhg_1u$ with $v\in V(\Sigma, R)$, $u\in U(\Sigma,R)$, $hg_1\in T(\Phi,R)G(\Delta,R)$, the factors v, u, and hg_1 being uniquely determined. Moreover, if $g_{\mu\mu} = 1$, then $g = vg_1u = eg_1$ with $e\in(E(\Phi,R),\mu)$.

Finally, if $g_{\mu\mu} = 1$ and $g_{\lambda\mu} = 0$ for $\lambda \neq \mu$, then $g = g_1\mu$ with $g_1\in G(\Delta,R)$, $u\in U(\Sigma,R)$.

It is easy to see that the Chevalley—Matsumoto decomposition is the first step towards the important Gauss decomposition

$$G(\Phi,R) = U(\Phi,R)V(\Phi,R)T(\Phi,R)U(\Phi,R)$$

We now suppose that α_k is an arbitrary root. Then the theorem remains valid if we make the following changes: instead of invertibility of the element $g_{\mu\mu}$ (i.e., of the principal minor of order one in the representation with principal weight μ such that $\mu-\alpha_k\in\Lambda(\pi)$) we require invertibility of the principal minor of the matrix $g\in(G(\Phi, R), \mu)$ that consists of those elements $g_{\lambda,\nu}$ for which the root α_k does not occur in the decomposition $\lambda-\nu = \Sigma\alpha_s$ with $\lambda, \nu\in\Lambda(\pi)$, $\alpha_s\in\Pi$. Note that if the root α_k has the property that there exists a base representation π with principal weight μ such that $\mu-\alpha_k\in\Lambda(\pi)$, then invertibility of the corresponding principal minor of the element g in an arbitrary base representation follows from the existence of the decomposition for g in the group $(G(\Phi, R),\mu)$ under the assumption $g_{\mu\mu}\in R^*$, since the factors u, v, hg_1 do not depend on a representation. The Chevalley—Matsumoto theorem admits a graphic illustration in terms of base diagrams: Namely, if the root α_k is thrown away, the diagram of the base representation aplits into several connected components, the last of which corresponds to an invertible minor. There is another illustration for classical groups: Almost all their fundamental representations, except the spinor one for $\Phi = B_n$ and the two semispinor representations for $\Phi = D_n$, are the exterior powers of natural representations of universal groups or their subrepresentations ($\Phi = C_n$), and a block form for a matrix $g\in(G(\Phi,R),\omega_1)$, $\Phi=A_n,B_n,C_n,D_n$, implies a certain form of its n-th exterior power, which belongs to $(G(\Phi,R),\omega_k)$, ($k\neq n$ for $\Phi = B_n$ and $k \neq n, n-1$ for $\Phi = D_n$).

3. The Stability Theorem

Our purpose is to prove a theorem about surjective stability for the K_1-functor for some inclusions of Chevalley groups associated with maximal standard inclusions of irreducible root systems, i.e., with those inclusions defined by connected subgraphs of a Dynkin diagram that could be obtained by removing one extreme fundamental root. As a corollary, we obtain stability for all maximal inclusions of root systems.

It is well-known [14] that in the above cases the root system Φ has exactly one class of subsystems of the type Δ, up to conjugation by an element of the Weyl group, except in the following two cases: The root system D_ℓ with $\ell \neq 2k$ has two conjugacy classes of subsystems of the type $A_{\ell-1}$ that are transformed into each other by an outer automorphism of order two, and the root system E_8 has the subsystems $A_7'\subset A_8$ and $A_7''\not\subset A_8$. Also, the subsystem A_3 of the system D_4 is transformed by an outer automorphism of order three into a subsystem of the type D_3, i.e., there are three conjugacy classes.

THEOREM 1. 1. Surjective stability for the K_1-functor holds under the condition ASR_4 for the following inclusions of root systems:

$$D_5,A_5 \longrightarrow E_6;\ E_6,D_6,A_6 \longrightarrow E_7; D_7,E_7,A_7',A_7'' \longrightarrow E_8;\ A_{n-1} \longrightarrow D_n$$

2. Under the condition ASR_3, the same holds for the inclusions $B_3,C_3 \longrightarrow F_4$, $A_{n-1} \longrightarrow B_n$.

Since stability for the K_1-functor is well known for various inclusions ($A_{n-1} \longrightarrow A_n$; $B_{n-1} \longrightarrow B_n$; $C_{n-1} \longrightarrow C_n$; $D_{n-1} \longrightarrow D_n$; $A_{n-1} \longrightarrow C_n$; $A_1,\tilde{A}_1 \longrightarrow G_2$) ; (see [26, 1, 2, 9-12, 33, 35], etc.), we collect the information on the above inclusions $\Delta\subset\Phi$ in Table 1.

COROLLARY 1. Under the following restrictions on the ring R, depending on the maximal standard inclusions of the root systems $\Delta \to \Phi$, we have a decomposition of the form $G(\Phi, R) = E(\Phi, R)G(\Delta, R)$ for the Chevalley group.

TABLE 1

Inclusion $\Delta \to \Phi$	Cond. on R	Inclusion $\Delta \to \Phi$	Cond. on R
$A_{n-1} \longrightarrow A_n$	SR_{n+1}	$D_6 \longrightarrow E_7$	ASR_4
$B_{n-1} \longrightarrow B_n$	ASR_n	$A_6 \longrightarrow E_7$	ASR_4
$A_{n-1} \longrightarrow B_n$	ASR_3	$D_7 \longrightarrow E_8$	ASR_4
$C_{n-1} \longrightarrow C_n$	SR_{2n}	$A_7' \longrightarrow E_8$	ASR_4
$\tilde{A}_{n-1} \longrightarrow C_n$	ASR_2	$A_7'' \longrightarrow E_8$	ASR_4
$D_{n-1} \longrightarrow D_n$	ASR_n	$E_7 \longrightarrow E_8$	ASR_4
$A_{n-1} \longrightarrow D_n$	ASR_4	$B_3 \longrightarrow F_4$	ASR_3
$D_5 \longrightarrow E_6$	ASR_4	$C_3 \longrightarrow F_4$	ASR_3
$A_5 \longrightarrow E_6$	ASR_4	$A_1 \longrightarrow G_2$	ASR_3
$E_6 \longrightarrow E_7$	ASR_4	$\tilde{A}_1 \longrightarrow G_2$	ASR_2

Here we denote an inclusion on the short roots by a tilde. We assume that $A_1 = C_1$ for the inclusion $C_{n-1} \to C_n$, and $\tilde{A}_1 = B_1$ for the inclusion $B_{n-1} \to B_n$.

