CHARACTERIZATION OF SOLVABLE GROUPS AND SOLVABLE RADICAL

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Abstract. We give a survey of new characterizations of finite solvable groups and the solvable radical of an arbitrary finite group which were obtained over the past decade. We also discuss generalizations of these results to some classes of infinite groups and their analogues for Lie algebras. Some open problems are discussed as well.

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1. Introduction

The present survey is motivated by a spectacular progress achieved during the past decade in revisiting some problems concerning finite solvable groups. Not only do we present several new theorems but rather put emphasis on new approaches with origin outside group theory (arithmetic geometry, dynamical systems, algebraic groups, computer algebra) which led to these results.

Let us quote the main results:

Theorem A ([BGGKPP1], [BGGKPP2]). *Let*

$$v_1(x,y) := x^{-2}y^{-1}x, \dots, v_{n+1}(x,y) := [xv_n(x,y)x^{-1}, yv_n(x,y)y^{-1}], \dots$$

A finite group G is solvable if and only if for some n it satisfies the identity $v_n(x,y) \equiv 1$.

Theorem A′ ([BWW]). *Let*

$$s_1(x,y) := x, \dots, s_{n+1}(x,y) := [y^{-1}s_n(x,y)y, s_n(x,y)^{-1}], \dots$$

A finite group G is solvable if and only if for some n it satisfies the identity $s_n(x,y) \equiv 1$.

Fritz Grunewald suddenly passed away on the 21st of March, 2010. Fritz was the one whose mathematical vision and intuition initiated this research. Everybody who had a privilege to work with him and who had been touched by his generous personality will never forget him. Let the memory of our friend be blessed. *B.K., E.P.*

Theorem B ([GGKP4], [GGKP5], [Gu1], [FGG])**.** *Let G be a finite group. Let C be a conjugacy class of G consisting of elements of prime order $p \geq 5$. Then C generates a solvable subgroup if and only if every pair of elements of C generates a solvable subgroup.*

Corollary C ([GGKP4], [GGKP5], [Gu1], [FGG], [LXZ])**.** *A finite group G is solvable if and only if every pair of conjugate elements of G generates a solvable subgroup.*

Various related topics are included in the paper. First, with some effort, one can extend the theorems quoted above to certain classes of infinite groups satisfying natural finiteness conditions. Second, one can try to get characterizations for the solvable radical of a finite group, or of an infinite group satisfying some finiteness conditions, in the spirit of Theorems A and A′. Third, most of group-theoretic statements under consideration admit natural analogues for finite-dimensional Lie algebras.

The paper consists of three parts. In the first one (Sections 2 and 3) we recall interrelations between Engel properties and nilpotency, explain parallels between the nilpotent and solvable cases and outline a method for reducing characterization problems for solvable groups to certain statements for simple groups.

Sections 4 and 5 constitute the core of the paper. Here new results are formulated and proofs are sketched. We discuss main ideas and methods dominating the whole area of research. Among those, one can single out two approaches for which we have chosen the nicknames "Engel-line" and "Thompson-line". They can also be christened as an explicit and implicit way of description, respectively (compare Theorems A and A′ with Corollary C to feel the difference). For the Engel-line, we focus our attention on explicitly written sequences and formulas allowing one to define quasi-Engel elements. The role these elements play for the solvability property is similar to that of Engel elements with respect to the nilpotency property. As to the Thompson-line, the emphasis is put on the so-called radical elements which, generally speaking, possess the property to generate a solvable subgroup together with arbitrary elements of the group. Thus, these elements are characterized not by explicit formulas but by their generation properties. In other words, we want to check the solvability property on subgroups with fixed (desirably, as small as possible) number of elements (desirably, satisfying certain additional properties, say, being conjugate). From such point of view, the Thompson-line can be regarded as a generalization of both Burnside's philosophy (if understood as checking certain properties of a group on the cyclic subgroups) and the Baer–Suzuki approach (consisting in checking properties of a group on its conjugacy classes). More details can be found in the body of the paper.

The last part of the paper (Sections 6 and 7) deals with various ramifications along with some open problems. In Section 6 we consider generalizations of the results previously obtained for finite groups to some classes of infinite groups. We also touch similar questions for finite-dimensional Lie algebras. In Section 7 we concentrate on numerous open problems which seem to us quite important and tempting.

2. From nilpotent groups to solvable groups

In this section we recall some well-known relations between the Engel and nilpotency properties. They are extremely important in various structure problems of group theory and have been extensively studied during several past decades. We regard this topic as a starting point of our interest in parallel questions related to the solvability property, where analogues of the Engel property are far less investigated.

2.1. **Engel properties and nilpotency.**

Let L be a finite-dimensional Lie algebra over a field k with Lie operation $[\ ,\ ]$. Define a sequence $\overrightarrow{e}$ of words in the free Lie algebra $W_2(x,y)$ by $e_1(x,y) = [x,y]$ and, by induction, $e_{n+1}(x,y) = [e_n(x,y),y]$. The starting point of the whole Engel theory is the following theorem [J1, Chap. II. Sec. 3]:

Theorem 2.1 (Engel)**.** *A finite-dimensional Lie algebra L is nilpotent if and only if it satisfies one of the identities $e_n(x,y) := [x,y,y,\dots,y] \equiv 0$.*

In a similar way, Zorn's theorem [Zo], [Hu, Satz III.6.3] characterizes nilpotent groups in the class of finite groups:

Theorem 2.2. *A finite group G is nilpotent if and only if it satisfies one of the identities $e_n(x,y) := [x,y,y,\dots,y] \equiv 1$, where $e_n(x,y)$ belongs to the free group $F_2(x,y)$.*

Here and throughout this paper the group commutator is $[x, y] = xyx^{-1}y^{-1}$, and $[x, y, y] = [[x, y], y]$, etc. So for finite groups the Engel property is equivalent to nilpotency.

The interest to Engel properties, in the context of infinite groups, has been revived in the mid-1950s by R. Baer, K. Gruenberg and B. Plotkin who paid attention to the following Burnside-type problem (see, e.g., [Plo5]):

Problem 2.3. Fix a natural n. Is a group G satisfying the identity $e_n(x, y) \equiv 1$ locally nilpotent? In other words, is every n-Engel group G locally nilpotent?

Although this problem is most likely to have a negative solution (for sufficiently large n), it remains open up to now.

Remark 2.4. Problem 2.3 has a positive solution for $n = 2$ ([Le]), $n = 3$ ([He]), and $n = 4$ ([HVL]). It also has a positive solution for many classes of groups, see [BM], [Gr1]–[Gr3], [KR], [Pla1], [Plo1]–[Plo6], [Wi1], [WZ], etc.

A more general problem consists in characterization of elements constituting the locally nilpotent (Hirsh–Plotkin) radical of an arbitrary group G (see, e.g., [Plo5], [Ro2]), which is another source for studying Engel properties. If G is finite (or, more generally, noetherian), its locally nilpotent radical coincides with the nilpotent radical, i.e., the Fitting subgroup of G, which, in turn, is described by Baer's theorem [Ba1]:

Definition 2.5. An element $g \in G$ is called (left)-Engel if for every $x \in G$ there exists $n = n(x, g)$ such that $e_n(x, g) = 1$.

Theorem 2.6 ([Ba1])**.** *The nilpotent radical of a noetherian group G coincides with the collection of all Engel elements of G.*

It has been proven that besides noetherian groups, the locally nilpotent radical coincides with the collection of all Engel elements for solvable groups [Gr1], [Plo1], radical groups [Plo2], groups with ascending normal series with locally noetherian quotients [Plo3], linear groups [Gr3], PI-groups and locally compact topological groups [Pla1], etc. The interested reader is referred to [Ro1], [Abd] for a comprehensive survey of Engel theory.

One should also note a result announced in [Blu] which provides an example of a non-Engel group (i.e., the group in which not all the elements are Engel) generated by Engel elements. Bludov proved that there exist groups in which a product of Engel elements is not necessarily an Engel element (cf. a question raised in [Plo5]). This means that the Engel elements do not necessarily constitute a subgroup. The example of [Blu] is based on Grigorchuk 2-groups [Gri].

The crucial fact allowing one to obtain fine Engel structure for a very wide class of groups, including finite and noetherian groups, is the following theorem:

Theorem 2.7 ([Plo3])**.** *Let g be an Engel element in an arbitrary group G, and suppose that the subgroup H_1 generated by g is not normal in G. Then there exists a sequence of subgroups $H_1 \subset H_2 \subset H_3 \dots$ such that*

(1) each H_i is nilpotent;

(2) H_i is normal in H_{i+1};

(3) H_{i+1} is generated by H_i and a conjugate of g not in H_i;

(4) the series breaks off at H_n if and only if H_n is normal in G.

Historically, there is a terminological ambiguity which may sometimes lead to confusion. Note that in [Plo1]–[Plo6] Engel elements are called nil-elements. If in Definition 2.5 the number $n = n(y)$ does not depend on $x \in G$, then y is usually called *bounded Engel* (in [Plo1]–[Plo6] such elements are called Engel elements). In what follows we use the terminology from Definition 2.5.

Engel-type results provide a tool for recognition of the nilpotency property in terms of explicit sequences in two variables defined by commutator formulas. Moreover, the Engel sequence is convergent in the profinite completion $\hat{F}$ of the free group $F_2(x, y)$ to the element (pro-identity) which defines the provariety of pronilpotent groups (see, e.g., [Alm], [AMSV], etc.). This characterization of nilpotency by two-variable formulas gives rise to a number of applications (see, e.g., the monographs [Ro2], [Hu], [Plo4], [AmSt]). In particular, this approach was used in the solution of the restricted Burnside problem by Kostrikin [Ko] and Zelmanov [Ze2], [Ze3].

2.2. **Solvability. Statement of the problem.** To adapt the Engel theory to the case where nilpotency is replaced with solvability, one should look for a replacement of Engel elements with similar ones, whose behaviour with respect to the solvability property is the same as the behaviour of Engel elements with respect to nilpotency.

Definition 2.8 ([Plo3], [BBGKP])**.** We say that a sequence $\overrightarrow{u}(x,y) = u_1, u_2, \dots, u_n, \dots$ of elements from $F_2(x,y)$ is **correct** if the following conditions hold:

(i) for every group G and any $a, g \in G$ we have $u_n(a,1) = 1$ and $u_n(1,g) = 1$ for all sufficiently large n;

(ii) if $a, g \in G$ are such that $u_n(a,g) = 1$, then for every $m > n$ we have $u_m(a,g) = 1$.

Thus, if the identity $u_n(x,y) \equiv 1$ is satisfied in G, then for every $m > n$ the identity $u_m(x,y) \equiv 1$ also holds in G.

Definition 2.9. For every correct sequence $\overrightarrow{u}$ in $F_2(x,y)$ define the class of groups $\Theta = \Theta(\overrightarrow{u})$ by the rule: a group G belongs to Θ if and only if there is n such that the identity $u_n(x,y) \equiv 1$ holds in G.

Definition 2.10. For every group G denote by $G(\overrightarrow{u})$ the subset of G defined by the rule: $g \in G(\overrightarrow{u})$ if and only if for every $a \in G$ there exists $n = n(a,g)$ such that $u_n(a,g) = 1$. Elements of $G(\overrightarrow{u})$ are viewed as Engel elements with respect to the given correct sequence $\overrightarrow{u}$. We call these elements $\overrightarrow{u}$**-Engel-like** or, for brevity, $\overrightarrow{u}$**-Engel elements**.

Examples 2.11.

(1) If $\overrightarrow{e} = e_1,\ e_2,\ \dots,$ where
$$e_1(x,y) = [x,y] = xyx^{-1}y^{-1}, \dots, e_n(x,y) = [e_{n-1}(x,y), y], \dots,$$
then $\Theta(\overrightarrow{e})$ is the class of all Engel groups. In the case of finite groups and in many other cases described above the class $\Theta(\overrightarrow{e})$ coincides with the class of nilpotent groups. Clearly, $\overrightarrow{e}$-Engel elements of any group G are usual Engel elements in G. In particular, if G is finite, the set $G(\overrightarrow{e})$ coincides with the nilpotent radical of G.

(2) If $\overrightarrow{u}$ is defined by the following correct sequence of words:
$$u_1 = xy^{-1}, u_2 = u_1(xy, yx) = [x,y], \dots, u_n = u_{n-1}(xy, yx), \dots,$$
then for finite groups the class $\Theta(\overrightarrow{u})$ coincides with the class of all finite nilpotent-by-two groups [BP].

Now, if we want to obtain a reasonable Engel-like theory for the solvability property, a major question is as follows:

Problem 2.12. Is there an explicit correct sequence of words $q_n(x,y)$ in $F_2(x,y)$ such that a finite group belongs to the class $\Theta = \Theta(\overrightarrow{q})$ if and only if it is solvable, and the solvable radical $\mathfrak{R}(G)$ of every finite group G coincides with $G(\overrightarrow{q})$?

In parallel to the Engel theory, the next question is:

Problem 2.13. Suppose Problem 2.12 has a positive answer. What classes of infinite groups possess the same characterization?

Remark 2.14. Note that the solvable (locally solvable) radical of G, that is the unique maximal (locally) solvable normal subgroup of G, may not exist in an arbitrary group G, see [BKN].

In the last two decades of the 20th century, there were obtained several results concerning characterization of solvable groups in terms of two-variable identities based on Engel words (see [Ni1], [Ni2], [Br], [BN], [Gup], [GH], etc.). Namely, in [Ni1], [Ni2] it was proved that if a finite group G satisfies for some n the identity $e_2 \equiv e_n$, where $\{e_i\}$ is the sequence of Engel words, then G is solvable. However, it is easy to find a solvable group satisfying no identity of the form $e_2 \equiv e_n$. For example, take G a finite nilpotent group of class 3 such that the identity $e_2 \equiv 1$ does not hold in G. Since $e_3 \equiv 1$, the group G cannot satisfy any identity of the form $e_2 \equiv e_m$. However, G is solvable.

In [BN] it was proved that the identity $e_3 \equiv e_n$ can hold in certain finite simple groups such as $\mathrm{PSL}(2,4)$, $\mathrm{PSL}(2,8)$, etc. Let us also mention a pioneer result of N. Gupta [Gup]: any finite group satisfying the identity $e_1 \equiv e_n$ is abelian.

The first real progress in solving Problem 2.12 was obtained by Brandl [Br]. He proved that there exists an implicit sequence in two variables $\{\lambda_n(x, y)\}$ (i.e., a countable set of words) such that a finite group G is solvable if and only if the identity $\lambda_n(x, y) \equiv 1$ holds in G for all but finitely many indices n. In the subsequent paper [BrW] a more explicit sequence was constructed. However, for each of these sequences there is no easily described relationship between their consecutive terms. A further progress based on the same streamline of ideas was recently obtained in [Wi5] (see Section 7, Theorem 7.3).

In fact, in [BrW] the question "whether finite solvable groups can be characterized by sequences of words in a small number of variables which are derived from a simple recursive definition" was raised. An explicit sequence with four variables and a sequence with three variables characterizing solvable groups were constructed in the same paper.

A kind of general passage from nilpotency to solvability is provided by the notion of a radical group.

Definition 2.15 ([Plo2])**.** A group G is called radical if it has an ascending normal series with locally nilpotent factors.

An arbitrary group G has the upper radical $\widetilde{HP}(G)$ (that is the unique maximal normal radical subgroup) which appears to be the result of iteration of the locally solvable radical $HP(G)$. The quotient group $G/\widetilde{HP}(G)$ is semisimple with respect to the property of being locally nilpotent, i.e. $HP(G/\widetilde{HP}(G)) = 1$, see [Plo2] for details. If G is finite, noetherian, or linear, $\widetilde{HP}(G)$ coincides with the solvable radical $\mathfrak{R}(G)$ [Sup].