Stability for the other maximal inclusions of root systems $\Delta \subset \Phi$ follows immediately from the inclusions $A_{n-1} \longrightarrow D_n \longrightarrow B_n$, $A_6 \longrightarrow A_7' \longrightarrow E_7$, $A_6 \longrightarrow A_7'' \longrightarrow E_7$, $A_7 \longrightarrow A_8 \longrightarrow E_8$, $B_3 \longrightarrow B_4 \longrightarrow F_4$. Thus, we obtain

COROLLARY 2. 1. Surjective stability for the K_1-functor for the inclusions $D_n \to B_n$ and $B_4 \to F_4$ holds when the condition ASR_3 holds.

2. The same is true for the inclusions $A_7', A_7'' \longrightarrow E_7$ and $A_8 \longrightarrow E_8$ when the condition ASR_4 holds.

Obviously, the conditions on the ring R given in Corollary 1 may be weakened in some cases or replaced by others. We will consider this question in more detail. It is easy to see that all conditions on the ring R are formulated in terms of stable and absolute stable rank.

The stable rank condition, which is historically the most popular for stabilization of Chevalley groups, proves to be suited to the case of the special linear group, or, more precisely, to the case of those classical groups whose rows and columns do not satisfy any equation, i.e., SL(n, R) and Sp(n, R). L. Vaserstein [11] discovered that a more complicated condition must hold for surjective stability of the K_1-functor for orthogonal groups. This condition is related to the equation satisfied by an arbitrary column of an orthogonal matrix. It was a remarkable observation of M. Stein [35] that the absolute stable rank condition takes into account the equations mentioned above, so it plays an important role whenever subrepresentations of an orthogonal group are considered. However, there is no doubt that, for example, in the case of exceptional Chevalley groups there are conditions of the type [11] giving a sharper estimate than Corollary 1. These conditions should take into account the numerous equations satisfied by the rows and the columns of matrices which belong to exceptional Chevalley groups. As those equations are cumbersome and the representation diagrams are very complex, the desired conditions might prove to be rather complicated.

COROLLARY 3. Under the following restrictions on the ring R, the inclusions of root systems induce isomorphisms:

ASR_3	$K_1(D_n, R) \approx K_1(B_n, R)$	$D_n \longrightarrow B_n$
ASR_4	$K_1(D_5, R) \approx K_1(E_6, R)$	$D_5 \longrightarrow E_6$
ASR_4	$K_1(E_6, R) \approx K_1(E_7, R)$	$E_6 \longrightarrow E_7$
ASR_4	$K_1(E_7, R) \approx K_1(E_8, R)$	$E_7 \longrightarrow E_8$
ASR_3	$K_1(D_4, R) \approx K_1(F_4, R)$	$D_4 \longrightarrow F_4$

In particular, all these groups are isomorphic to $SK_1(R)$ under the condition ASR_3, which is a slight generalization of Matsumoto's result on the same isomorphism for a Dedekind ring.

The proof follows immediately from Theorem 1 and the lemma (see [35]) on injective stability for the K_1-functor under the following conditions: $D_n \to B_n$ under SR_n, $D_5 \to E_6$

under SR_4, $E_6 \to E_7$ under SR_5, $E_7 \to E_8$ under SR_6, and $D_4 \to F_4$ under SR_4. Finally, the isomorphisms

$$K_1(A_3,R) \approx K_1(D_3,R) \approx K_1(D_4,R) \approx K_1(D,R) \approx K_1(A,R) = SK_1(R)$$

that are valid under the condition ASR_3 imply the latter assertion of Corollary 3.

We note that if the ring R satisfies the condition ASR_2, then for any root system Φ, we have $K_1(\Phi, R) = 1$, since under the condition SR_2 the functor SK_1 is trivial and $K_1(\Phi, R) \cong SK_1(A, R)$.

In fact, it follows from the proof of the theorem that a stronger assetion is true under the conditon ASR_2; namely, the Chevalley group admits the Gauss decomposition, i.e.,

$$G(\Phi,R)=U(\Phi,R)T(\Phi,R)V(\Phi,R)U(\Phi,R)$$

(It is known that the Gauss decomposition holds for the groups $G(A_n, R)$ and $G(C_n, R)$ even under the condition SR_2).

It follows that under the condition ASR_2, Theorem 1 of [5] is valid for any root system Φ, i.e., if the ring R satisfies the condition ASR_2 and some additional insignificant assumptions, then we have the standard description of parabolic subgroups of the group $G(\Phi, R)$.

4. Proof of the Theorem

First we will sketch the proof. We will consider base representations of the Chevalley group $G(\Phi, R)$ with the highest weight μ, where μ is a fundamental weight chosen according to the root α_k excluded from the root system Φ under the inclusion $\Delta \to \Phi$.

It follows directly from the Chevalley—Matsumoto theorem that the proof of the theorem will be completed whenever for every $g \in (G(\Phi, R), \mu)$, we can choose an $e \in (E(\Phi, R), \mu)$ such that $(eg)_{\mu,\mu} \in R^*$ or, equivalently, in another notation, $\mu(eg) = 1$. In order to do this, we use only stable computations, i.e., those concerning only one row or a column of the matrix g. We carry out these computations using Lemma 1 and diagrams of base representations, these last providing a graphic illustration of transformations of a fixed column of the matrix $\pi(g)$ under the action of elementary root unipotents $x_\alpha(t)$, $\alpha \in \Phi$, $t \in R$.

We will frequently use the following two facts in the proof without a reference.

If $\mu(g) = 1$, it follows from the Chevalley—Matsumoto theorem that there exists an $e \in (E(\Phi, R), \mu)$ such that $\lambda_i(eg) = 0$ for $\lambda_i \neq \mu$. Furthermore, if $\lambda_i \neq \mu$ and $\lambda_i(g) = 1$, there exists an $e \in (E(\Phi, R), \mu)$ such that $\mu(eg) = 1$.

4.1. The Inclusions $D_5 \to E_6$ and $A_5 \to E_6$

LEMMA 2. Let $g \in (G(D_5,R),\omega_4)$. Then, under the condition ASR_4, there exists an $e \in (E(D_5,R),\omega_4)$ such that $\mu(eg) = 1$.

PROOF. The diagram of the base representation (D_5, ω_4) is depicted in Fig. 2, and the enumeration of weights is given in Fig. 3.

It is known [35] that if condition ASR_5 holds, we have $G(D_5,R)=E(D_5,R)G(D_4,R)$, so there exists an $e \in E(D_5, R)$ such that the element $g_1 = e^{-1}g$ belongs to $G(D_4, R)$. We are interested in the form of the element g_1 in the representation with the highest weight ω_4. We have $D = \langle D-\{\alpha_1\}\rangle$. Hence we have $(g_1)_{\lambda,\nu} = 0$ for the matrix $g_1 \in (G(D_4,R),\omega_4)$ whenever α_1 occurs in the decomposition $\lambda-\nu=\sum\alpha_k$, $\alpha_k \in \Pi$. This corresponds to the splitting of g_1 into blocks according to the splitting of the base representation diagram after removing the root α_1. Thus,

$$g_1 = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

where $g_{11}, g_{22} \in (G(D_4,R),\omega_4)$. Applying the condition ASR_4 to g_{11}, we find that there exists an $e_1 \in (G(D_4,R),\omega_1) \longrightarrow (G(D_5,R),\omega_4)$ such that $e_2 = e_1e^{-1}$ implies $\mu(e_2g) = 1$.

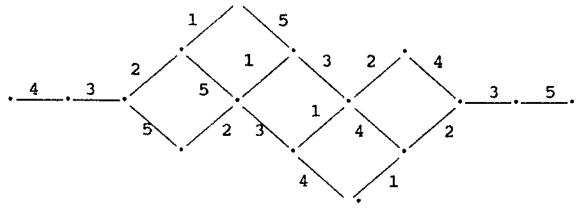


Fig. 2

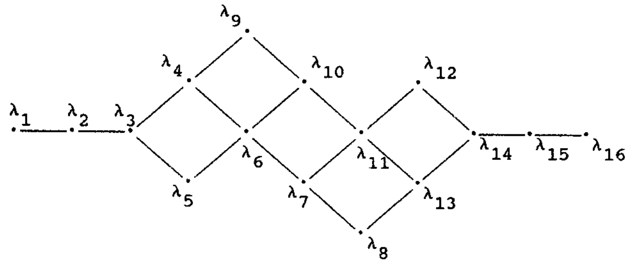


Fig. 3

REMARK. Since $\langle D_5-\{\alpha_4\}\rangle=A_4$ Lemma 2 claims precisely that surjective stability holds for the inclusion $A_4 \to D_5$ under the condition ASR_4.

The Case $D_5 \to E_6$. Let $g\in G(E_6, R)$, and consider the group $(G(E_6, R), \omega_6)$.

The representation diagram has the form

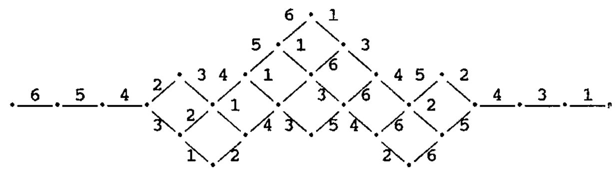


Fig. 4

We will numerate the weights according to the inclusion $D_5 \to E_6$ (we take $\lambda_i = i$),

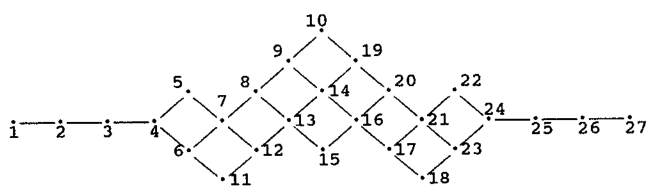


Fig. 5

We have a unimodular row $(\lambda_1(g),\ldots,\lambda_{27}(g))$. We will use the inclusion $(G(A_3 \to E_6,R),\omega_1)$. From now on, we assume that the weight ω_k belongs to Δ for the inclusion $\Delta \to E_6$, since the representation E_6 is fixed. Consider the weights λ_{27}, λ_{26}, λ_{25} and λ_{24} of $(G(A_3 \to E_6, R), \omega_1)$. By the condition SR_4, there exists an $e\in(E(A_3 \to E_6, R), \omega_1)$ such that $\langle\lambda_i(eg)\rangle = R$, $i = 1, \ldots, 26$. We put $g_1 = e_1g$. Let $\mathfrak{a}=\langle\lambda_i(g_1)\rangle$, $i=1,\ldots,10$. Then the row $(\lambda_{11}(g_1),\ldots,\lambda_{26}(g_1))$ is unimodular modulo the ideal $\mathfrak{a}$ (i.e., its image in $R/\mathfrak{a}$ is unimodular) and, moreover, the group $(G(D_5 \to E_6,R),\omega_4)$ acts on this row. By Lemma 2, there exists an element $e_2\in(E(D_5 \to E_6, R), \omega_4)$ such that $\lambda_{11}(e_2g_1)\equiv1 (\mathrm{mod}\,\mathfrak{a})$ and $\lambda_i(e_2g)\equiv0 (\mathrm{mod}\,\mathfrak{a})$ for $i = 12, \ldots, 26$. As $\alpha_1\notin D_5 \to E_6$ on the weights $\lambda_1, \ldots, \lambda_{26}$, it follows from Lemma 1 that $\langle\lambda_i(e_2g_1)\rangle=\mathfrak{a}$, $i=1,\ldots,10$. It follows from these congruence relations that the row $(\lambda_1(g_2),\lambda_2(g_2),\ldots,\lambda_{11}(g_2))$ is unimodular, where $g_2 = e_2g_1$. The group $(G(D_5 \to E_6, R), \omega_1)$ acts on the weights $\lambda_1, \ldots, \lambda_{10}$. Consider the weights $\lambda_7, \ldots, \lambda_{10}$. By the condition ASR_4, there exists an $e_3\in(G(A_3 \to E_6,R),\omega_1)$ such that any maximal ideal containing $\langle\lambda_7(g_2),\ldots,\lambda_9(e_3g_2)\rangle$ also contains $\langle\lambda_7(g_2), \ldots, \lambda_{10}(g_2)\rangle$. Since $\alpha_2\notin A_3 \to E_6$ on the weights $\lambda_7, \ldots, \lambda_{10}$, we have $\langle\lambda_1(e_3g_3),\ldots,\lambda_6(e_3g_2)\rangle=\langle\lambda_1(g_2),\ldots,\lambda_6(g_2)\rangle$ and

$\lambda_{11}(e_3g_2)=\lambda_{11}(g_2)$. Hence, any maximal ideal $\mathfrak{m}$ containing $\langle\lambda_1(e_3g_2), \ldots, \lambda_9(e_3g_2), \lambda_{11}(e_3g_2)\rangle$ also contains the ideal $\langle\lambda_1(g_2), \ldots, \lambda_{10}(g_2), \lambda_{11}(g_2)\rangle$. So the row $(\lambda_1(g_3), \ldots, \lambda_9(g_3), \lambda_{11}(g_3))$, with $g_3 = e_3g_2$, is unimodular. Applying the condition ASR_4 to the inclusion $(G(A_3 \longrightarrow E_6, R), \omega_1)$ on the weights $\lambda_1, \ldots, \lambda_4$, we obtain, as above, that there exists $e_4 \in (E(A_3 \longrightarrow E_6, R), \omega_1)$ such that the