Although Theorems A and A$'$ give characterizations of the solvability property for finite groups in terms of correct two-variable sequences $\overrightarrow{q(x,y)}$ and the corresponding classes $\Theta(\overrightarrow{q})$, Problem 2.12 in full generality is still open (see Section 7, Problem 7.2). We should mention that for the solvability property there is no tool parallel to that of Theorem 2.7, which works in the nilpotent case. So the main efforts are focused on the class of finite groups and the passage to the semisimple group $G/\mathfrak{R}(G)$. In the latter case the whole classification theory of finite simple groups works.

The proof of Theorem A involves surprisingly diverse methods of algebraic geometry, arithmetic geometry, group theory, and computer algebra (note, however, a paper of Bombieri [Bom] which can serve as an inspiring example of such an approach; a more recent illustration of striking efficiency of arithmetic-geometric approach to group-theoretic problems can be found in [BS1], [BS2]). We want to emphasize a special role played by problem-oriented software (particularly, the packages SINGULAR and MAGMA): not only proofs but even the precise statements of our results would hardly have been found without extensive computer experiments.

3. From solvable groups to simple groups

In this short section we describe a method allowing one to move certain problems in the theory of solvable groups to some other problems in the theory of simple groups. Note that this is in contrast to the class of nilpotent groups (and some other classes intermediate between nilpotent and solvable, such as supersolvable groups).

Although the method is fairly standard and has been repeatedly used (see, e.g., [BrW]), we present it, for the sake of completeness and reader's convenience, in two slightly different setups: in the problem of characterization of finite solvable groups by identities and in the problem of characterization of the solvable radical of a finite group. In both cases the original problem is reduced to another one requiring some classification of finite simple groups: a classification of minimal nonsolvable groups (due to J. Thompson) in the first case, and full classification in the second one.

3.1. **Characterization of finite solvable groups by identities.** We describe here the initial steps of the approach taken in both [BGGKPP2] and [BWW]: the first two steps are identical (they are considered in Sections 3.1.1 and 3.1.2, respectively, and the difference in the 3rd step is explained in Section 3.1.3). Correspondingly, the notation $u_n(x, y)$ will be used to designate either the sequence defined in [BGGKPP2]:

$$v_1(x, y) := x^{-2}y^{-1}x, \dots, v_{n+1}(x, y) := [xv_n(x, y)x^{-1}, yv_n(x, y)y^{-1}], \dots \tag{3.1}$$

or the sequence defined in [BWW]:

$$s_1(x, y) := x, \dots, s_{n+1}(x, y) := [ys_n(x, y)y^{-1}, s_n(x, y)^{-1}], \dots \tag{3.2}$$

Note that for both sequences $u_n(x, y) = 1$ implies $u_m(x, y) = 1$ for all $m > n$.

3.1.1. *1st step.* Recall that we want to show that each of sequences (3.1) and (3.2) characterizes finite solvable groups, i.e. a finite group G is solvable if and only if for some n the identity $u_n(x,y) \equiv 1$ holds in G. By construction, the "only if" direction is obvious. To prove the converse one, assume the contrary: there exists a finite group satisfying the identity $u_n(x,y) \equiv 1$ which is not solvable. Let G denote a *minimal counter-example*, i.e. a finite nonsolvable group of smallest order satisfying the identity $u_n(x,y) \equiv 1$.

The first observation is as follows: G is a *simple* group all of whose proper subgroups are solvable.

Indeed, if H is a normal subgroup of G, then both H and G/H are solvable (because any identity holding in G is inherited by all its subgroups and quotients and G is a minimal counter-example), hence G is solvable too (as an extension of solvable groups), contradiction.

Therefore we can make use of J. Thompson's list of minimal nonsolvable simple groups [Th]:

- $G = PSL(2,p)$, $p = 5$ or $p \equiv \pm 2 \pmod 5$, $p \neq 3$,
- $G = PSL(2, 2^p)$,
- $G = PSL(2, 3^p)$, p is an odd prime,
- $G = Sz(2^p)$, p is an odd prime,
- $G = PSL(3,3)$.

We thus have to prove that none of our identities holds in any of groups of Thompson's list.

3.1.2. *2nd step.* We now want to use the most important structure property of sequences (3.1) and (3.2): $u_n = 1$ implies $u_m = 1$ for all $m > n$. Thanks to this property, to prove the needed statement it is enough to solve an equation

$$1 \neq u_n(x,y) = u_{n+k}(x,y)$$

in $G \times G$ where G runs over Thompson's list.

This simple observation allows us to move from identities to equations.

3.1.3. *3rd step.* The approaches of [BGGKPP2] and [BWW] split here. We only explain main ideas postponing details to Section 4.

In [BGGKPP2], a clever choice of the first word $v_1(x,y)$ (suggested by computer), allowed one to prove that the simplest equation $v_1(x,y) = v_2(x,y)$ has a nontrivial solution in $G \times G$ for all G belonging to Thompson's list. A streamline of the proof is as follows. For each G from Thompson's list choose a matrix representation over some finite field $\mathbb{F}_q$. View matrix entries as variables. Regard solutions of the equation $v_1(x,y) = v_2(x,y)$ as $\mathbb{F}_q$-points on the corresponding algebraic $\mathbb{F}_q$-variety V. It remains to prove, for each G, the existence of a nontrivial (i.e. such that $v_n(x,y) \neq 1$) rational point on V.

In the $PSL(2,q)$-case, the above mentioned clever choice of the initial word leads to a dimension jump, and we get a curve. It remains to prove that it is absolutely irreducible, compute its genus, and apply Weil's estimate.

The Suzuki case is much harder. It requires Lefschetz's trace formula for operators on affine varieties (Zink–Pink–Fujiwara, former Deligne's conjecture) and estimates for ℓ-adic Betti numbers of these varieties.

All in all, here we move from group theory to arithmetic geometry.

The approach taken in [BWW] can be reformulated in the language of dynamical systems. Instead of considering a particular equation $s_n(x,y) = s_{n+k}(x,y)$, let us look for nontrivial periodic points of the dynamical system on $G \times G$ arising from the word map $(y,u) \mapsto (y, [yuy^{-1}, u^{-1}])$, or, in other words, for invariant sets disjoint from the "forbidden set" $I_G = G \times \{1\}$. Once the existence of such a set is established, we are done. This dynamical alternative is described in more detail in Section 4 and [BGK].

3.2. **Characterization of the solvable radical.** Suppose we want to characterize the solvable radical $\mathfrak{R}(G)$ of a finite group G with the help of some property $\mathcal{P}$ of $(s+1)$-tuples of elements of G as follows: $\mathfrak{R}(G) = S(G) := \{g \in G :$ for every s-tuple $x_1, \dots, x_s$ of elements of G the property $\mathcal{P}(x_1, \dots, x_s, g)$ holds$\}$. Then, under certain assumptions on $\mathcal{P}$, one can reduce the validity of such a characterization to the following statement:

$$\text{for every almost simple group } G \text{ we have } S(G) = \{1\}. \tag{3.3}$$

As a sample of such an argument, consider the case where the property $\mathcal{P}$ reads off as "the group $\langle x_1, \dots, x_s, g\rangle$ is solvable" (cf. [GGKP3]). Note that essentially the same argument can be applied for various ramifications of the property $\mathcal{P}$ as above (cf. [GPS], [GKPS], [GGKP1]–[GGKP2], [GGKP4]–[GGKP5], [Gu1], [FGG]).

Here is the argument. Assume that (3.3) holds and show that $S(G) = \mathfrak{R}(G)$. As the inclusion $\mathfrak{R}(G) \subseteq S(G)$ is obvious, we only have to establish the opposite inclusion $S(G) \subseteq \mathfrak{R}(G)$.

3.2.1. *1st step.* It is easy to see that the set $S(G/\mathfrak{R}(G))$ is in one-to-one correspondence with the collection of cosets $S(G)/\mathfrak{R}(G) := \{s\mathfrak{R}(G) : s \in S(G)\}$. Thus, factoring out $\mathfrak{R}(G)$, we may assume that G is semisimple (i.e. $\mathfrak{R}(G) = 1$). We have to prove that $S(G) = 1$. Henceforth let G denote a minimal counter-example, i.e. a semisimple group of smallest order with $S(G) \neq 1$. Let $1 \neq g \in S(G)$.

3.2.2. *2nd step.* According to [Ro2, 3.3.16], any finite semisimple group G contains a unique maximal normal centreless completely reducible subgroup $\mathrm{CR}(G)$ (by definition, CR means a direct product of finite non-abelian simple groups) which is called the CR-radical of G. Denote it by V. This is a characteristic subgroup of G. It is known that the centralizer of V in G is trivial [Ro2, proof of 3.3.18(i)]. We call the product of the isomorphic factors in the decomposition of V *an isotypic component* of G. Thus $V = H_1 \times \cdots \times H_t$, where H_i is an isotypic component.

3.2.3. *3rd step.* Let us show that $t = 1$ (i.e. there is only one isotypic component). Assume the contrary, i.e. $V = N_1 \times N_2$, where $N_1 \cap N_2 = 1$. Consider $\bar{G} = G/N_1$ and denote $\bar{\mathfrak{R}} = \mathfrak{R}(G/N_1)$. Denote by $\bar{g}$ and $\bar{\bar{g}}$ the images of g in $\bar{G}$ and $\bar{G}/\bar{\mathfrak{R}}$, respectively. Since $\bar{G}/\bar{\mathfrak{R}}$ is semisimple and $\bar{\bar{g}} \in \mathfrak{R}(\bar{G}/\bar{\mathfrak{R}})$, we have $\bar{\bar{g}} = 1$ (because G is a minimal counter-example), and hence $\bar{g} \in \mathfrak{R}(G/N_1)$. Consider $V/N_1 \simeq N_2$. Then $V/N_1 \subset G/N_1$ is semisimple, and therefore $V/N_1 \cap \mathfrak{R}(G/N_1) = 1$. Since $\bar{g} \in \mathfrak{R}(G/N_1)$, we have $[\bar{g}, \bar{v}] = \bar{1}$ for every $\bar{v} \in V/N_1$. Hence $[g, v] \in N_1$ for every $v \in V$. Similarly, $[g, v] \in N_2$ for every $v \in V$. Therefore $[g, v] = 1$. Hence g centralizes every $v \in V$. Since the centralizer of V in G is trivial, we get $g = 1$. Contradiction.

So we may assume that g acts as an automorphism $\tilde{g}$ on $V = H_1 \times \cdots \times H_n$, where all H_i, $1 \leq i \leq n$, are isomorphic nonabelian simple groups.

3.2.4. *4th step.* Let us show that $\tilde{g}$ cannot act on V as a nonidentity element of the symmetric group S_n. Denote by σ the element of S_n corresponding to $\tilde{g}$.

By definition, the subgroup $\Gamma = \langle g, x_i g x_i^{-1} \rangle$, $i = 1, \ldots, s$, is solvable for any choice of $x_i \in G$. Evidently, the subgroup $\langle [g, x_1], [g, x_2] \rangle$ lies in Γ.

Suppose $\sigma \neq 1$, and so $\sigma(k) \neq k$ for some $k \leq n$. Take $\bar{x}_1$ and $\bar{x}_2$ of the form $\bar{x}_i = (1, \ldots, x_i^{(k)}, \ldots, 1)$, where $x_i^{(k)} \neq 1$ lies in H_k $(i = 1, 2)$. Then we may assume $(\bar{x}_i)^\sigma = (x_i^{(k)}, 1, \ldots, 1)$, and so $[g, \bar{x}_i] = (\bar{x}_i)^\sigma \bar{x}_i^{-1} = (x_i^{(k)}, 1, \ldots, (x_i^{(k)})^{-1}, \ldots, 1)$.

By a theorem of Steinberg, see [St1], H_k is generated by two elements, say a and b. On setting $x_1^{(k)} = a$, $x_2^{(k)} = b$, we conclude that the group generated by $[g, \bar{x}_1]$ and $[g, \bar{x}_2]$ cannot be solvable because the first components of these elements, a and b, generate the simple group H_k. Contradiction with solvability of Γ.

So we can assume that g acts as an automorphism of a simple group H.

3.2.5. *5th step.* Consider the extension of the group H with the automorphism $\tilde{g}$. Denote this almost simple group by G_1. As $g \in S(G)$, we have $g \in S(G_1)$, and hence $g = 1$. Contradiction.

So we achieved our goal by reducing the original problem to some statement on almost simple groups. Several ways for proving such statements will be discussed in Section 5. Here we present some crucial facts from the theory of finite simple groups (some of them recent enough) which allowed us to put the things up to the end.

3.2.6. *Some facts from the theory of finite simple groups.* First of all, we want to emphasize that we freely make use of the classification of finite simple groups (CFSG) which states that apart from the alternating groups and groups of Lie type there are only 26 sporadic groups (listed, say, in [CCNPW]). We also rely on the latest computer version of [ATLAS] and apply it to our computations.

The first basic fact we repeatedly use is the following 2-generation theorem which was first noticed by R. Steinberg for the groups of Lie type and then proved as a result of long-lasting efforts of many mathematicians.

Theorem 3.1. *Every finite simple group can be generated by two elements.*

We also need a stronger result, the so-called "one-and-a-half generation" theorem, which was proved for all almost simple groups in [GK] by probabilistic methods (cf. [Sh]) and in [Ste] for all simple groups using only their structural properties.

Theorem 3.2. *Let G be a finite almost simple group with socle L, and let $1 \neq g \in G$. Then there exists $x \in G$ such that $\langle x, g\rangle$ contains L. In particular, if G is simple then there exists $x \in G$ such that $G = \langle x, g\rangle$.*

Yet another version of generation theorems, proved in [GS], is of great importance for what follows.

Theorem 3.3. *For every element $g \neq 1$ of an almost simple group G, $L \leq G \leq Aut(L)$ (L is a simple group), denote by $\alpha(g)$ the minimal number of L-conjugates of g which generate the group $\langle L, g\rangle$. Then $\alpha(g)$ can be estimated from above by*

- *n, if L is a classical group such that the dimension n of its natural representation is at least 5, unless $L = Sp_n(q)$ with q even, g is a transvection and $\alpha(g) = n + 1$;*
- *$\ell+3$, if L is an exceptional group of untwisted Lie rank ℓ except possibly for the case $L = F_4(q)$ with g an involution where $\alpha(g) \leq 8$;*
- *$n - 1$ if $L = A_n$.*

For small groups the estimates are as follows:

- *if $L = PSL(2, q)$, $q \leq 4$, $g \in G$ is of prime order r, then $\alpha(g) \leq 3$ unless that either*
  (a) g is a field automorphism of order 2 and $\alpha(g) \leq 4$ except for $\alpha(g) = 5$ for $q = 9$, or
  (b) $q = 5$, g is a diagonal automorphism of order 2 and $\alpha(g) = 4$;
  moreover, if r is odd, then $\alpha(g) = 2$ unless $q = 9$, $r = 3$ and $\alpha(g) = 3$;
- *if $L = PSL(3, q)$ and g is of prime order, then $\alpha(g) \leq 3$ unless g is an involutory graph-field automorphism and $\alpha(g) \leq 4$;*
- *if $L = PSU(3, q)$, $q > 2$ and g is of prime order, then $\alpha(g) \leq 3$ unless g is an inner involution and $\alpha(g) = 4$;*
- *if $L = PSU(4, q)$ and g is of prime order, then $\alpha(g) \leq 4$ unless that either*
  (a) g is an involutory graph automorphism and $\alpha(g) \leq 6$, or
  (b) $q = 2$ and g is a transvection with $\alpha(g) \leq 5$.