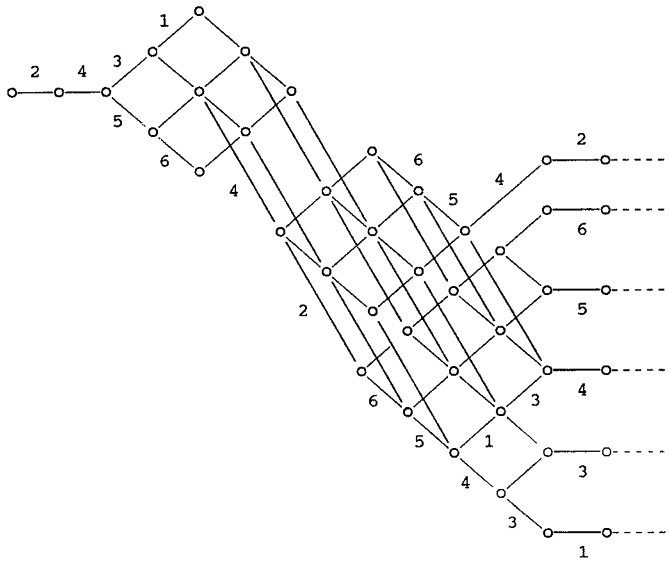


Fig. 6

row $(\lambda_2(e_4g_3), \ldots, \lambda_9(e_4g_3), \lambda_{11}(e_4g_3))$ is unimodular. We put $e_4g_3 = g_4$. Then the group $(\lambda_2(g_4), \ldots, \lambda_9(g_4)$ acts on the unimodular row $\lambda_{11}(g_4), \ldots, \lambda_{18}(g_4))$ $(G(D_5 \longrightarrow E_6, R), \omega_5)$. However, Lemma 2 is also valid for the representation (D_5, ω_5) as the dual of (D_5, ω_4). Hence, there exists $e_5 \in (E(D_5 \longrightarrow E_6, R), \omega_5)$ such that $\lambda_2(e_5, g_4)=1$.

The Case $A_5 \to E_6$. We have $A_5 = \langle E_6 - \{\alpha_2\}\rangle$. The representation (E_6, ω_2) is a base represent ation, and its dimension is equal to 78. The diagram of (E_6, ω_2) is depicted at Fig. 6. By the above, surjective stability for the inclusion $D_5 \to E_6$ holds subject to the condition ASR_4. Hence, for an element $g \in G(E_6, R)$, we have $g = eg_1$ with $e \in E(E_6, R)$, $g_1 \in G(D_5, R)$. This means that there exists $e \in (E(E_6, R), \omega_2)$ such that the element $g' = eg$ has the form

$$g' = \begin{pmatrix} g'_{11} & 0 & 0 \\ 0 & g'_{22} & 0 \\ 0 & 0 & g'_{33} \end{pmatrix}$$

where $g'_{11} \in (G(D_5, R), \omega_4)$, $g'_{22} \in (G(D_5, R), \omega_2)$ (the adjoint representation), and $g'_{33} \in (G(D_5, R), \omega_5)$. Applying Lemma 2 to the element g'_{11}, we find that there exists an $e_2 \in (D_5\ E_6, R), \omega_4)$ such that $\mu(e_2g') = 1$.

REMARK. We could also prove the inclusion $A_5 \to E_6$ by using the inclusions

$$\begin{array}{ccccc} A_4 & \longrightarrow & D_5 & \longrightarrow & E_6 \\ & \searrow & A_5 & \nearrow & \end{array}$$

However, the diagram of the base representation (E_6, ω_2) might be interesting itself, which determined the choice of the proof.

4.2. The Inclusions $E_6 \to E_7$ and $E_7 \to E_8$

The Case $E_6 \to E_7$. We have $E_6 = \langle E_7 - \{\alpha_7\}\rangle$. Thus, we are interested in the base representation (E_7, ω_7) of dimension 56. The corresponding diagram is given in Fig. 7.

The representation (E_7, ω_7) consists of the two dual representations (E_6, ω_6) and (E_7, ω_7) "glued together" by the root α_7. We enumerate the weights of the representation

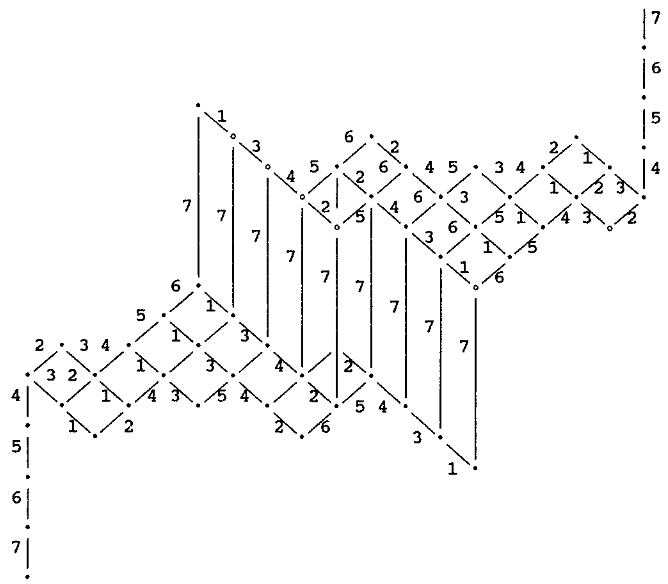


Fig. 7

(E_7, ω_7) according to the inclusion $E_6 \to E_7$. The weights inside (E_6, ω_6) and (E_6, ω_1) are enumerated in the same way as in the previous subsection.

Let $g \in (G(E_7,R),\omega_7)$. We have $\langle\lambda_i(g)\rangle = R$, $i = 1, \ldots, 56$. We will use the inclusion $(G(A_3 \to E_7, R), \omega_1)$ on the weights λ_{56}, λ_{55}, λ_{54} and λ_{53}. By the condition SR_4, there exists an $e_1 \in (E(A_3 \longrightarrow E_7,R),\omega_1)$ such that $\langle\lambda_i(e_1 g)\rangle = R$, $i = 1, \ldots, 55$. We set $g_1 = e_1 g$. Let $\langle\lambda_i(g_1)\rangle = \mathfrak{a}$, $i = 1, \ldots, 28$. The row $(\lambda_{29}(g_1), \ldots, \lambda_{55}(g_1))$ is unimodular modulo the ideal $\mathfrak{a}$ and, moreover, the group $(G(E_6 \to E_7, R), \omega_1)$ acts on it. By the condition ASR_4, there exists an $e_2 \in (E_6 \longrightarrow E_7,R),\omega_1)$ such that for elements of the matrix $g_2 = e_2 g_1$, we have $\lambda_{29}(g_2) \equiv 1 \pmod{\mathfrak{a}}$ and $\lambda_i(g_2) \equiv 0 \pmod{\mathfrak{a}}$ for $i = 30, \ldots, 55$. Furthermore, since $\alpha_7 \notin (E_6 \longrightarrow E_7)$ on the weights λ_i, with $i = 29, \ldots, 55$, we have $\langle\lambda_i(g_2)\rangle = \mathfrak{a}$, $i = 1, \ldots, 28$, so the row $(\lambda_1(g_2), \ldots, \lambda_{29}(g_2))$ is unimodular. We set $\mathfrak{a}_1 = \langle\lambda_i(g_2)\rangle$, $i = 25, \ldots, 28$, and $\mathfrak{a}_2 = \langle\lambda_i(g_2)\rangle$, $i = 1, \ldots, 24$. Then we have $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$. Using the condition ASR_4 and the inclusion $(G(A_3 \longrightarrow E_7,R),\omega_1)$ on the weights $\lambda_{25}, \ldots, \lambda_{28}$, we find that there exists an $e_3 \in (E(A_3 \longrightarrow E_7,R),\omega_1)$ such that any maximal ideal containing $\mathfrak{a}_1' = \langle\lambda_{25}(e_3 g_2), \ldots, \lambda_{27}(e_3 g_2)\rangle$ also contains $\mathfrak{a}_1$. Since $\alpha_2, \alpha_5 \notin A_3 \to E_7$ on the above weights, we have $\langle\lambda_i(e_3 g_2)\rangle = \mathfrak{a}_2$, $i = 1, \ldots, 24$, so any maximal ideal containing $\lambda_i(e_3 g_2)$ for $i = 1, \ldots, 27$ also contains $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{a}$. It remains to note that since $\lambda_i(g_2) \equiv 0 \pmod{\mathfrak{a}}$, we again have $\lambda_{29}(e_3 g_2) \equiv 1 \pmod{\mathfrak{a}}$. Hence we obtain the unimodular row $(\lambda_1(g_3), \ldots, \lambda_{27}(g_3), \lambda_{29}(g_3))$ with $g_3 = e_3 g_2$. Let $\mathfrak{a}_3 = \langle\lambda_i(g_3)\rangle$, $i = 1, \ldots, 11, 29$. The row $(\lambda_{12}(g_3), \ldots, \lambda_{27}(g_3))$ is unimodular modulo $\mathfrak{a}_3$, and the group $(G(D_5 \longrightarrow E_7,R),\omega_4)$ acts on it. This situation already occurred in the previous case, so there exists an $e_4 \in (E(D_5 \longrightarrow E_7,R),\omega_4)$ such that $\lambda_{12}(e_4 g_3) \equiv 1 \pmod{\mathfrak{a}_3}$ and $\lambda_i(e_4 g_3) \equiv 0 \pmod{\mathfrak{a}_3}$, $i = 13, \ldots, 27$. As $\alpha_1 \notin D_5 \longrightarrow E_7$ on the weights λ_i with $i = 12, \ldots, 27$, we again have $\langle\lambda_i(e_4 g_3)\rangle = \mathfrak{a}_3$, $i = 1, \ldots, 12, 29$. Letting $g_4 = e_4 g_3$, we obtain the unimodular row $(\lambda_1(g_4), \lambda_2(g_4), \ldots, \lambda_{12}(g_4), \lambda_{29}(g_4))$. Using the condition ASR_4 and the inclusion $(G(A_3 \longrightarrow E_7,R),\omega_1)$ on the weights λ_9, λ_{10}, λ_{11}, λ_{29} and λ_1, λ_2, λ_3, λ_4, we find, as above, that there exists an element $e_5 \in (G(A_3 \to E_7,R),\omega_1)$ such that any maximal ideal containing $\lambda_2(e_5 g_4)$, $\lambda_3(e_5 g_4)$, $\ldots$, $\lambda_{11}(e_5 g_4)$ also contains $\mathfrak{a}_3$. Since α_2 does not belong to the inclusion $A_3 \to E_7$ on the above weights, we have $\lambda_{12}(e_5 g_4) = \lambda_{12}(g_4)$, i.e., $\lambda_{12}(e_5 g_4) \equiv 1 \pmod{\mathfrak{a}_3}$. Hence, the row $(\lambda_2(g_5), \ldots, \lambda_{28}(g_5))$, with $g_5 = e_5 g_4$, is unimodular. The group $(G(E_6 \longrightarrow E_7,R),\omega_6)$ acts on it, so, by the condition ASR_4, we find that there exists an $e_6 \in (E(E_6 \longrightarrow E_7,R),\omega_6)$ such that $\lambda_2(e_6 g_5) = 1$.

The Case $E_7 \to E_5$. We have $E_7 = \langle E_8 - \{\alpha_8\}\rangle$, and the proof is similar to the previous case. As the present case is considerably more complicated, we give a sketch of the proof. The representation (E_8, ω_8) has dimension 248. After deleting the root α_8, the structure of the remaining blocks can be easily understood. The thick arcs at Fig. 8 mean that, in fact, the representations are connected by the 27 arcs associated with the root α_8.