Finally, we shall need the following theorem of Gow.

Theorem 3.4 ([Gow])**.** *Let G be a finite simple group of Lie type. Let $C \subset G$ be a conjugacy class consisting of regular semisimple elements. Then for every semisimple element $1 \neq g \in G$ there exist $x \in C$ and $z \in G$ such that $g = [x, z]$.*

4. Engel-line

Our goal in this section is to sketch proofs of the characterizations of finite solvable groups obtained in [BGGKPP1]–[BGGKPP2] and [BWW] (see Theorems A and A′ in the Introduction). The first part of the proof (common for the two characterizations) has been described in Section 3.1. Recall that we are now reduced to proving the existence of nontrivial solutions for certain equations in the minimal nonsolvable simple groups from J. Thompson's list. We maintain the notation of Section 3.1. In particular, $v_n(x, y)$ and $s_n(x, y)$ stand for the sequences introduced in Theorems A and A′, respectively. Let G be one of the groups $PSL(2, q)\,(q > 3)$, $Sz(2^n)\,(n \geq 3$ odd), $PSL(3, 3)$. The needed characterizations (Theorems A, A′) are consequences of the following theorems.

Theorem 4.1 ([BGGKPP1], [BGGKPP2])**.** *There exist $x, y \in G$ such that*

$$v_1(x, y) = v_2(x, y) \neq 1. \tag{4.1}$$

Theorem 4.2 ([BWW])**.** *There exist $x, y \in G$ and positive integers n, k such that*

$$s_n(x, y) = s_{n+k}(x, y) \neq 1. \tag{4.2}$$

Below we outline the proofs mainly following the original papers, combining arguments thereof with some ideas from [BGK] which allow one to make exposition more unified and consistent.

4.1. **Proof of Theorem 4.1.** For small groups from the list it is an easy computer exercise to verify Theorem 4.1. There are for example altogether 44928 suitable pairs x, y in the group $\mathrm{PSL}(3,3)$. So henceforth we assume that G is either $PSL(2,q)$ or $Sz(q)$.

Recall the general idea of our proof (see Section 3.1). For a group G from the list, using a matrix representation over $\mathbb{F}_q$ we interpret solutions of the equation $v_1(x,y) = v_2(x,y)$ as $\mathbb{F}_q$-rational points of an algebraic variety. Lang–Weil type estimates for the number of rational points on a variety defined over a finite field guarantee in appropriate circumstances the existence of such points for big q. Of course we are faced here with the extra difficulty of having to ensure that $v_1(x,y) \neq 1$ holds. This is achieved by taking the x, y from appropriate Zariski-open subsets only. See Sections 4.1.1, 4.1.2 for more details.

4.1.1. *The case $G = PSL(2,q)$.* We shall explain here a more general setup which will also shed some light on the somewhat peculiar choice of the word v_1 in Theorem A.

Let w be a word in x, x^{-1}, y, y^{-1}. Let G be a group and x, $y \in G$. Define

$$v_1^w(x,y) := w, \qquad \text{and inductively} \qquad v_{n+1}^w(x,y) := [\,x\,v_n^w(x,y)\,x^{-1},\ y\,v_n^w(x,y)\,y^{-1}\,]. \tag{4.3}$$

Let $R := \mathbb{Z}[t,\, a,\, b,\, c,\, d]$ be the polynomial ring over $\mathbb{Z}$ in five variables. Consider further the two following 2×2-matrices over R.

$$x = x(t) = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}, \quad y = y(a,b,c,d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\mathfrak{a}$ be the ideal of R generated by $\det(y) - 1$ and by the 4 polynomials arising from the matrix equation $v_1^w(x,y) = v_2^w(x,y)$, and let $\mathcal{V}^w \subset \mathbb{A}^5$ be the corresponding closed set of 5-dimensional affine space. Let further $\mathfrak{a}_0$ be the ideal of R generated by $\det(y) - 1$ and by the matrix entries arising from the equation $v_1^w(x,y) = 1$, and let $\mathcal{V}_0^w \subset \mathbb{A}^5$ be the corresponding closed set. Our approach aims at showing that $\mathcal{V}^w \setminus \mathcal{V}_0^w$ has points over finite fields. We have therefore searched for words w satisfying $\dim(\mathcal{V}^w) - \dim(\mathcal{V}_0^w) \geq 1$. We have only found the following words with this property:

$$x^{-2}y^{-1}x,\ y^{-1}xy,\ yx^{-1}y^{-1},\ yxy^{-1},\ x^{-1}yxy^{-1}x,\ x^{-1}yx^{-1}y^{-1}x. \tag{4.4}$$

The extra freedom one might get by introducing variables for the entries of x does not lead to more suitable results. Indeed, $GL(2)$ acts (by conjugation) on the corresponding varieties, and every matrix of determinant 1 except ± 1 is conjugate (over any field) to a matrix with entries like x.

For the last 5 of the words in (4.4) the corresponding closed sets $\mathcal{V}^w$ do not have absolutely irreducible components which are not contained in $\mathcal{V}_0^w$, and in fact the analogue of Theorem 4.1 is not true for them. For the first word $w = x^{-2}y^{-1}x$ the closed set $\mathcal{V}^w$ has two irreducible components. One of them is $\mathcal{V}_0^w$, the second (denoted by $\mathcal{S}$) has dimension 2 and is absolutely irreducible.

Let $\varphi\colon \mathcal{S} \to \mathbb{A}^1 \setminus \{0\}$, $\varphi(x,y) = a$ denote the projection onto the first entry of y, and put $\mathcal{C} := \varphi^{-1}(1)$. This means that we consider y of the special form $y = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix}$. According to Weil's bound, the number of $\mathbb{F}_q$-rational points of the curve $\mathcal{C}$ is at least $q + 1 - 2p_a\sqrt{q} - d$ where d is the degree and p_a the arithmetic genus of $\overline{\mathcal{C}}$, the projective closure of $\mathcal{C}$. Computations give $d = 10$ and $p_a = 12$. This implies that for $q > 593$ there exist enough $\mathbb{F}_q$-rational points on $\mathcal{C}$ to prove Theorem 4.1 in the case of the groups $PSL(2,q)$.

4.1.2. *The case of the Suzuki groups.* To prove Theorem 4.1, the Suzuki groups $G = \mathrm{Sz}(q)$ ($q = 2^n$, n odd) provide the most difficult case. In contrast to the $PSL(2)$-case, where each group $PSL(2,p)$ can be viewed as the fibre at the place p of the $\mathbb{Z}$-scheme $PSL(2,\mathbb{Z})$, such a realization does not exist for $G = \mathrm{Sz}(q)$. Note that although $\mathrm{Sz}(q)$ is contained in $GL(4,q)$, it is not an algebraic subgroup. In fact the group $\mathrm{Sz}(q)$ is defined with the help of a field automorphism of $\mathbb{F}_q$ (the square root of the Frobenius), and hence the standard matrix representation for $\mathrm{Sz}(q)$ contains entries depending on q. We shall describe now how our problem can still be treated by methods of algebraic geometry.

Let $R := \mathbb{F}_2[a,\, b,\, c,\, d,\, a_0,\, b_0,\, c_0,\, d_0]$ be the polynomial ring over $\mathbb{F}_2$ in eight variables. Let $\pi\colon R \to R$ be its endomorphism defined by $\pi(a) = a_0$, $\pi(a_0) := a^2, \ldots, \pi(d) := d_0$, $\pi(d_0) := d^2$. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_2$ and consider $a, \ldots, d_0$ as the coordinates of eight-dimensional affine space $\mathbb{A}^8$ over $\mathbb{F}$. The endomorphism π defines an algebraic bijection $\alpha\colon \mathbb{A}^8 \to \mathbb{A}^8$. The square of α is the Frobenius automorphism on $\mathbb{A}^8$. Let $p \in \mathbb{A}^8$ be a fixed point of α^n, then its coordinates are in $\mathbb{F}_{2^n}$ if n is odd and in $\mathbb{F}_{2^{n/2}}$ if n is even.

Consider further the two following matrices in $GL(4, R)$:

$$x = \begin{pmatrix} a^2a_0 + ab + b_0 & b & a & 1 \\ aa_0 + b & a_0 & 1 & 0 \\ a & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} c^2c_0 + cd + d_0 & d & c & 1 \\ cc_0 + d & c_0 & 1 & 0 \\ c & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices x, y also define maps from $\mathbb{A}^8$ to $GL(4, \mathbb{F})$. It can easily be checked that the matrices corresponding to a fixed point of α^n (n odd and $n \geq 3$) lie in $\mathrm{Sz}(2^n)$.

Let $\mathfrak{a}$ be the ideal of R generated by the 16 polynomials arising from the matrix equation $v_1(x, y) = v_2(x, y)$, and let $\mathcal{V} \subset \mathbb{A}^8$ be the corresponding closed set. By a computer computation we find

Proposition 4.3. *We have*

(i) $\dim(\mathcal{V}) = 2$;
(ii) $\pi(\mathfrak{a}) = \mathfrak{a}$.

Using Proposition 4.3, we see that α defines an algebraic map $\alpha\colon \mathcal{V} \to \mathcal{V}$. Our task now becomes to show that α^n (n odd and $n \geq 3$) has a nonzero fixed point on the surface $\mathcal{V}$. Our basic tool is the Lefschetz trace formula resulting from Deligne's conjecture proved by Fujiwara [Fu]. To apply the Lefschetz trace formula we need to study the geometric structure of $\mathcal{V}$, find its irreducible components and their singular loci, etc. All this was done using computer algebra packages (SINGULAR and MAGMA). In fact, we have

Proposition 4.4. *Let $\mathfrak{a}'$ be the ideal quotient of $\mathfrak{a}$ by $a^3c_0^2$, and let $\mathcal{V}' \subset \mathbb{A}^8$ be the corresponding closed set. Then $\mathcal{V}'$ is a unique 2-dimensional irreducible component of $\mathcal{V}$. We have $\alpha(\mathcal{V}') = \mathcal{V}'$, and $\mathcal{V}'$ is absolutely irreducible.*

Let now $\mathcal{U} \subset \mathcal{V}'$ be the complement in $\mathcal{V}'$ of the closed set given by the equation $cc_0 = 0$. We have

Proposition 4.5. *The two-dimensional affine variety $\mathcal{U}$ is smooth, α-invariant, and absolutely irreducible, and we have $b^1(\mathcal{U}) \leq 675$ and $b^2(\mathcal{U}) \leq 2^{22}$.*

Here $b^i(\mathcal{U}) = \dim H^i_{\text{ét}}(\mathcal{U}, \overline{\mathbb{Q}}_\ell)$ are the ℓ-adic Betti numbers ($\ell \neq 2$). The estimates contained in Proposition 4.5 are derived from results of Adolphson–Sperber [AdSp] and Ghorpade–Lachaud [GL] permitting to bound the Betti numbers of an affine variety in terms of the number of variables, the number of defining polynomials and their degrees. Note that since $\mathcal{U}$ is affine, we have $b^3(\mathcal{U}) = b^4(\mathcal{U}) = 0$. Since $\mathcal{U}$ is nonsingular, the ordinary and compact Betti numbers of $\mathcal{U}$ are related by the Poincaré duality, and we have $b^i_c(\mathcal{U}) = b^{4-i}(\mathcal{U})$.

Let $\mathrm{Fix}(\mathcal{U}, n)$ be the number of fixed points of α^n acting on $\mathcal{U}$. From the Lefschetz trace formula applied to $\mathcal{U}$ and from Deligne's estimates for the eigenvalues of the endomorphism induced by α on étale cohomology we get

$$|\mathrm{Fix}(\mathcal{U}, n) - 2^n| \leq b^1(\mathcal{U})\, 2^{3n/4} + b^2(\mathcal{U})\, 2^{n/2}.$$

An easy estimate shows that $\mathrm{Fix}(\mathcal{U}, n) \neq 0$ for $n > 48$. The cases $n < 48$ are checked with the help of MAGMA.

Remark 4.6. More sequences for which Theorem 4.1 holds were produced in [Ri]. We conjecture after long computer experiments that Theorem 4.1 holds for any sequence formed like in (4.3) from any initial word not of the form $u_1 = (x^{-1}y)^k$ ($k \in \mathbb{N}$).

4.2. **Proof of Theorem 4.2.** As in the previous section, the case $G = PSL(3, 3)$ is settled by a direct computation. The cases $G = PSL(2, q)$ and $G = Sz(q)$ are treated separately. We shall explain the case $G = PSL(2, p)$ ($p > 3$ prime) in some detail referring the interested reader to the original papers for complete proofs in the remaining cases.

The main idea consists in considering a dynamical system on $G \times G$ arising after iterating the self-map $\varphi\colon G \times G \to G \times G$ defined by $\varphi(y, u) = (y, [y^{-1}uy, u^{-1}])$. More precisely, taking $s_0(x, y) = x$, $s_{n+1}(x, y) = \mathrm{pr}_2(\varphi(y, s_n(x, y)))$, we arrive at the sequence of words in $F_2(x, y)$ appearing in the statement of Theorem A$'$. Any periodic point of this dynamical system which lies outside the "forbidden" set $G \times \{1\}$ gives a needed nontrivial solution of equation (4.2). To prove the existence of such a periodic point, one can make use of the trace method going back to classical works of Vogt, Fricke, Klein, Magnus (see, e.g., [Ho], [CMS] for modern exposition). Here are the main steps of the proof, the idea of which is implicitly contained in [BWW] and is presented in full detail and in much more general context in [BGK].

4.2.1. *1st step.* It is convenient to replace $G = PSL(2,p)$ with its simply connected cover $\mathcal{G} = SL(2,p)$. Again, we shall consider the corresponding dynamical system on $\mathcal{G}\times\mathcal{G}$ looking for orbits outside the forbidden set $\mathcal{G} \times \{1\}$.

4.2.2. *2nd step.* Recall the following classical fact (cited from [Ho]):

Theorem 4.7. *Let $F = \langle a_1, \dots, a_n\rangle$ denote the free group on n generators. Let us embed F into $SL(2,\mathbb{Z})$ and denote by tr the trace character. If u is an arbitrary element of F, then the character of u can be expressed as a polynomial*

$$tr(u) = P(t_1, \dots, t_n, t_{12}, \dots, t_{12\dots n})$$

with integer coefficients in the $2^n - 1$ characters $t_{i_1 i_2 \dots i_\nu} = tr(a_{i_1} a_{i_2} \dots a_{i_\nu})$, $1 \le \nu \le n$, $1 \le i_1 < i_2 < \dots < i_\nu \le n$. □

Note that the theorem remains true for the group $\mathcal{G} = SL(2,p)$ (and, more generally, for $SL(2,R)$ where R is any commutative ring, see [CMS]).

4.2.3. *3rd step.* We include the dynamical system as above into the following commutative diagram:

$$\begin{array}{ccc} \mathcal{G}\times\mathcal{G} & \xrightarrow{\tilde{\varphi}} & \mathcal{G}\times\mathcal{G} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathbb{A}^3_{s,u,t} & \xrightarrow{\psi} & \mathbb{A}^3_{s,u,t}, \end{array} \tag{4.5}$$

where the trace map ψ, whose existence follows from Theorem 4.7, is given explicitly by $\psi(s,u,t) := (f_1(s,u,t), f_2(s,u,t), t)$ with $f_1(s,u,t) = tr(\varphi(x,y))$, $f_2(s,u,t) = tr(\varphi(x,y)y)$.

4.2.4. *4th step.* We prove that the projection π is a surjective map (note that in [BGK] such a surjectivity theorem is proven under very general assumptions).