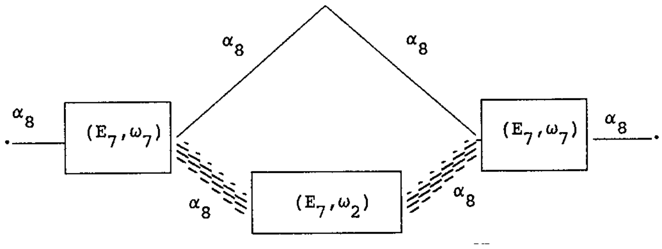


Fig. 8

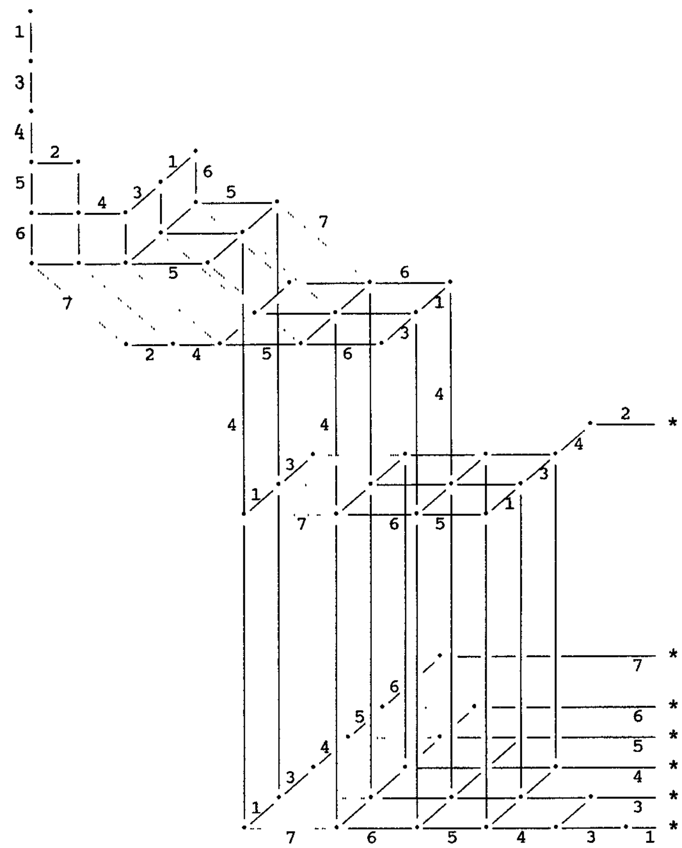


Fig. 9. * denotes the zero weight.

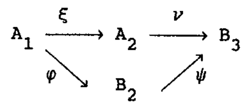
Let $g\in(G(E_8,R),\omega_8)$. In order to obtain an invertible element $\lambda(eg)\in R^*$ by multiplying g by an element $e\in(E(E_8,R),\omega_8)$, we must be able to obtain invertible elements in each of the blocks. This was done for $(E_7,\ \omega_7)$ while studying the inclusion $E_6\to E_7$ in the previous subsection. The case $(E_7,\ \omega_2)$ corresponding to the inclusion $D_6\to E_7$ will be dealt with in §4.4 (note that a proof similar to that for the inclusion $A_5\to E_6$ works in this case). The most difficult problem is to obtain an invertible element for the pair $(E_7,\ \omega_7)$, $(E_7,\ \omega_2)$. Basically, we do this in the same way as for $E_6\to E_7$, by using the representation $(G(D_7\to E_8,\ R),\ \omega_2)$ and the diagram $(E_7,\ \omega_2)$ (see Fig. 9). We omit the calculations, because they are cumbersome.

4.3. The Inclusion $B_3\to F_4$

LEMMA 3. Suppose the condition ASR_3 is satisfied in the ring R.

1. If $g\in(G(B_3,R),\omega_3)$, then there exists an $e\in(E(B_3,R),\omega_3)$ such that $\mu(eg)=1$.
2. If $g\in(G(C_3,R),\omega_2)$, then there exists an $e\in(E(C_3,R),\omega_2)$ such that $\mu(eg)=1$.

Proof 1. The assertion of the lemma is equivalent to surjective stability of the K_1-functor for the inclusion $A_2\to B_3$. We will use the diagram



and the isomorphisms $C_1 \cong A_1$ and $B_2 \cong C_2$. The inclusions ζ, ν, φ and ψ induce homomorphisms of the K_1-functors. The homomorphism corresponding to $\varphi \cdot \psi$ is an epimorphism under the conditions SR_4 and ASR_3. Therefore, under the condition ASR_3, the homomorphism corresponding to ν is also an epimorphism, by virtue of Proposition 1.5 of [35].

2. The diagram of the base representation (C_3, ω_2) has the form

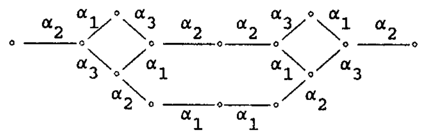


Fig. 10

This is an adjoint representation of dimention 14. Let $g \in (G(C_3, R), \omega_2)$. We will use the inclusion $C_2 \to C_3$. We have $g = eg_1$, with $e \in E(C_3,R), g_1 \in G(C_2,R)$. As $C_2 = \langle C_3 - \{\alpha_3\} \rangle$ the matrix g_1 has the form

$$g_1 = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

It suffices to apply the condtion ASR_3 to g_{11} in $G(C_2, R)$.

<u>The Case $B_3 \to F_4$</u>. We have $g \in (G(B_3,R),\omega_3)$, The base representation $\mu(eg)=1$ has the diagram

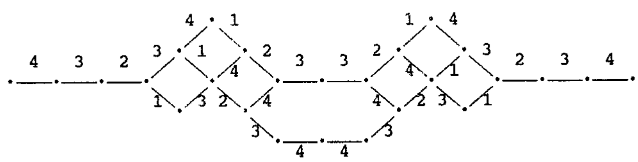


Fig. 11

We enumerate the weights in the following manner:

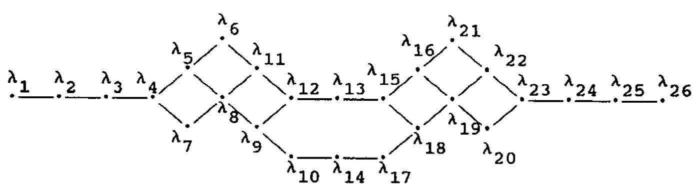


Fig. 12

Let $g \in (G(F_4,R),\omega_4)$. We have $\langle \lambda_i(g) \rangle = R$, $i=1,\dots,26$. Let $\mathfrak{a} = \langle \lambda_i(g) \rangle$, $i=1,\dots,20$. The row $(\lambda_2(g), \dots, \lambda_{26}(g))$ is unimodular modulo the ideal $\mathfrak{a}$, and the group $(G(C_3 \to F_4,R),\omega_1)$ acts on it. Hence, even under the condition SR_6, there exists an $e \in (E(C_3 \to F_4,R),\omega_1)$ such that for $g_1 = e_1 g$, we have $\lambda_{21}(g_1) \equiv 1 \pmod{\mathfrak{a}}$ and $\lambda_i(g_1) \equiv 0 \pmod{\mathfrak{a}}$ for $i = 22, \dots, 26$. Since $\alpha_1 \notin C_3 \to F_4$ on the weights λ_i, with $i = 21, \dots, 26$, we have $\langle \lambda_i(g_1) \rangle = \mathfrak{a}$, $i=1,\dots,20$, so the row $(\lambda_1(g_1),\dots,\lambda_{21}(g_1))$ is unimodular. We let $\mathfrak{a}_1 = \langle \lambda_1(g_1),\dots,\lambda_{16}(g_1),\lambda_{21}(g_1) \rangle$. Therefore, the row $(\lambda_{17}(g_1),\dots,\lambda_{20}(g_1))$ is unimodular modulo $\mathfrak{a}_1$, and the group $(G(C_2 \to F_4,R),\omega_1)$ acts on it. By virtue of the condition SR_4, there exists an $e_2 \in (E(C_3 \to F_4,R),\omega_1)$ such that for $g_2 = e_2 g_1$, we have $\lambda_{17}(g_2) \equiv 1 \pmod{\mathfrak{a}_1}$. As $\alpha_1, \alpha_4 \notin C_2 \to F_4$ on these weights, we have $\langle \lambda_i(g_2) \rangle = \mathfrak{a}_1$, $i = 1, \dots, 16, 21$. Hence, the row $(\lambda_1(g_2),\dots,\lambda_{17}(g_2),\lambda_{21}(g_2))$ is unimodular. We now apply the condition ASR_3 to the inclusion

$(G(A_2 \to F_4, R), \omega_1)$ on the weights λ_{15}, λ_{16}, λ_{21}. Let $\mathfrak{a}_3 = \langle\lambda_i(g_2)\rangle$, $i = 15, 16, 17$. Then there exists an $e_3 \in (E(A_2 \to F_4, R), \omega_1)$ such that any maximal ideal $\mathfrak{m}$ containing $\langle\lambda_{15}(e_3 g_2), \lambda_{16}(e_3 g_2), \rangle$ also contains $\mathfrak{a}_3$. As $\alpha_4 \notin C_2 \to F_4$ on these weights, we again obtain the same ideal generated by the rest of $\lambda_i(e_3 g_2)$ from the unimodular row. We therefore have the unimodular row $(\lambda_1(g_3), \ldots, \lambda_{17}(g_3))$. We set $\mathfrak{a}_4 = \langle\lambda_i(g_3)\rangle$, $i=1,..,6$. The group $(G(C_3 \to F_4, R), \omega_2)$ acts on the row $(\lambda_7(g_3), \ldots, \lambda_{20}(g_3))$. By Lemma 3, there exists an $e_4 \in (E(C_3 \to F_4, R), \omega_2)$ such that $\lambda_7(e_4 g_3) \equiv 1 \pmod{\mathfrak{a}_4}$ and $\lambda_i(e_4 g_3) \equiv 0 \pmod{\mathfrak{a}_4}$ for $i = 8, \ldots, 20$. Let $g_4 = e_4 g_3$. Since $\alpha_2 \notin C_2 \to F_4$ on the weights λ_i with $i = 7, \ldots, 20$, we obtain the unimodular row $(\lambda_1(g_4), \ldots, \lambda_7(g_4))$. We now apply the condition ASR_3 and the inclusion $(G(A_2 \to F_4, R), \omega_1)$ on the weights λ_4, λ_5, λ_6. Since $\lambda_i(g_4) \equiv 0 \pmod{\mathfrak{a}_4}$ for $i = 8, \ldots, 20$, we find that there exists an $e_5 \in (E(A_2 \to F_4, R), \omega_1)$ such that the raw $(\lambda_1(e_5 g_4), \ldots, \lambda_5(e_5 g_4), \lambda_7(e_5 g_4))$ is unimodular. We set $g_5 = e_5 g_4$. The inclusion $A_2 \to F_4$ on the weights λ_1, λ_2, λ_3, together with the condition ASR_3, enables us to find an $e_6 \in (E(A_2 \to F_4, R), \omega_1)$ such that $\langle\lambda_i(e_6 g_5)\rangle = R$, $i = 2, \ldots, 5, 7$. Note that the group $(G(B_3 \to F_4, R), \omega_3)$ acts on the weights λ_i with $i = 2, \ldots, 5, 7, \ldots, 10$. Now, by Lemma 3, there exists an $e_7 \in (E(B_3 \to F_4, R), \omega_2)$ such that $\lambda_2(e_7 e_6 g_5) = 1$.

4.4. Completion of the Proof

We recall the method of the proof of Theorem 4.1 of [35]. Suppose we are given two homomorphisms ν_1, ν_2 of Chevalley groups $G(\Delta, R) \to G(\Phi, R)$ that carry root subgroups of $G(\Phi, R)$ into root subgroups of $G(\Delta, R)$. We also assume that $\Delta \neq A_{n-1}$ for $\Phi = D_n$, $n = 2k$, and $\Delta \neq A_7$ for $\Phi = E_8$. Then there exists an inner automorphism χ of the group $G(\Phi, R)$ such that $\nu_1 = \chi \cdot \nu_2$. As a rule, we take for ν_1 the standard homomorphism associated with an inclusion of root systems $\Delta \to \Phi$. We denote the induced homomorphisms of K_1-functors by the same letters with a tilde. Suppose that ν_2 is equal to the composite map

$$G(\Delta, R) \xrightarrow{\phi} G(\Phi', R) \xrightarrow{\psi} G(\Phi, R)$$

Therefore, if $\tilde{\nu}_2$ is an epimorphism subject to a certain condition on R, then $\tilde{\psi}$ is also an epimorphism. If $\tilde{\nu}_2$ is a monomorphism, then $\tilde{\phi}$ is also a monomorphism.

<u>The Inclusions $A_6 \to E_7$ and $D_6 \to E_7$.</u> Consider the diagrams

$$\begin{array}{ccccc} A_5 & \xrightarrow{\phi} & A_6 & \xrightarrow{\psi} & E_7 \\ & \searrow^{\xi} & E_6 & \nearrow_{\delta} & \end{array} \qquad \begin{array}{ccccc} D_5 & \xrightarrow{\phi_1} & D_6 & \xrightarrow{\psi_1} & E_7 \\ & \searrow^{\xi_1} & E_6 & \nearrow_{\delta_1} & \end{array}$$

Under the conditon ASR_4, the homomorphisms $\tilde{\xi}$, $\tilde{\delta}$ and $\tilde{\xi}_1$, $\tilde{\delta}_1$ are epimorphic, and hence so are $\tilde{\psi}$ and $\tilde{\psi}_1$.