4.2.5. *5th step.* We show that ψ has a fixed point lying outside the projection of the above mentioned forbidden set. This is proved by viewing diagram (4.5) as the special fibre at the place p of the corresponding commutative diagram of morphisms of $\mathbb{Z}$-schemes (denoted by the same letters):

$$\begin{array}{ccc} \mathcal{G}\times\mathcal{G} & \xrightarrow{\tilde{\varphi}} & \mathcal{G}\times\mathcal{G} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathbb{A}^3_{\mathbb{Z}} & \xrightarrow{\psi} & \mathbb{A}^3_{\mathbb{Z}}, \end{array} \tag{4.6}$$

where $\mathcal{G} = SL(2,\mathbb{Z})$. We then consider the scheme W of fixed points of ψ. It turns out to be a curve consisting of three irreducible components: a straight line, corresponding to the forbidden set, and two absolutely irreducible curves, each of those is a curve of genus one with three punctures. We then use the Weil estimates for either of these two components to show that W has rational points for p big enough (small p's are treated separately by straightforward computations). Again, this argument is presented in [BGK] in greater generality.

4.2.6. *6th step.* We put everything together. Let a be a fixed point under ψ which lies outside the image of the forbidden set (it exists by Step 5). Then the fibre at a is fixed under φ (by commutativity of diagram (4.5)) and nonempty (by the surjectivity theorem of Step 4). As the fibre is finite, it contains a needed φ-periodic point outside the forbidden set, and we are done.

5. Burnside–Thompson line

Since this Section is related to the Burnside philosophy, we start with a short historical note (see [Ko], [VL], [O3], [Ze1], [Ze2], [Ze3], [KS] and references therein). The General Burnside problem asks: is a torsion group locally finite? In 1964 Golod constructed infinite finitely generated residually finite torsion groups, thus giving a negative answer to the general Burnside problem. Another Burnside problem deals with the identity $x^n \equiv 1$; namely, is every group of finite exponent locally finite? A negative solution for the Burnside problem was obtained by P. Novikov and Adian [NA], and later by A. Ol'shanskiĭ [O2] and Rips (unpublished). A recent new approach by T. Delzant and M. Gromov [DG] is described in [Co].

Denote by $B(r,n)$ the free group with r generators in the Burnside variety $x^n \equiv 1$. The restricted Burnside problem asks whether the group $B(r,n)$ has a unique maximal finite quotient. The final (positive) solution of the restricted Burnside problem was obtained by Zelmanov [Ze2], [Ze3]. It is mostly based on studying infinite-dimensional Lie algebras and Engel or Engel-like identities (Kostrikin, Kostrikin–Zelmanov). These three Burnside problems have various applications and give rise to numerous questions of Burnside type.

5.1. **Burnside-type problems related to Thompson-type properties: two-generated subgroups.** Widely speaking, one can say that classical Burnside problems ask to what extent finiteness of cyclic (i.e. one-generator) subgroups determines finiteness of arbitrary finitely generated subgroups of a group. In the Burnside–Thompson approach, investigation of global properties of groups relies on the investigation of their two-generated subgroups. This streamline is especially relevant with respect to nilpotency or solvability of groups. For example, Zorn's theorem [Zo] implies that a finite group G is nilpotent if and only if every two-generated subgroup of G is nilpotent. Baer proved [Ba2] that a finite group G is supersolvable if and only if every two-generated subgroup of G is supersolvable.

A similar result for two-generated subgroups with respect to the solvability property is provided by a remarkable theorem of J. Thompson [Th], [Fl1]:

Theorem 5.1. *A finite group G is solvable if and only if every two-generated subgroup of G is solvable.*

The proof of this theorem does not use the full strength of classification of finite simple groups (though the original proof in [Th] heavily relies on their properties). The next step in the same direction is to study the situation with the solvable radical of a finite group and to obtain for it some counter-parts of theorems on the nilpotent radical of a finite group.

Baer's theorem provides an explicit characterization of the nilpotent radical of a finite group in terms of Engel elements. It also implies an implicit characterization of the nilpotent radical in terms of two-generated subgroups. We formulate this fact as a small proposition. For any $x, y \in G$ denote by $\langle y^{\langle x\rangle}\rangle$ the minimal normal subgroup in $\langle x, y\rangle$ containing y.

Proposition 5.2. *Let G be a finite group. The nilpotent radical $\mathfrak{N}(G)$ of G coincides with the collection of all $g \in G$ such that for any $x \in G$ the subgroup $\langle g^{\langle x\rangle}\rangle$ is nilpotent.*

Proof. Let $g \in \mathfrak{N}(G)$. Take an arbitrary $x \in G$ and consider $H = \mathfrak{N}(G) \cap \langle x, g\rangle$. We have $H \le \mathfrak{N}(G)$, so H is nilpotent. On the other hand, H is a normal subgroup in $\langle x, g\rangle$ and $g \in H$. Since $\langle g^{\langle x\rangle}\rangle$ is the minimal normal subgroup containing g, we have $\langle g^{\langle x\rangle}\rangle \le H$. Since H is nilpotent, $\langle g^{\langle x\rangle}\rangle$ is nilpotent too.

Conversely, suppose that g has the property that the subgroup $\langle g^{\langle x\rangle}\rangle$ is nilpotent for any $x \in G$. Evidently, for any $x \in G$ the commutator $[x, g]$ belongs to $\langle g^{\langle x\rangle}\rangle$. Since $\langle g^{\langle x\rangle}\rangle$ is nilpotent, the Engel series $[[x, g], g, \dots, g]$ terminates at 1. Thus g is an Engel element and therefore, according to Baer's theorem, belongs to $\mathfrak{N}(G)$. □

Definition 5.3. Let G be a group. We say that $g \in G$ is a radical element if for any $x \in G$ the subgroup generated by x and g is solvable.

For the solvable radical of a finite group the following extension of Thompson's theorem holds:

Theorem 5.4 ([GKPS]). *Let G be a finite group. The solvable radical $\mathfrak{R}$ of G coincides with the collection of all radical elements in G.*

Remark 5.5. If we go back to the nilpotent case, it turns out that the radical (with respect to nilpotency) elements do not cover the whole nilpotent radical (Fitting subgroup) of G: the collection of elements $g \in G$ such that for any $x \in G$ the subgroup generated by x and g is nilpotent coincides with the hypercentre of G [RM]. (We thank R. Meier for this remark.)

The proof of Theorem 5.4 is rather short since, in contrast to the proof of Theorem 5.1, it invokes the classification of finite simple groups. The whole difficulty is hidden in the "one-and-a-half generation theorem" for almost simple groups, see Theorem 3.2.

Proof. Using arguments from Subsection 3.2, one can reduce the proof of Theorem 5.4 to the following statement:

Lemma 5.6. *Let G be a finite almost simple group. Then G does not contain nontrivial radical elements.*

The latter fact is a direct consequence of the "one-and-a-half generation" theorem [GK]. □

The result of Theorem 5.4 was conjectured by P. Flavell in 1997 [Fl4] (see also [Fl5]). After obtaining in [Fl1] a short proof of Thompson's theorem, he raised a natural question what happens if we keep one of the generators fixed and conjectured that we arrive at the solvable radical.

Theorem 5.4 also trivially implies another conjecture from the same paper: a Sylow p-subgroup P of the finite group G is contained in the solvable radical of G if and only if $\langle P, x\rangle$ is solvable for all $x \in G$. Indeed, this theorem implies the same result for any subgroup of a finite group.

Remark 5.7. Even for the above mentioned weaker conjecture of Flavell there is still no classification-free proof (note, however, a certain step in this direction made in [Wa]).

It is clear that an element $g \in G$ is radical if and only if the group $\langle g^{\langle x\rangle}\rangle$ is solvable for all x. So Theorem 5.4 can be reformulated in the same terms as Proposition 5.2:

Proposition 5.8. *Let G be a finite group. The solvable radical $\mathfrak{R}(G)$ of G coincides with the collection of all $g \in G$ such that for any $x \in G$ the subgroup $\left\langle g^{\langle x\rangle}\right\rangle$ is solvable.*

Here is another consequence of Theorem 5.4:

Corollary 5.9. *Let G be a finite group, let $g \in G$, and let $\left\langle g^G\right\rangle$ denote the minimal normal subgroup of G containing g. Then $\left\langle g^G\right\rangle$ is solvable if and only if the subgroup $\left\langle g^{\langle x\rangle}\right\rangle$ is solvable for all $x \in G$.*

This corollary can be viewed as a hint to possible generalizations of Burnside-type questions with respect to a particular element of a group. Namely, instead of trying to deduce properties of the whole group G assuming certain properties of two-generated subgroups, we now fix an element $g \in G$, assume that its behaviour with respect to any element $x \in G$ is prescribed, and ask if the normal closure of g in G satisfies the same properties. This local-global behaviour is a kind of Burnside-type problem. We shall now put this staff in a setting more suitable for generalizations.

Let $\mathfrak{X}$ be a class of groups. Let G be a group.

Definition 5.10. An element $g \in G$ is called locally $\mathfrak{X}$-radical if $\langle g^{\langle x\rangle}\rangle$ belongs to $\mathfrak{X}$ for every $x \in G$. An element $g \in G$ is called globally $\mathfrak{X}$-radical if $\left\langle g^G\right\rangle$ belongs to $\mathfrak{X}$.

So we have local and global properties. Obviously, if the class $\mathfrak{X}$ is closed under taking subgroups, then a globally $\mathfrak{X}$-radical element is automatically locally $\mathfrak{X}$-radical.

The main problem is to determine classes $\mathfrak{X}$ for which the converse property holds. Proposition 5.2 and Corollary 5.9 state that if $\mathfrak{X}$ is the class of nilpotent/solvable groups and G is finite, then every locally $\mathfrak{X}$-radical element is globally $\mathfrak{X}$-radical.

Remark 5.11. An attempt to generalize further the approach described above on so-called Fitting pseudovarieties of groups was undertaken in [AMSV].

5.2. **Finite groups: general situation.** Let S be a set of finite simple groups. Denote by $\mathfrak{X} = \mathfrak{X}(S)$ the class of finite groups G such that all composition factors of G belong to S. It is easy to see that such an $\mathfrak{X}$ is closed under taking normal subgroups, homomorphic images and extensions. On the other hand, if a class $\mathfrak{X}$ is closed under these three operations and S is the set of all simple groups in $\mathfrak{X}$, then $\mathfrak{X} = \mathfrak{X}(S)$.

It is clear that such an $\mathfrak{X}$ is a radical class. This means that in every finite group G there is a unique maximal normal subgroup $\mathfrak{X}(G)$ belonging to $\mathfrak{X}$. We want to characterize elements constituting $\mathfrak{X}(G)$.

We will use basic properties of the generalized Fitting subgroup $F^*(G)$ of a finite group G, see [As].

Theorem 5.12 ([GKPS])**.** *Let $\mathfrak{X}$ be a class of finite groups closed under homomorphic images, normal subgroups and extensions (equivalently, let $\mathfrak{X}$ be a class of finite groups with composition factors in some set S of simple groups).*

- *If G is a finite group, then every locally $\mathfrak{X}$-radical element belongs to $\mathfrak{X}(G)$.*
- *If, in addition, $\mathfrak{X}$ is closed under taking subgroups, then $\mathfrak{X}(G)$ coincides with the set of all $\mathfrak{X}$-locally radical elements.*

Remark 5.13. Theorem 5.4 is a particular case of Theorem 5.12 if $\mathfrak{X}$ is the class of solvable groups. In this case S consists of all cyclic groups of prime order. We can also consider the classes of p-groups (p a fixed prime), π-groups (π a fixed finite set of primes), and other interesting classes. However, Theorem 5.12 does not cover the class of nilpotent groups where we have to use Baer's theorem for $\mathfrak{X}(G) = \mathfrak{N}(G) = F(G)$, the Fitting subgroup.

Proof. If $\mathfrak{X}$ is closed under taking subgroups, then $\langle g^G\rangle$ in $\mathfrak{X}$ implies that $\langle g^{\langle x\rangle}\rangle$ is in $\mathfrak{X}$, and so we see that the second statement follows from the first. We shall prove the first implication, i.e., we have to prove that if G is a finite group and $g \in G$ with $\langle g^{\langle x\rangle}\rangle$ in $\mathfrak{X}$ for all $x \in G$, then $\langle g^G\rangle$ is in $\mathfrak{X}$.

So assume that G is a minimal counter-example to the first statement. This means that there exists a locally $\mathfrak{X}$-radical element g in G with $\langle g^G\rangle$ not in $\mathfrak{X}$. Consider properties of this group G.

Reduction 1: Take an arbitrary locally $\mathfrak{X}$-radical element g in G such that $\langle g^G\rangle$ is not in $\mathfrak{X}$. Show that $G = \langle g^G\rangle$. If not, set $H = \langle g^G\rangle$ and suppose that $H < G$. Take an arbitrary element $h \in H$. Then $h = \prod g_i$ where all g_i are conjugate to g. Hence all g_i are locally $\mathfrak{X}$-radical elements. Since $H < G$, all elements g_i lie in the radical $\mathfrak{X}(H)$. Then, clearly, h lies in $\mathfrak{X}(H)$. Therefore, $\langle g^G\rangle = H = \mathfrak{X}(H)$ and H is in $\mathfrak{X}$, a contradiction, and we may assume that $G = \langle g^G\rangle$.

Reduction 2. G has a unique minimal normal subgroup. If not, G has two normal subgroups N_1 and N_2 with trivial intersection. Consider $N_1N_2/N_2 \simeq N_1$. Since G/N_2 lies in $\mathfrak{X}$, we have $N_1 \in \mathfrak{X}$. Since $G/N_1 \in \mathfrak{X}$, we have $G \in \mathfrak{X}$, a contradiction. In particular, we may assume that every two normal subgroups in G have a nontrivial intersection.

Reduction 3: $\mathfrak{X}(G) = 1$.

If not, pass to $G/\mathfrak{X}(G)$ and so by induction $g\mathfrak{X}(G)$ is in $\mathfrak{X}(G/\mathfrak{X}(G)) = 1$, whence $g \in \mathfrak{X}(G)$.

Reduction 4: $F(G) = Z(G)$.

Assume the contrary, i.e. $F(G) \neq Z(G)$. Since each Sylow subgroup of $F(G)$ is normal in G, it follows that $F(G)$ is a p-group for some prime p. Since $G = \langle g^G\rangle$, it suffices to show that g commutes with $F(G)$. If not, then taking $x \in F(G)$ with $[g, x] \neq 1$ shows that $1 \neq g^{-1}g^x \in (\langle g^{\langle x\rangle}\rangle \cap F(G))$, and so $\langle g^{\langle x\rangle}\rangle \in \mathfrak{X}$ has a composition factor of order p. Thus $F(G) \in \mathfrak{X}$, and $\mathfrak{X}(G) \geq F(G) \neq 1$, a contradiction.

We now complete the proof. Since G is not abelian and $G/Z(G)$ acts faithfully on $F^*(G)$, there is a nontrivial component Q of G (otherwise, $F^*(G) = F(G) = Z(G)$ and G is abelian).

If $Z(G) \neq 1$, then $Z(G) \cap Q \neq 1$ (otherwise the normal closure of Q would be a minimal normal subgroup).

We may assume that g does not commute with Q (for if g commuted with every component of G, then so would $G = \langle g^G\rangle$, a contradiction). Let $H = \langle Q, g\rangle$. Set N to be the (central) product of the distinct conjugates of Q under $\langle g\rangle$. Then N is clearly perfect and H/N is cyclic (generated by g). Also, $Z(N) = \Phi(N) \leq \Phi(H)$ (where Φ is the Frattini subgroup), and $N/Z(N)$ is a minimal normal subgroup of H/N. So applying [GKPS, Lemma 3.4] to $H/Z(N)$, we see that $H = \langle g, h\rangle$ for some $h \in N$.