Similarly, for the inclusions A'_7, $A''_7 \to E_8$ and $D_7 \to E_8$, we have the diagrams

$$\begin{array}{ccccc} A_6 & \xrightarrow{\phi} & A'_7 & \xrightarrow{\psi} & E_8 \\ & \searrow^{\xi} & E_7 & \nearrow_{\delta} & \end{array} \qquad \begin{array}{ccccc} A_6 & \xrightarrow{\phi} & A''_7 & \xrightarrow{\psi} & E_8, \\ & \searrow^{\xi} & E_7 & \nearrow_{\delta} & \end{array}$$

$$\begin{array}{ccccc} D_6 & \xrightarrow{\phi} & D_7 & \xrightarrow{\psi} & E_8 \\ & \searrow^{\xi} & E_7 & \nearrow_{\delta} & \end{array}$$

<u>The Inclusion $C_3 \to F_4$.</u> We have the diagram

$$\begin{array}{cccccc} C_2 \cong & B_2 & \xrightarrow{\phi} & C_3 & \xrightarrow{\psi} & F_4 \\ & & \searrow^{\xi} & B_3 & \nearrow_{\delta} & \end{array}$$

The homomorphism $\tilde{\xi}\circ\tilde{\delta}$ is epimorphic under the condition ASR_3, and hence so is $\tilde{\psi}$.

The Inclusion $A_{n-1} \to B_n$. We have

$$\begin{array}{ccccc} A_2 & \xrightarrow{\ \phi\ } & B_3 & \xrightarrow{\ \psi\ } & B_4 \\ & {\scriptstyle\xi}\searrow & A_{n-1} & \nearrow{\scriptstyle\delta} & \end{array}$$

By virtue of Lemma 3, the homomorphism $\tilde{\phi}$ is epimorphic under the condition ASR_3, and hence so is $\tilde{\delta}$.

The Inclusion $A_{n-1} \to C_n$. We have

$$\begin{array}{ccccc} \tilde{A}_1 & \xrightarrow{\ \phi\ } & C_2 & \xrightarrow{\ \psi\ } & C_n \\ & {\scriptstyle\xi}\searrow & A_{n-1} & \nearrow{\scriptstyle\delta} & \end{array}$$

The homomorphism $\tilde{\phi}$ is epimorphic under ASR_2, since it corresponds to the inclusion $A_1 \to B_2$ on the long roots. Thus, $\tilde{\delta}$ is an epimorphism too.

The Inclusion $A_{n-1} \to D_n$. We have

$$\begin{array}{ccccc} A_4 & \xrightarrow{\ \phi\ } & D_5 & \xrightarrow{\ \psi\ } & D_n \\ & {\scriptstyle\xi}\searrow & A_{n-1} & \nearrow{\scriptstyle\delta} & \end{array}$$

By Lemma 2, $\tilde{\phi}$ is an epimorphism under ASR_3. Hence, so is $\tilde{\delta}$. Here we could also use the inclusions $A_3 \to D_4$, $D_3 \to D_4$, and the fact that they are transformed into each other by an inner homomorphism of D_4.

5. Some Irregular Inclusions

We can also state the stability problem for homomorphisms of Chevalley groups associated with nonstandard inclusions of root systems. We provide some information on surjective stability for such inclusions in the following assertion.

PROPOSITION. Under the following assumptions on the ring R, we have a decomposition of the form

$$G(\Phi,R)=E(\Phi,R)G(\Phi_\rho,R)$$

where Φ_ρ is a twisted root system:

$G_2 \to D_4$	for	ASR_3
$B_n \to D_{n+1}$	for	ASR_n
$F_4 \to E_6$	for	ASR_4
$C_n \to A_{2n-1}$	for	SR_3

PROOF. The cases $G_2 \to D_4$ and $B_n \to D_{n+1}$ are well known [35]. A similar proof could be given for the inclusion $F_4 \to E_6$. Consider the diagram

$$\begin{array}{ccccccc} G(D_4,R) & \xrightarrow{\ \phi\ } & G(B_4,R) & \xrightarrow{\ \xi\ } & G(F_4,R) & \xrightarrow{\ \psi\ } & G(E_6,R) \\ & {\scriptstyle\xi}\searrow & & G(D_5,R) & & \nearrow{\scriptstyle\nu} & \end{array}$$

Let $\omega_i=\omega_i(1)=x_i(1)x_{-i}(1)x_i(1)$. We explicitly write out the action of the homomorphism $\phi\circ\xi\circ\psi$ on the unipotent root elements $x_i(t)$. For $\phi{:}G(D_4\to B_4,R)$, we have

$$
\begin{array}{lcl}
x_i(t) & \longrightarrow & x_i(t), \quad i=1,2,3 \\
x_3(t) & \longrightarrow & \omega_4 x_3(t)\omega_4^{-1}
\end{array}
$$

For $\xi\colon G(B_3 \to F_4, R)$, we obtain

$$
\begin{array}{lcl}
x_1(t) & \longrightarrow & \omega_4\omega_3 x_2(t)(\omega_4\omega_3)^{-1} \\
x_2(t) & \longrightarrow & x_1(t) \\
x_3(t) & \longrightarrow & x_2(t) \\
x_4(t) & \longrightarrow & x_3(t)
\end{array}
$$

and, finally, for $\psi\colon G(F_4 \to E_6, R)$, we have

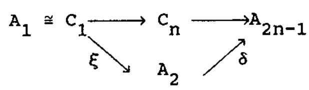
$$
\begin{array}{lcl}
x_1(t) & \longrightarrow & x_2(t) \\
x_2(t) & \longrightarrow & x_4(t) \\
x_3(t) & \longrightarrow & x_3(t)x_5(t) \\
x_4(t) & \longrightarrow & x_1(t)x_6(t)
\end{array}
$$

The composite homomorphism $\psi\circ\xi\circ\phi$ acts according to the rule

$$
\begin{array}{lcl}
x_1(t) & \longrightarrow & \omega_3\omega_5\omega_1\omega_6 x_1(t)(\omega_3\omega_5\omega_1\omega_6)^{-1} \\
x_2(t) & \longrightarrow & x_2(t) \\
x_3(t) & \longrightarrow & x_4(t) \\
x_1(t) & \longrightarrow & \omega_3\omega_5 x_1(t)(\omega_3\omega_5)^{-1}
\end{array}
$$

and it takes root subgroups to root subgroups. Proceeding to the homomorphisms of K_1-functors, we find that the homomorphism $\tilde{\psi}\circ\tilde{\xi}\circ\tilde{\phi}$ is epimorphic under the condition ASR_4. Then $\tilde{\xi}\circ\tilde{\phi}$ and $\tilde{\psi}$ are also epimorphic. In particular, we have surjective stability for $B_4 \to F_4$ under ASR_4.

The proof for the inclusion $C_n \to A_{2n-1}$ follows directly from the diagram



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