We claim that $H = J := \langle g^{\langle h\rangle}\rangle$. Clearly, J is normal in H because it is normalized by $g \in J$ and by h (by definition). Clearly, H/J is abelian (since $g \in J$ and so $[h, g] \in J$). Thus, J contains $[H, H] = N$. Now $H = \langle Q, g\rangle = \langle N, g\rangle$, so once $N < J$, since g is in J, we have $H = J$.

Since $H = \langle g^{\langle h\rangle}\rangle$, all composition factors of H are $\mathfrak{X}$-groups. If $Z(G) \neq 1$, then $Z(G)$ is an $\mathfrak{X}$-group, and if $Z(G) = 1$, then Q is an $\mathfrak{X}$-group, Q lies in $\mathfrak{X}(G)$. In either case, $\mathfrak{X}(G) \neq 1$, a contradiction. □

Note one more result of Flavell [Fl5] of the same flavour:

Theorem 5.14. *Let g be an element of the finite group G. Then $\left\langle g^G\right\rangle$ is solvable of Fitting height at most 2 if and only if the subgroup $\left\langle g^{\langle x\rangle}\right\rangle$ has this property for all $x \in G$.*

5.3. **Baer–Suzuki-type theorem for solvable groups.** The classical Baer–Suzuki theorem [Ba1], [Suz2], [AL] states that

Theorem 5.15 (Baer–Suzuki)**.** *The nilpotent radical of a finite group G coincides with the collection of $g \in G$ satisfying the property: for every $x \in G$ the subgroup generated by g and xgx^{-1} is nilpotent.*

The Baer–Suzuki theorem allows one to improve the characterization of the nilpotent radical given in Proposition 5.2 in the best possible way: instead of considering the subgroup $\langle g^{\langle x \rangle} \rangle$, it is enough to consider the subgroup $\langle g, g^x \rangle$ because its nilpotency for any $x \in G$ already guarantees $g \in \mathfrak{N}(G)$. In particular, this theorem implies

Corollary 5.16. *A finite group G is nilpotent if and only if in each conjugacy class of G every two elements generate a nilpotent subgroup.*

Within last years a lot of efforts have been undertaken in order to describe the solvable radical of a finite group and to establish a sharp analogue of the Baer–Suzuki theorem with respect to the solvability property (see [Fl2], [Fl3], [GGKP1], [GGKP2]). In particular, the following problem raised by Flavell is parallel to the Baer–Suzuki result:

Problem 5.17. Let G be a finite group with solvable radical $\mathfrak{R}(G)$. What is the minimal number k such that $g \in \mathfrak{R}(G)$ if and only if the subgroup generated by $x_1 g x_1^{-1}, \dots, x_k g x_k^{-1}$ is solvable for every $x_1, \dots, x_k$ in G?

Problem 5.17 was solved in the independent papers [GGKP3] and [FGG]:

Theorem 5.18. *The solvable radical of a finite group G coincides with the collection of $g \in G$ satisfying the property: for any three elements $a, b, c \in G$ the subgroup generated by the conjugates $g, aga^{-1}, bgb^{-1}, cgc^{-1}$ is solvable.*

Thus, an element of a finite group belongs to the solvable radical if and only if *any four* its conjugates generate a solvable group. This statement may be viewed as a theorem of Baer–Suzuki type with respect to the solvability property. Theorem 5.18 is sharp and provides the best possible characterization: in the symmetric groups S_n $(n \geq 5)$ any triple of transpositions generates a solvable subgroup.

Remark 5.19. Looking for a solution of Problem 5.17, Flavell proved [Fl1] that an element g belongs to the solvable radical $\mathfrak{R}(G)$ if and only if the subgroup generated by $x_1 g x_1^{-1}, \dots, x_k g x_k^{-1}$ is solvable for every $x_1, \dots, x_k$ in G, and one can choose $k = 10$. His proof does not rely on the classification of finite simple groups. In the paper [FGG] he managed to reduce the value of k to $k = 7$. Finally, in [Fl3, Cor. E] Flavell established the assertion of Theorem 5.18 under the additional assumption that g is a $\{2,3\}'$-element, i.e. he proved that a $\{2,3\}'$ element $g \in G$ belongs to the solvable radical of G if and only if every four conjugates of g generate a solvable group. The proofs of all these results use only classification-free methods.

The original Flavell's estimate $k = 10$ was improved in [GGKP1], [GGKP2] by $k = 8$. The proof went through yet another description of $\mathfrak{R}(G)$ in terms of commutators and heavily relied upon the classification of finite simple groups. Both proofs of the sharp Theorem 5.18 (in [GGKP3] and [FGG]) also rely on the classification of finite simple groups.

5.4. **Outline of the proof of Theorem 5.18.** Our proof follows [GGKP2] and [GGKP3].

Notational conventions. Whenever possible, we maintain the notation of [GGKP2] which mainly follows [St], [Ca1]. In particular, we adopt the notation of [Ca1] for twisted forms of Chevalley groups (so unitary groups are denoted by $PSU_n(q^2)$ and not by $PSU_n(q)$). However, the classification of outer automorphisms follows [GLS, p. 60], [GL, p. 78]. In order to avoid widespread misunderstandings we recall this classification. Let us call the subdivision of automorphisms of Chevalley groups into inner, diagonal, field, and graph automorphisms in the sense of [St], [Ca1], the usual one.

In the classification of finite simple groups a slightly different subdivision of automorphisms is used. Let G be an adjoint Chevalley group, untwisted or twisted (the cases where G is a Suzuki or a Ree group are treated separately). Denote by $\mathrm{Aut}\,(G)$ the group of automorphisms of G. Then ([GLS, Definition 2.5.13]):

1. *Inner-diagonal automorphisms* coincide with usual inner-diagonal automorphisms.

2. *Field automorphisms* are as follows:

2.1. If G is untwisted, then a “field” automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a usual field automorphism.

2.2. If $G = {}^dG$ is a twisted group, then a “field” automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a usual field automorphism of order relatively prime to d.

2.3. If G is a Suzuki or a Ree group, then a “field” automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a usual field automorphism.

3. *Graph automorphisms* are as follows:

3.1. If G is untwisted, then a "graph" automorphism is an $\mathrm{Aut}\,(G)$-conjugate of a graph-inner-diagonal usual automorphism with nontrivial graph part, except for the cases B_2, F_4, G_2 with the characteristics of the ground field $p = 2, 2, 3$, respectively, in which cases there are no "graph" automorphisms.

3.2. If $G = {}^dG$ is a twisted group, then a "graph" automorphism is an element of $\mathrm{Aut}\,(G)$ whose image modulo the group of inner-diagonal automorphisms has order divisible by d.

3.3. If G is a Suzuki or a Ree group, then there are no graph automorphisms.

4. *Graph-field automorphisms* are as follows:

4.1. If G is untwisted, then a "graph-field automorphism" is an $\mathrm{Aut}\,(G)$-conjugate of a usual graph-field automorphism where both components are nontrivial, except for the cases B_2, F_4, G_2 with the characteristics of the ground field $p = 2, 2, 3$, respectively, in which cases all conjugates of usual graph-field automorphisms with nontrivial graph part are considered as "graph-field" automorphisms.

4.2. If $G = {}^dG$ is a twisted group, then there are no graph-field automorphisms.

4.3. If G is a Suzuki or a Ree group, then there are no graph-field automorphisms.

In particular, in this sense a "graph" automorphism may be a composition of an automorphism of the Dynkin diagram with an inner-diagonal automorphism, or (in the case of a twisted form ${}^dL$ of a simple group L) a field automorphism of order divisible by d.

Recall that a finite group G is almost simple if it contains a unique normal simple group L such that $L \le G \le \mathrm{Aut}\,(L)$.

Definition 5.20. Let $k \ge 2$ be an integer. We say that $g \in G$ is a *k-radical element* if for any $x_1, \dots, x_k \in G$ the subgroup $H = \langle x_1 g x_1^{-1}, \dots, x_k g x_k^{-1} \rangle$ is solvable.

The main step in our proof of Theorem 5.18 is

Theorem 5.21. *Let G be a finite almost simple group. Then G does not contain nontrivial 4-radical elements.*

The reduction from Theorem 5.18 to Theorem 5.21 is fairly standard, and follows the scheme described in Subsection 3.2. Namely, let $S(G)$ be the set of all 4-radical elements of the group G. Obviously, $\mathfrak{R}(G)$ lies in $S(G)$, and we have to prove the opposite inclusion. We may assume that G is semisimple (i.e., $\mathfrak{R}(G) = 1$), and we shall prove that G does not contain nontrivial 4-radical elements. Assume the contrary and consider a minimal counter-example, i.e. a semisimple group of smallest order with $S(G) \ne \{1\}$. Then a slightly modified method from Subsection 3.2 implies the result.

Let G be an almost simple group, $L \le G \le \mathrm{Aut}\,(L)$. The proof of Theorem 5.21 splits into two cases.

Case 1. If $G = L$ is simple, then Theorem 5.21 is an immediate consequence of [GGKP2, Theorem 1.15]. Here we formulate this theorem in the following form:

Lemma 5.22. *For any $g \in L$ there exist three elements a, b, c from L such that the commutators $[g, a], [g, b], [g, c]$ generate a nonsolvable subgroup. In particular, the subgroup $\langle g, aga^{-1}, bgb^{-1}, cgc^{-1} \rangle$ is nonsolvable too.*

Proof. We provide the reader with a scheme of the proof referring to [GGKP2] for details. Let G be a simple group. For every $g \in G$ denote by $\rho(g) = n$ the smallest number of elements $x_1, x_2, \dots, x_n$ in G such that the subgroup $\langle [g, x_1], \dots, [g, x_n] \rangle$ is not solvable. Denote by $\rho(G)$ the biggest $\rho(g)$, where g runs over G. In these terms we have the following result:

Theorem 5.23. *If G is a finite nonabelian simple group, then $\rho(G) \le 3$. If G is a group of Lie type over a field K with $\mathrm{char}\, K \ne 2$ and $K \ne \mathbb{F}_3$, or a sporadic group not isomorphic to Fi_{22}, Fi_{23}, then $\rho(G) = 2$.*

The proof of Theorem 5.23 goes through inspection of all simple groups.

1. *Alternating groups.* Let $G = A_n$, $n \ge 5$. Then $\rho(G) = 2$.

Proof. For $n = 5, 6, 7$ the statement can be checked in a straightforward manner, so assume $n \geq 8$. Let us proceed by induction. Let $g \in G$, $g \neq 1$. First suppose that g can be written in the form

$$g = \sigma\tau, \quad \sigma \in A_m, \quad \sigma \neq 1, \quad 5 \leq m < n, \tag{5.1}$$

where σ and τ are disjoint (and thus commute). Then by induction hypothesis there exist $\sigma_1, \sigma_2 \in A_m$ such that the subgroup generated by $[\sigma, \sigma_1]$ and $[\sigma, \sigma_2]$ is not solvable. Take $x_i = \sigma_i\tau$, $i = 1, 2$. Then $[g, x_i] = [\sigma, \sigma_i]$, and we are done.

Suppose g cannot be represented in the form (5.1). Then we have one of the following cases: either n is odd and $g = (12 \dots n)$, or n is even and $g = (12 \dots n-2)(n-1, n)$. In any of these cases we take $x_1 = (123)$ and $x_2 = (345)$ and get $\langle [x_1, g], [x_2, g] \rangle \cong A_5$. □

2. *Groups of Lie type,* $\mathrm{char}(K) \neq 2$ *and* $|K| \neq 3$. Let first G be a group of rank 1.

Let G be one of the groups $A_1(q)$ $(q \neq 2, 3)$, ${}^2A_2(q^2)$ $(q \neq 2)$, ${}^2B_2(2^{2m+1})$ $(m \geq 1)$, ${}^2G_2(3^{2m+1})$ $(m \geq 0)$. Then $\rho(G) = 2$.

The proof is quite technical and involves calculations based on different canonical decompositions in G (see details in [GGKP2, Proposition 4.1]). The uniform part of the proof relies on a theorem of Gow [Gow] regarding conjugacy classes of semisimple elements in Chevalley groups (see Theorem 3.4, compare with [EG1]). Certain steps require, however, explicit matrix representations for the groups of rank 1 (see [HB], [KLM]). As usual, groups over small fields are considered separately.

Suppose now that the rank of G is greater than 1.

Let G be a Chevalley group of rank > 1 over a field K with $\mathrm{char}\, K \neq 2, K \neq \mathbb{F}_3$. *Then $\rho(G) = 2$.*

In this case we use the Levi decomposition of G together with arguments from [GS1] (based on the notions of generalized Bruhat cells and generalized Coxeter elements) in order to reduce to rank 1 case.

3. *Groups of Lie type,* $\mathrm{char}(K) = 2$ *or* $|K| = 3$.

Let G be a nonsolvable Chevalley group over a field K, where either $\mathrm{char}\, K = 2$ *or* $K = \mathbb{F}_3$. *Then $\rho(G) \leq 3$.*

The proof goes by reduction to the case of groups of rank at most 3 and involves more technicalities. In particular, the case $G = {}^2F_4(q)$ requires separate consideration. Some groups of small ranks are treated by MAGMA.

4. *Sporadic groups.*

Let G be a sporadic simple group. Then $\rho(G) = 3$ for $G = Fi_{22}, Fi_{23}$ and $\rho(G) = 2$ for all the remaining groups.

The proof goes case by case. Apart from theoretical arguments, MAGMA calculations for rechecking them (in all the cases except for the M=Monster) are used. In particular, to prove whether a subgroup under consideration is not solvable, the Hall criterion: a group H is nonsolvable if and only if it contains nonidentity elements a, b, c of pairwise coprime orders such that $abc = 1$, is quite convenient.

Both in the theoretical proof and in the computer-aided one, we rely on the ATLAS classification of conjugacy classes of maximal cyclic subgroups [ATLAS]. The proof for M also relies on the classification of conjugacy classes. However, all other arguments are theoretic. We reduce the statement to checking the elements g of prime orders p each of those (except for $p = 41$) can be included in some proper simple subgroup of M [No]. The case $p = 41$ is treated separately using the fact that the normalizer $N_M(\langle g \rangle) = 41 \cdot 40$ is a maximal subgroup of M [ATLAS].

We also use the fact that Fischer's groups Fi_{22}, Fi_{23} are generated by 3-transpositions (see [Fi]) and thus contain a nontrivial 2-radical element. □

Case 2. Let G be almost simple but not simple, i.e., $L < G \leq Aut(L)$, L is simple. We only have to consider outer automorphisms g of L and find three elements a, b, c from L such that the commutators $[g, a], [g, b], [g, c]$ generate a nonsolvable subgroup.

For alternating and sporadic groups, the group of outer automorphisms is rather small. So these groups are treated with straightforward considerations. If L is a group of Lie type, the cases when g is an inner-diagonal, field, graph, or graph-field automorphism are considered separately. The first case is treated in [GGKP2] (see the discussion at the end of Section 4 of this paper for groups of small Lie rank), so we only need to complete the induction arguments. Field, graph, and graph-field automorphisms are treated using their classification. Here we mainly follow the approach of [GS], see Subsection 3.2.6, Theorem 3.3, which either gives the appropriate values of $\alpha(g)$ or provides tools for the careful case-by-case analysis of particular groups and particular types of automorphisms (see [GS], [GGKP3] for details).

5.5. **Baer–Suzuki-type theorem for solvable groups. Elements of prime order bigger than 3.** As we have seen from Theorem 5.18, the sharp analogue of Baer–Suzuki theorem for solvable groups requires four conjugates generating a solvable group. It is clear that elements of small order are troublesome which prevents from decreasing the number of conjugates in Theorem 5.18. For example, if we take an involution $g \in G$, $g^2 = 1$, then any two conjugates of g generate a dihedral group, which is metabelian. Hence two conjugates are not enough to characterize the solvable radical.

Flavell [Fl2] observed that if we take a transvection g in the group $SL_n(3)$, $n \geq 3$, i.e., an element of order 3, then we have a similar phenomenon. Indeed, let C be the conjugacy class containing g. The element g and an arbitrary xgx^{-1} generate a group acting trivially on a subspace of codimension at most 2. Hence $\langle g, xgx^{-1}\rangle$ is solvable since it has a normal abelian subgroup A such that $\langle g, xgx^{-1}\rangle /A$ is isomorphic to a subgroup of the solvable group $GL_2(3)$. However, $SL_n(3)$, $n \geq 3$, is not solvable and is generated by C.

So he mentioned in [Fl2] that one can expect a better analogue of the Baer–Suzuki theorem to hold for the elements in $\mathfrak{R}(G)$ of prime order greater than 3. The following result confirms this expectation:

Theorem 5.24 ([GGKP4], [GGKP5] and [Gu1])**.** *Let G be a finite group. An element g of prime order $\ell > 3$ belongs to $\mathfrak{R}(G)$ if and only if for every $x \in G$ the subgroup $H = \langle g, xgx^{-1}\rangle$ is solvable.*

Theorem 5.24 has an immediate consequence (Theorem B of the Introduction):

Corollary 5.25 ([GGKP4], [GGKP5], [Gu1], [FGG])**.** *Let G be a finite group. Let C be a conjugacy class of G consisting of elements of prime order $p \geq 5$. Then C generates a solvable subgroup if and only if every pair of elements of C generates a solvable subgroup.*

Combining this with Burnside's $p^\alpha q^\beta$-theorem, we obtain the following corollary (Corollary C of Introduction) which is parallel to Corollary 5.16:

Corollary 5.26 ([GGKP4], [GGKP5], [Gu1], [LXZ])**.** *A finite group G is solvable if and only if in each conjugacy class of G every two elements generate a solvable subgroup.*

Remark 5.27. Note a result of the same spirit [Shu] whose proof heavily relies on Corollary 5.26: a finite group is solvable with Fitting height at most h if and only if every pair of conjugate elements generates a solvable subgroup whose Fitting height is at most h.

Remark 5.28. The proof of Theorem 5.24 uses the classification of finite simple groups. The proof of Corollary 5.26 can be obtained without classification using only the above mentioned J. Thompson's characterization of the minimal nonsolvable groups. Indeed, suppose G is minimal counter-example to Corollary 5.26. Then G must be a simple group due to the minimality condition, G must be a minimal simple group in the sense of Thompson for the same reason, and G must contain an element g of prime order ≥ 5 because of Burnside's $p^\alpha q^\beta$-theorem. So if Theorem 5.24 is valid for the minimal simple groups, then we come up with a contradiction. Thus we do not need the classification in order to prove Corollary 5.26. Moreover, as noticed in [LXZ], the statement of Corollary 5.26 can be obtained by a direct inspection of the groups on Thompson's list (so one can dispense with going through a much harder assertion of Corollary 5.25).

Remark 5.29. Another characterization of finite solvable groups was obtained in [KL]: a finite group G is solvable if and only if for every x, y in G, with x a p-element for some odd prime p and y a 2-element, the subgroup generated by x, y is solvable. See [GuL] for results of the similar spirit. Further generalizations were obtained in [GT]: a finite group G is solvable if and only if $x_1x_2x_3 \neq 1$ for all nontrivial p_i-elements x_i of G for distinct primes p_i, $i = 1, 2, 3$ (and, moreover, it is enough to assume $p_1 = 2$ and $p_2 \in \{3, 5\}$). On the other hand, as proved in [DHP], a finite group G is solvable if for all conjugacy classes C_1 and C_2 in G there exist $x \in C_1$ and $y \in C_2$ such that the subgroup generated by x, y is solvable. A generalization of the latter theorem to a class $\mathfrak{X}$ of finite groups closed under subgroups, quotients and extensions is considered in [DGHP]. These results should be compared with [GLL], [GM1], [FMP] where a conjecture of Bauer–Catanese–Grunewald about the existence of so-called unmixed Beauville structures in finite simple groups is established. As a by-product, it was shown that in each finite simple group G there exist two conjugacy classes C_1 and C_2 such that G is generated by any pair of elements from these classes (see also [GM2]).

Remark 5.30. The proofs of Theorem 5.24 given in [GGKP5] and in [Gu1] are quite different. The proof in [GGKP5] was growing up among the ideas of the theory of algebraic groups while the one from [Gu1] relies on deep facts from classification of finite simple groups and related topics.

In fact, Guest also obtained a further refinement of Theorem 5.24 (see [Gu1, Theorem A^*]). Namely, he proved

Theorem 5.31 ([Gu1]). *Let G be a finite almost simple group with socle G_0. Suppose that g is an element of odd prime order p in G. Then one of the following holds.*

(i) There exists $x \in G$ such that $\langle g, xgx^{-1} \rangle$ is not solvable.

(ii) $p = 3$ and (g, G_0) belongs to a list of exceptions given in Table 1. Moreover, there exist x_1, x_2 in G such that the subgroup $\langle g, x_1gx_1^{-1}, x_2gx_2^{-1} \rangle$ is not solvable, unless $G_0 \cong PSU_n(2), PSp_{2n}(3)$. In any case, there exist x_1, x_2, x_3 in G such that $\langle g, x_1gx_1^{-1}, x_2gx_2^{-1}, x_3gx_3^{-1} \rangle$ is not solvable.

G_0	g
$PSL_n(3)$, $n > 2$	*transvection*
$PSp_{2n}(3)$, $n > 1$	*transvection*
$PSU_n(3)$, $n > 2$	*transvection*
$PSL_n(2)$, $n > 3$	*reflection of order 3*
$P\Omega_n^\epsilon(3)$, $n > 6$	*g a long root element*
$E_l(3)$, $F_4(3)$, ${}^2E_6(3)$, ${}^3D_4(3)$	*g a long root element*
$G_2(3)$	*g a long or short root element*
$G_2(2)' \cong PSU_3(3)$	*transvection*

Table 1. List of exceptions to Theorem 5.31

See [Gu2] for further refinements.

5.6. **Outline of the proof of Theorem 5.24.** The proof sketched below follows the scheme of [GGKP5]. We reduce Theorem 5.24 to the following statement:

Theorem 5.32. *Let G be a finite almost simple group, and let $g \in G$ be of prime order > 3. Then there is $x \in G$ such that the subgroup generated by g and xgx^{-1} is not solvable.*

The reduction is fairly standard and follows the scheme of Subsection 3.2, so the rest of the section is devoted to an outline of the proof of Theorem 5.32.

We refer to the property stated in Theorem 5.32 as Property **(NS)** (for "nonsolvable"):

(NS) For every $g \in G$ of prime order > 3 there is $x \in G$ such that the subgroup generated by g and xgx^{-1} is not solvable.

We prove, using case-by-case analysis, that every almost simple group G, where $L \leq G \leq \operatorname{Aut} L$, satisfies **(NS)**.

5.6.1. *Alternating groups and sporadic groups.*

Lemma 5.33. *Let $L = A_n$, $n \geq 5$, be the alternating group on n letters. Then G satisfies* (**NS**).

Proof. Clearly it is enough to consider the alternating groups: as $\operatorname{Aut}(A_n) = S_n$ for $n \neq 6$ and $[\operatorname{Aut}(A_6) : A_6] = 4$, any element of odd order in $\operatorname{Aut}(A_n)$ lies in A_n. So let $G = A_n$, $n \geq 5$. For $n = 5$ the proof is straightforward, so we may proceed by induction. We may thus assume that g acts without fixed points, so $n = k\ell$, where ℓ stands for the order of g, and g is a product of k disjoint cycles of length ℓ. If $k = 1$, we can conjugate $g = (12 \ldots \ell)$ by a 3-cycle $z = (123)$ to see that $\langle g, zgz^{-1} \rangle = A_\ell$. For $k > 1$, we conjugate g by a product of k 3-cycles. ☐

Lemma 5.34. *Let L be a sporadic simple group. Then G satisfies* (**NS**).

Proof. As the group of outer automorphisms of any sporadic group is of order at most 2, it is enough to treat the case where G is a simple sporadic group. Here the proof goes, word for word, as in [GGKP1, Prop. 9.1]. Namely, case-by-case analysis shows that any element $g \in G$ of prime order $\ell > 3$ is either contained in a smaller simple subgroup of G, or its normalizer is a maximal subgroup of G. In the latter case it is enough to conjugate g by an element x not belonging to $N_G(\langle g \rangle)$ to ensure that $\langle g, xgx^{-1} \rangle = G$. □

5.6.2. *Groups of Lie rank 1.* We start with the following well-known fact.

Proposition 5.35. *Let G be a finite almost simple group of Lie type, and let $g \in G$ be an element of prime order $\ell > 3$. Then g is either an inner-diagonal or a field automorphism of L.*

Proof. See [GLS, p. 82, 7-3] and [LLS, Proposition 1.1]. □

Proposition 5.36. *Suppose that the Lie rank of L is 1. Then G satisfies* (**NS**)*.*

Proof. Let $g \in G$ be of prime order > 3. We check that there is $x \in L$ such that the subgroup of G generated by g and g^x is not solvable.

In the cases $L = PSL_2(q)$, $q \geq 4$, and $L = PSU_3(q^2)$, $q > 2$, it is enough to use arguments from [GS].

If L is a Suzuki group ${}^2B_2(2^{2m+1})$, $m \geq 1$, or a Ree group ${}^2G_2(3^{2m+1})$, $m > 0$, we only have to consider the cases where $g \in G$ is a semisimple element of L (since g is of prime order > 3 and cannot be unipotent), or a field automorphism (since every outer automorphism is a field automorphism).

1. *Suzuki–Ree case, g is semisimple.* Suppose g is a semisimple element of order greater than 3. Then g is regular [Ca1], [KLM]. We then use the description of maximal subgroups of G and the structure of the normalizers of maximal tori ([Suz1], [SS], [We], [LSS], [LN], [We], [LSS]) along with Gow's theorem [Gow].

2. *Ree case, g is a nontrivial field automorphism.*

The Ree groups ${}^2G_2(3^{2m+1})$, $m > 0$, are not minimal and contain a subgroup isomorphic to $PSL_2(q)$, normalized but not centralized by g. Thus there exists $x \in L$ such that $\langle g, g^x \rangle$ is not solvable.

3. *Suzuki case, g is a nontrivial field automorphism.*

For the Suzuki case $L = {}^2B_2(2^{2m+1})$, $m \geq 1$, where an explicit construction for x required in **(NS)** is a hard task, we use a slightly modified counting method from [GS, Lemma 3.1]. □

5.6.3. *Groups of type ${}^2F_4$.* If $g \in G$ is a unipotent element of prime order > 3, then it is easy to reduce the problem to the case of Lie rank 1. The same idea works if g is a field automorphism of L. The case where g is semisimple is treated similarly to Case 1 of Proposition 5.36 using the list of [Ma] and Gow's theorem [Gow].

5.6.4. *General case.* In order to complete the proof of Theorem 5.24, we have to prove

Theorem 5.37. *Let L be a simple group of Lie type of Lie rank ≥ 2, $L \neq {}^2F_4(q^2)$, and let $L \leq G \leq Aut L$. Then G satisfies* (**NS**)*.*

If $g \in G$ is a field automorphism of L, then it normalizes but does not centralize a smaller rank group, and we are done. Thus, in view of classification of automorphisms of prime order (see Proposition 5.35), it is enough to prove the following:

Theorem 5.38. *Let $L = L(K)$ be a simple group of Lie type, $rank(G) \geq 2$, K a finite field, $char(K) = p$. Let σ be a diagonal automorphism of L, and let $G = \langle \sigma, L \rangle$. Let $g \in G$ be of prime order $q > 3$. Then G satisfies* (**NS**)*, i.e., there exists $x \in G$ such that the group $\langle g, xgx^{-1} \rangle$ is not solvable.*

In contrast with the rank 1 case, in the proof of Theorem 5.37 we avoid considerations related to the specific subgroup structure of the groups in question and use some basic results on algebraic groups instead.

Let $g \in G$ be of prime order $\ell > 3$. Our aim is to prove property (**NS**) using a kind of induction by parabolic subgroups of G and the corresponding Levi subgroups. For the sake of convenience, we replace induction by studying the minimal counter-example to (**NS**).

This means that we suppose that the property (**NS**) does not hold for some group G. We may assume for G the following property (**MC** stands for "minimal counter-example"):

MC:

(a) G is a finite almost simple group which does not satisfy (**NS**);

(b) $[G, G] = L$ is a simple group of Lie type different from ${}^2F_4$;

(c) if H is a group satisfying conditions (a) and (b), then the order of $[H, H]$ is greater than or equal to the order of G.

Throughout below $g \in G$ is an element of prime order $\ell > 3$ such that the group $\langle g, xgx^{-1}\rangle$ is solvable for every $x \in L$ (such an element exists according to hypothesis (a)). We will consequently study the properties of **(MC)**.

Recall that the group L can be represented in the form

$$L = [\mathbb{G}(K), \mathbb{G}(K)] = \mathbb{G}_{\rm sc}(K)/Z(\mathbb{G}_{\rm sc}(K))$$

where $\mathbb{G}_{\rm sc}$ is a simple, simply connected linear algebraic group defined over a finite field K and $\mathbb{G} = \mathbb{G}_{\rm ad}$ is the corresponding *adjoint* group.

We prove that for every pair $\langle \sigma, L\rangle$ there exists a reductive group $\mathfrak{G}$ over K satisfying the following conditions:

- the derived group $\mathfrak{G}'$ is simply connected;
- $\mathfrak{G}'(K)/Z(\mathfrak{G}'(K)) = L$;
- there is $\tau \in \mathfrak{G}(K)$ such that $\langle \tau, \mathfrak{G}'(K)\rangle/Z(\langle \tau, \mathfrak{G}'(K)\rangle = \langle \sigma, L\rangle$.

We then note that the group G lies in $G_{ad}(K)$.

Now we are able to study the minimal counter-example. The first reduction is as follows:

Lemma 5.39. *The element $g \in G = \langle \sigma, L\rangle \le \mathbb{G}(K)$ does not belong to any proper parabolic subgroup of $\mathbb{G}(K)$.*

Proof. We use parabolic induction whose base is Theorem 5.36. □

The next two reductions are as follows:

Lemma 5.40. *The element g does not normalize any unipotent subgroup V of $\mathbb{G}(K)$.*

Lemma 5.41. *The element $g \in G = \langle \sigma, L\rangle \le \mathbb{G}(K) \le \mathbb{G}$ is a regular semisimple element of $\mathbb{G}$.*

The proof of Lemma 5.40 uses a construction of Borel–Tits ([BT],[BuW], or just [GLS, Theorem 3.1.3]). The proof of Lemma 5.41 relies on Lemma 5.40 and general facts from [Ca1] regarding centralizers of g.

Recall that a Coxeter element w_c of the Weyl group $W = W(\Phi)$ with respect to Π is a product (taken in any order) of the reflections w_α, $\alpha \in \Pi$, where each reflection occurs exactly once.

Using arguments from [GS] and [EG2], one can prove that g is of the form $g = v\dot{w}_c$ where w_c is the Coxeter element of the Weyl group of $G_{ad}(K)$ and $v \in U \le G$.

By [GGKP2], for any $g = v\dot{w}_c$ there is $x \in G$ such that $[g, x] = u \in U$.

Put $H = \langle g, xgx^{-1}\rangle$. Suppose H is solvable. Denote by $H_{p\ell}$ a $p\ell$-Hall subgroup of H, and let A be a maximal abelian normal subgroup of $H_{p\ell}$. Denote by A_p the p-Sylow subgroup of A. One can prove that g does not normalize any unipotent subgroup of G. This, in particular, implies that $A_p = 1$, because A_p is normalized by g. Hence $|A| = \ell^s$.

The next fact about the structure of the minimal counter-example is as follows. Let $F \lneq G_{ad}$ be a reductive subgroup defined over K, and suppose $g \in F = F(K)$. Consider F^0, the identity component of F. Then F^0 is a K-defined connected reductive group. Suppose that F^0 is not a torus. Then $g \in F$ leads to a contradiction and therefore:

The element g cannot lie in any proper reductive subgroup of G_{ad} other than a torus. This leads to a crucial reduction:

Proposition 5.42. *If Γ is a minimal counter-example, $g \in \Gamma$, then the reductive group $\mathfrak{G}$ corresponding to G is either the linear group GL_ℓ or the unitary group U_ℓ.*

Each of these cases is considered separately and in both of them it turns out that g is represented by a monomial matrix, which leads to a contradiction by Lemma 5.33. □

6. Generalizations and analogues: linear groups, PI-groups, noetherian groups, Lie algebras

In this section, we obtain extensions of our earlier results for some classes of infinite groups. Throughout, let $v_n(x, y)$ and $s_n(x, y)$ denote the sequences introduced in Theorems A and A′, respectively.

6.0.5. *Linear groups.*

Theorem 6.1 ([BGGKPP2])**.** *Suppose that G is a subgroup of $GL(r, K)$ where K is a field. Then G is solvable if and only if it satisfies an identity $v_n(x, y) \equiv 1$ (respectively $s_m(x, y) \equiv 1$) for some n, m.*

Proof. The "only if" part is obvious. The "if" part is an immediate consequence of Theorems A and A′ combined with Platonov's theorem [Pla1] stating that every linear group over a field satisfying a nontrivial identity is solvable-by-finite. (In fact, the assertion of the theorem also follows from the Tits alternative [Ti].) □

Remark 6.2. The sequences $v_n(x, y)$ and $s_n(x, y)$ can be replaced in Theorem 6.1 by one of the sequences from [Ri].

Let $Solv(G)$ denote the sets of elements introduced in Theorem 5.4 and Theorem 5.18. Namely, $Solv(G)$ is either

(∗) the set of $g \in G$ such that $\langle g, x\rangle$ is solvable for any $x \in G$, or

(∗∗) the set of $g \in G$ such that $\left\langle g, aga^{-1}, bgb^{-1}, cgc^{-1}\right\rangle$ is solvable for any $a, b, c \in G$.

Theorem 6.3 ([GKPS], [GGKP3], [FGG])**.** *Suppose that $G \subset GL(n, K)$ where K is a field. Then $\mathfrak{R}(G) = Solv(G)$.*

Proof. First of all, every element from $\mathfrak{R}(G)$ belongs to $Solv(G)$ since $\mathfrak{R}(G)$ is a characteristic subgroup of G.

We shall prove the opposite inclusion, i.e., $Solv(G) \subseteq \mathfrak{R}(G)$. Let H be the subgroup generated by $Solv(G)$. It is enough to show that H is solvable. Take a finitely generated subgroup $H_1 = \langle g_1, \dots, g_s\rangle$ where $g_i \in Solv(G)$, $i = 1, \dots, s$. Then H_1 is approximated by finite linear groups $G_\alpha = H_1/N_\alpha$, $\cap N_\alpha = 1$ in dimension n [Ma]. Each G_α is finite and is generated by the images of g_i which lie in $Solv(G_\alpha)$. Thus G_α is solvable by Theorems 5.4 and 5.18.

Therefore H_1 can be embedded into a cartesian product D of finite solvable groups G_α. The solvability class of G_α is bounded by the rank of the linear group G. Since the class of solvable groups of fixed solvability class is closed under cartesian products, we conclude that D is solvable, hence so is H_1. We now observe that every finitely generated subgroup of H lies in some H_1 and is thus solvable. This means that H is locally solvable. It remains to apply a theorem of Zassenhaus [Za] saying that any locally solvable linear group is solvable. □

The same scheme applied to Corollary 5.26 implies the following result:

Theorem 6.4 ([GGKP4], [Gu1])**.** *Let $G \subset GL(n, K)$ where K is a field. Then G is solvable if and only if in each conjugacy class of G every two elements generate a solvable subgroup.* □

Theorem 5.24 also remains true for linear groups. In particular, it implies

Proposition 6.5 ([FGG])**.** *Let $G \subset GL(n, K)$ where the characteristic of the field K is either 0 or $p > 3$. A unipotent element g belongs to $\mathfrak{R}(G)$ if and only if every two conjugates of g generate a solvable subgroup.* □

Theorem 6.1 is true not only for linear groups but also for other classes of groups with the Tits alternative (see [BFH], [Mc], [Iv], [BG], etc.). For example, subgroups of mapping class groups with the law $v_n(x, y) \equiv 1$ or $s_n(x, y) \equiv 1$ are solvable in view of the following result:

Theorem 6.6. [Mc, Theorem A] *Let G be a subgroup of a mapping class group Γ. Either G contains an abelian subgroup of finite index, or G contains a nonabelian free group.*

The same situation holds for the group of outer automorphisms of a free group $\mathrm{Out}\,(F_n)$.

Theorem 6.7. [BFH, Theorem 7.0.1] *Suppose that H is a subgroup of $Out(F_n)$ that does not contain a free subgroup of rank 2. Then there exist a finite index subgroup H_0 of H, a finitely generated free abelian group A, and a map $\Phi\colon H_0 \to A$ such that $Ker(\Phi)$ is a UPG (unipotent, polynomial growth) subgroup.*

The fact similar to Theorem 6.1 follows from [BFH, Theorem 1.0.3] which states that a UPG subgroup of $\mathrm{Out}\,(F_n)$ that does not contain a free subgroup of rank 2 is solvable. Note that the groups $\mathrm{Out}\,(F_n)$, $n > 3$, provide an example of nonlinear and non-PI groups which satisfy the Tits alternative.

6.0.6. *PI-groups.*

Definition 6.8. A group G is called a PI-group if G is a subgroup of the group of invertible elements of an associative PI-algebra over a field.

Below we collect some useful facts about PI-groups (also known as PI-representable groups [Pi]), see [Plo6] and references therein.

Linear groups are a particular case of PI-groups since every matrix satisfies a polynomial identity (see [J1], [Row]). PI-groups provide a class of groups with positive solutions of Burnside-type problems. Namely, every torsion PI-group is locally finite (see [Pr], [To2]). Every nil-PI-group is locally nilpotent, the set of all nil-elements coincides with the locally nilpotent radical $HP(G)$ (see [Pla2], [To1], [To2], [Plo5], [Plo6]). Thus, a PI-group G is locally nilpotent if and only if every two-generated subgroup is nilpotent or, moreover, in each conjugacy class of G every two elements generate a nilpotent subgroup.

Every PI-group G has a unique maximal locally solvable normal subgroup $\mathfrak{R}(G)$ called the locally solvable radical of G, and the locally solvable radical of a finitely generated PI-group is solvable [Pi]. (For arbitrary groups the locally solvable radical may not exist, and for arbitrary PI-groups the locally solvable radical is not necessarily solvable.)

Every PI-group G is an extension of a locally nilpotent group by a linear group over a cartesian sum of fields. More precisely, G has the following invariant series: $1 \lhd H_0 \lhd H \lhd G$ where H is a locally nilpotent normal subgroup, H_0 is generated by the nilpotent normal subgroups in G, H/H_0 is nilpotent and G/H is a linear group over a cartesian sum of fields [Plo6].

Recall that every group has the upper radical $\widetilde{HP}(G)$ (see Definition 2.15).

Theorem 6.9 ([GKPS], [Ala], cf. [Plo6])**.** *If G is a PI-group, then $\mathfrak{R}(G) = \widetilde{HP}(G) = Solv(G)$.*

Proof. Case 1. If $G \le \mathrm{GL}_n(P)$, where P is a field, then $\mathfrak{R}(G) = Solv(G)$ by Theorem 6.3.

Case 2. If $G \le \mathrm{GL}_n(K)$, where $K = \bigoplus_s P_s$ is a cartesian sum of fields, then a reduction to the previous case is straightforward.

Case 3. General case. Let us first show that $\mathfrak{R}(G) \subseteq Solv(G)$ for the case $(*)$. Let $g \in \mathfrak{R}(G)$, $h \in G$. We shall check that $G_0 = \langle g, h\rangle$ is solvable. We have $g \in \mathfrak{R}(G) \cap G_0$ and, consequently, $g \in \mathfrak{R}(G_0)$. By [Pi], the locally solvable radical of a 2-generated group is solvable. So the group G_0 is solvable as a cyclic extension of a solvable group. The inclusion for the case $(**)$ holds trivially.

Coincidence of the radicals $\mathfrak{R}(G) = \widetilde{HP}(G)$ mostly relies on the fact that in every PI-group G there is a locally nilpotent subgroup H such that G/H lies in $\mathrm{GL}_n(K)$ where K is a cartesian sum of fields.

We are now able to prove that $Solv(G) \subseteq \mathfrak{R}(G)$. Let $g \in Solv(G)$. Denote by $\bar{g} \in G/H$ its image under the natural projection. Then $\bar{g} \in Solv(G/H)$ and, hence, $\bar{g} \in \mathfrak{R}(G/H)$ due to Case 2. Thus $\bar{g} \in \widetilde{HP}(G/H)$. Since H is locally nilpotent, then $g \in \widetilde{HP}(G) = \mathfrak{R}(G)$. □

Theorem 6.9 has a consequence which can be viewed as a natural generalization of Thompson's theorem:

Corollary 6.10. *A PI-group G is locally solvable if and only if every two-generated subgroup of G is solvable. In particular, a finitely generated PI-group G is solvable if and only if every two-generated subgroup of G is solvable.*

Remark 6.11. In linear groups the locally solvable radical is solvable. From Theorem 6.9 it follows that in PI-groups the locally solvable radical is solvable modulo the locally nilpotent radical. Indeed, $\widetilde{HP}(G)/HP(G)$ is solvable [Plo6], and $\widetilde{HP}(G) = \mathfrak{R}(G)$ by Theorem 6.9.

Corollary 6.12. *If G is a PI-group, then the normal subgroup $\langle g^G \rangle$ is locally solvable if and only if the element g is radical. In particular, if G is a finitely generated PI-group or a linear group, then $\langle g^G \rangle$ is solvable if and only if the element g is radical.*

Theorem 6.13 ([Ala])**.** *If a PI-group G satisfies an identity $v_n(x,y) \equiv 1$ (respectively $s_m(x,y) \equiv 1$) for some n, m, then G is locally solvable.*

Proof. Suppose that G satisfies an identity $v_n(x,y) \equiv 1$ (or $s_m(x,y) \equiv 1$). Then the group G/H, where H is a locally nilpotent normal subgroup, also satisfies this identity and lies in $\prod_s \mathrm{GL}_n(P_s)$ where all P_s are fields. By [Pla1], all the images of G/H in $GL_n(P_s)$ are solvable-by-finite and by Theorem 6.1 they are solvable. It can be seen that the derived lengths of all images are uniformly bounded. Hence, the group G/H is also solvable. For PI-groups, the extension of a locally solvable group by locally solvable is also locally solvable (this is not the case for arbitrary groups, see [Mi]). So the group G is locally solvable, since H is a locally solvable group, and G/H is solvable. □

Corollary 6.14. *A finitely generated PI-group G is solvable if and only if it satisfies an identity $v_n(x,y) \equiv 1$ (respectively $s_m(x,y) \equiv 1$) for some n, m.*

6.0.7. *Noetherian groups.* As we know, theorems describing the nilpotency property in terms of Engel elements are valid not only for finite but also for arbitrary noetherian groups. So it is quite natural to ask whether the same is true with respect to various characterizations of the solvability property described above. This problem in full generality is still open. However, for Hirsch groups (i.e., for finite extensions of polycyclic groups) the counter-parts of Theorems 6.1 and 6.3 hold ([Ala]). Indeed, these groups are solvable-by-finite and thus conditions guaranteeing solvability in the finite case determine solvability of the whole group. Note that not every noetherian group is polycyclic-by-finite because of Tarski monsters constructed by A. Olshanskii [O1].

6.0.8. *Lie algebras.* Let L be a finite-dimensional Lie algebra over a field k. Denote by $[,]$ the Lie operation. For $t \in L$ the linear operator $\mathrm{ad}\, t\colon L \to L$ is defined by $(\mathrm{ad}\, t)x = [t,x]$.

By the *solvable radical* of L we mean the largest solvable ideal $\mathfrak{R}$ of L (Bourbaki [Bou] and Jacobson [J1] call $\mathfrak{R}$ the radical of L). By the *nilpotent radical* of L we mean the largest nilpotent ideal N of L (Jacobson [J2] calls N nil radical, and Bourbaki [Bou] calls it just the largest nilpotent ideal).

Here we collect facts about explicit two-variable sequences which are related to solvability of finite-dimensional Lie algebras. We start with definitions which are similar to the case of groups.

Let $W_2 = W(x,y)$ be the free two-generator Lie algebra. An element g of a Lie algebra L is called Engel if for every $a \in L$ there exists $n = n(a,g)$ such that $e_n(a,g) = [a,g,g,\dots,g] = 0$ (i.e. $(\mathrm{ad}\, g)^n a = 0$). If every element of L is Engel, then L is called unbounded Engel. If L satisfies the identity $e_n(x,y) \equiv 0$, it is called Engel.

Definition 6.15 ([Plo3], [BBGKP])**.** We say that a sequence $\overrightarrow{u} = u_1, u_2, u_3, \dots, u_n, \dots$ of elements from W_2 is **correct** if the following conditions hold:

(i) For every Lie algebra L and elements $a, g \in L$ we have $u_n(a,0) = 0$ and $u_n(0,g) = 0$ for all sufficiently large n.

(ii) if a, g are elements of L such that $u_n(a,g) = 0$, then for every $m > n$ we have $u_m(a,g) = 0$.

Thus, if the identity $u_n(x,y) \equiv 0$ is satisfied in L, then for every $m > n$ the identity $u_m(x,y) \equiv 0$ also holds in L.

Definition 6.16. For every correct sequence $\overrightarrow{u}$ in $W_2(x,y)$, define the class of Lie algebras $\Theta = \Theta(\overrightarrow{u})$ by the rule: a Lie algebra L belongs to Θ if and only if there is n such that the identity $u_n(x,y) \equiv 0$ holds in L.

Definition 6.17. For every Lie algebra L denote by $L(\overrightarrow{u})$ the subset of L defined by the rule: $g \in L(\overrightarrow{u})$ if and only if for every $a \in L$ there exists $n = n(a,g)$ such that $u_n(a,g) = 0$. Elements of $L(\overrightarrow{u})$ are viewed as Engel elements with respect to the given correct sequence $\overrightarrow{u}$. We call these elements $\overrightarrow{u}$**-Engel-like** or, for brevity, $\overrightarrow{u}$**-Engel elements**.

The correct sequence $\overrightarrow{w}$, where

$$w_1(x,y) = [x,y], \dots, w_n(x,y) = [[w_{n-1},x],[w_{n-1},y]], \dots$$

and $[\ ,\ ]$ stands for the Lie bracket in a Lie algebra, determines the class $\Theta(\overrightarrow{w})$ of finite dimensional solvable Lie algebras over an infinite field k, $\mathrm{char}(k) \neq 2,3,5$. Indeed:

Theorem 6.18. [GKNP] *Let $w_1(x,y) = [x,y]$, $w_{n+1}(x,y) = [[w_n(x,y),x],[w_n(x,y),y]]$. Then a finite-dimensional Lie algebra L defined over an infinite field of zero characteristic or positive characteristic greater than 5 is solvable if and only if for some n the identity $w_n(x,y) \equiv 0$ holds in L.*

Proof. Obviously, if L is solvable, then it satisfies an identity of the form $w_n(x,y) \equiv 0$ since for any $X, Y \in L$ the value $w_n(X,Y)$ belongs to the corresponding term of the derived series. Conversely, suppose that L satisfies the identity $w_n \equiv 0$. We have to show that L is solvable.

1st step. The identity $w(x,y) \equiv 0$ also holds in the Lie algebra $\overline{L} = L \otimes_k \overline{k}$ defined over an algebraic closure $\overline{k}$ of k, so one can assume that k is algebraically closed.

2nd step. If L is not solvable, then $L^{ss} = L/\mathfrak{R}(L)$ is semisimple and nonzero. If $\mathrm{char}(F) = 0$, denote by $\{E_\alpha, H_\alpha, E_{-\alpha}\}$ the standard basis of sl_2. Then $[E_\alpha, E_{-\alpha}] = H_\alpha$, $[H_\alpha, E_\alpha] = 2E_\alpha$, $[H_\alpha, E_{-\alpha}] = -2E_{-\alpha}$. Set $x = E_\alpha$, $y = E_{-\alpha}$. Then

$$\begin{array}{rclrclrcl} w_1 &=& H_\alpha, & w_1' &=& 2E_\alpha, & w_1'' &=& -2E_{-\alpha}, \\ w_2 &=& -4H_\alpha, & \dots, \end{array}$$

i.e. $w_n = mH_\alpha$ with $m \neq 0$. Thus for any n we have $w_n(E_\alpha, E_{-\alpha}) \neq 0$.

3rd step. Let now $\mathrm{char}(k) = p > 5$. In the restricted case (see [SF, 2.1] for the definition), we can use the classification theorem of [BlW] in order to mimic the proof in characteristic zero. If L is not restricted, one needs more subtle arguments using [Blo, Th. 9.3] and [We, Cor. 1.4]. Details can be found in [GKNP]. □

Remark 6.19. A. Premet informed us that one can modify the proof to be valid for all $p > 2$.

Define yet another correct sequence $\overrightarrow{v}$ by $v_1(x,y) = x$ and, by induction, $v_{n+1}(x,y) = [v_n(x,y),[x,y]]$. Then $v_{n+1}(x,y) = (-\mathrm{ad}\,[x,y])^n x = e_n(x,[x,y])$.

Theorem 6.20. *Let L be a finite-dimensional Lie algebra over a field k of characteristic zero. Then L is solvable if and only if for some n the identity $v_n(x,y) \equiv 0$ holds in L.*

Proof. If L is solvable, then, since $\mathrm{char}(k) = 0$, $L' = [L,L]$ is nilpotent [J2, Cor. II.7.1]. Hence every pair z,t of elements of L' satisfies $(\mathrm{ad}\,t)^m z = 0$ where $m = \dim L'$. On putting $z = [x,[x,y]]$, $t = [x,y]$, we get $v_{m+2}(x,y) = 0$. In the opposite direction, the proof repeats the arguments of the previous theorem. □

Theorem 6.21 ([BBGKP])**.**
 (i) *Let L be a finite-dimensional Lie algebra over a field k of characteristic zero. The solvable radical $\mathfrak{R}(L)$ coincides with the set of all $\overrightarrow{v}$-Engel elements of L.*
 (ii) *Let L be a finite-dimensional Lie algebra over an algebraically closed field k of characteristic zero. The solvable radical $\mathfrak{R}(L)$ coincides with the set of all $\overrightarrow{w}$-Engel elements of L.*

Remark 6.22. The sequence $\overrightarrow{v}$ is strictly adjusted to the case of Lie algebras over a field of characteristic zero. Indeed, the key point in the proof of Theorem 6.20 and of item 1 of Theorem 6.21 is the fact that if L is solvable then $[L,L]$ is nilpotent. This is no longer true in positive characteristic. For an explicit counter-example to the corresponding statements in positive characteristic see [BBGKP], Example 3.10.

As for the sequence $\overrightarrow{w}$, in [BBGKP] the following statement is proved

Theorem 6.23. *Let L be a finite dimensional Lie algebra over an uncountable field k of characteristic zero. Then its solvable radical R coincides with the set of all $\overrightarrow{w}$-Engel elements of L.*

Theorem 6.23 also does not hold in positive characteristic because simple Lie algebras may contain nonzero $\overrightarrow{w}$-Engel elements:

Example 6.24. Let $L = W(1;1)$ be the Witt algebra defined over a field k of characteristic p. Recall (see, for example, [SF, 4.2, p. 148]) that L is of dimension p with multiplication table defined on a basis $\{e_{-1}, e_0, e_1, \dots, e_{p-2}\}$ as follows:

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & \text{if } -1 \leq i+j \leq p-2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

If $p > 2$, the algebra L is simple [SF, Thm. 2.4(1) on p. 149]. However, if $p > 3$, it contains nonzero $\overrightarrow{w}$-Engel elements. Indeed, let $g = e_{p-2}$, and let $x = \alpha_{-1}e_{-1} + \dots + \alpha_{p-2}e_{p-2}$ be an arbitrary element of L. From formulas (6.1) it follows that

$$w_1(x,g) = \alpha_{-1}e_{p-3} + \alpha_0 e_{p-2}.$$

For $p > 3$ this implies $[w_1(x, g), g] = 0$ and hence $w_2(x, g) = 0$. Thus y is a $\overrightarrow{w}$-Engel element, and statement (ii) of Theorem 6.21 and Theorem 6.23 do not hold for L.

Theorem 6.21 implies that (cf. Theorem 5.4):

Theorem 6.25. *Let L be a finite-dimensional Lie algebra defined over a field k of characteristic zero. Then the radical $\mathfrak{R}(L)$ coincides with the set of elements $g \in L$ with the following property: for any $x \in L$ the subalgebra generated by x and g is solvable.*

Proof. If $y \in R(L)$, then for any $x \in L$ the subalgebra generated by x and y contains a solvable ideal with one-dimensional quotient and is therefore solvable.

Let now x be an arbitrary element of L. Since the subalgebra generated by x and y is solvable, it satisfies the identity $v_n(x, y) \equiv 0$ for some n, and we are done. □

Remark 6.26. In view of Remark 6.22, the proof presented above cannot work in positive characteristic. The idea to use "one-and-a-half generation" for Lie algebras, which looks more promising in view of [Io] where this property was proved for any simple Lie algebra over $\mathbb{C}$, breaks down because of the recent results of J.-M. Bois. It turns out that although one-and-a-half generation holds for the modular analogues (for $p > 3$) of the classical simple Lie algebras, as well as for the graded Cartan type Lie algebra $W(1, \underline{\mathbf{n}})$ [Boi1], this is no longer true for the simple graded Cartan type Lie algebras of the remaining types [Boi1], [Boi2].

So it seems that the following result, which holds in arbitrary sufficiently large characteristic, is the limit of our hopes:

Theorem 6.27. *Let $p > 3$ be a prime, and let F be an infinite field of characteristic p. Let L be a finite-dimensional Lie algebra over F. Assume that every pair of elements in L generate a solvable Lie algebra. Then L is solvable.*

The proof of Theorem 6.27 relies on the following result of Schue [Sc] (see also Premet–Strade [PS]):

Lemma 6.28. *Let $p > 3$ be a prime, and let F be an algebraically closed field of characteristic p. Let L be a finite-dimensional Lie algebra over F such that every proper subalgebra of L is solvable. Then $L/R(L) \cong sl(2)$, where $R(L)$ is the solvable radical of L.*

Remark 6.29. As in the case of finite groups, many properties of Lie algebras close to solvability can be checked on two-generated subalgebras (see [BTV] for some results of this flavour).

7. Problems

The problems below are mostly formulated and discussed in [GPS], [BGK].

7.1. **Engel-like sequences and finite groups.** We believe that one of the most important conceptual questions left open after discovery of two-variable sequences characterizing finite solvable groups (Theorems A and A′) is the following: for a sequence of words in the free group on two generators, to what extent the property to characterize the class of finite solvable groups is a property of general position, and what type of the dynamic behaviour of the corresponding word maps is typical? This question is of "nonbinary" type and does not admit the answer of type "yes-no".

More precisely, a possible goal is to prove (or disprove) that for a sufficiently wide class of correct sequences the property to characterize the class of finite solvable groups holds in "general position" and is determined by its dynamics in the free group.

Question 7.1. Suppose that a sequence $\overrightarrow{u} = u_1, u_2, \dots, u_n, \dots$ of elements of F_2 satisfies the following conditions:

(i) $u_n(a, 1) = u_n(1, g) = 1$ for all sufficiently big n, every group G, and all elements $a, g \in G$;

(ii) if G is any group and a, g are elements of G such that $u_n(a, g) = 1$, then for every $m > n$ we have $u_m(a, g) = 1$;

(iii) no element of $\overrightarrow{u}$ equals 1 in F_2;

(iv) there exists N such that for all $n > N$ the word $u_n(x, y)$ belongs to the n-th derived subgroup $F_2^{(n)}$ of F_2.

Is it true that if a finite group G satisfies an identity $u_n(x, y) \equiv 1$ for some n, then it is solvable?

Extensive MAGMA computations show strong numerical evidence of a positive answer to Question 7.1, at least for the class of sequences $\overrightarrow{u}$ studied in [Ri]: $u_0 := f, \dots, u_n := [gu_ng^{-1}, hu_nh^{-1}], \dots,$ where f, g, h stand for some words from F_2.

The situation with a description of the solvable radical of a finite group in terms of correct sequences still remains unclear. The main problem is as follows:

Problem 7.2. Is there an explicit correct sequence of words $q_n(x, y)$ in $F_2(x, y)$ such that the following two conditions hold:

(i) a finite group G is solvable if and only if for some n the identity $q_n(x, y) \equiv 1$ holds in G (i.e., a finite solvable group belongs to the class $\Theta = \Theta(\overrightarrow{u})$);

(ii) the radical $\mathfrak{R}(G)$ of a finite group G coincides with the set of q-Engel elements, i.e. the set of $g \in G$ such that $q_n(x, g) = 1$ for all $x \in G$ and some $n = n(x, g)$ (in other words, $\mathfrak{R}(G) = G(\overrightarrow{u})$)?

Although (ii) implies (i), we state these problems separately since question (ii) is much harder (recall that Theorems A and A′ provide sequences satisfying (i); however, it is not clear if they are suitable for Problem 7.2(ii)).

Recently J. Wilson proved [Wi5] that there exists a sequence q_n of words in two variables satisfying (i) and (ii). However, this sequence is not explicit in the sense that it does not have a simple recursive definition, as required in Problem 7.2. More precisely, the proof in [Wi5] is based on the result from [BrW] where an implicit sequence q_n satisfying (ii) has been constructed. This sequence can also be used in order to characterize $\mathfrak{R}(G)$ for finite (or, more generally, linear) groups. In fact, q_n can be chosen correct, and the following stronger existence theorem is proved:

Theorem 7.3 ([Wi5]). *Let G be a finite (or linear) group. There exists a profinitely convergent sequence $q_n = q_n(x, y)$ of words in the free group $F_2(x, y)$ such that $\mathfrak{R}(G) = G(\overrightarrow{q})$, i.e., the radical $\mathfrak{R}(G)$ of a finite group G coincides with the set of $\overrightarrow{q}$-Engel elements.*

Profinite convergence of q_n means that it has a limit in the profinite completion $\hat{F}_2(x, y)$ of the free group $F_2(x, y)$. Theorem 7.3 is also valid for PI-groups ([Ala]).

Remark 7.4. The statement of Problem 7.2 should be compared with [Lu, Prop. 3.4], where it is proved that for any integer $d \geq 2$ the free prosolvable group $\hat{F}_d(S)$ can be defined by a single profinite relation.

Remark 7.5. Here are some other miscellaneous characterizations of the solvability property and of elements of $\mathfrak{R}(G)$, where G is finite.

A finite group is solvable if and only if no nontrivial element g is a product of 56 commutators of pairs of conjugates of g [Wi2].

Solvability can be characterized in terms of short identities [CLLS]. Namely, suppose that a finite group G satisfies the identity

$$x_1^{e_1} y_1^{f_1} \cdots x_r^{e_r} y_r^{f_r} = 1,$$

with $gcd(e_i, |G|) = 1$ and $gcd(f_i, |G|) = 1$ for all i. If $r < 30$ then G is solvable [CLLS].

The elements of the radical $\mathfrak{R}(G)$ of a finite group G can be characterized by a formula (independent of G) in the first-order logic [Wi4].

From the probabilistic point of view, a finite group G is solvable if with probability $> 11/30$ two randomly chosen elements generate a solvable subgroup [GW], see also [Wi3].

Another probabilistic characterization has been obtained in [NS]. Let G be a finite solvable group of order m and let $w(x_1, \dots, x_n)$ be a group word. Then the probability that $w(g_1, \dots, g_n) = 1$ (where $(g_1, \dots, g_n)$ is a random n-tuple in G) is at least $p^{-(m-t)}$, where p is the largest prime divisor of m and t is the number of distinct primes dividing m. This contrasts with the case of a nonsolvable group G, for which Abért [Abe] has shown that the corresponding probability can take arbitrarily small positive values as n tends to infinity.

A counter-part of Problem 7.2 for Lie algebras is as follows:

Problem 7.6. Let L be a finite-dimensional Lie algebra over an algebraically closed field k of characteristic $p > 2$. Is there an explicit correct sequence of words $q_n(x, y)$ in the free Lie algebra $W_2(x, y)$ such that the solvable radical $\mathfrak{R}(L)$ coincides with the set of all $\overrightarrow{q}$-Engel elements of L?

7.2. **Burnside-type problems related to Engel-like sequences.** Theorem 6.18 on finite-dimensional Lie algebras leads to a similar question in the infinite-dimensional case. Namely, the remarkable Kostrikin–Zelmanov theorem on locally nilpotent Lie algebras [Ko], [Ze2], [Ze3] and Zelmanov's theorem [Ze1] give rise to the following Burnside-type problems for Lie algebras.

Problem 7.7. Suppose that L is a Lie algebra over a field k, the w_n are defined by the formulas of Theorem 6.18, and there is n such that the identity $w_n(x,y) \equiv 0$ holds in L. Is it true that L is locally solvable? If k is of characteristic 0, is it true that L is solvable?

It would be of significant interest to consider similar problems for groups. Recall that G is called an Engel group if there is an integer n such that the Engel identity $e_n(x,y) \equiv 1$ holds in G. Suppose a sequence $q_n(x,y)$ is chosen as in Theorem A, i.e., $q_n(x,y) = u_n(x,y)$, or as in Theorem A′, i.e., $q_n(x,y) = s_n(x,y)$. We call G a *quasi-Engel* group (with respect to the sequence $q_n(x,y)$, or just quasi-Engel) if there is an integer n such that the identity $q_n(x,y) \equiv 1$ holds in G.

The following problems imitate the analogous problems for Engel groups.

Problem 7.8. Is every quasi-Engel group locally solvable?

Remark 7.9. The answer to this question is most likely to be negative, as it is expected for the question about local nilpotency of Engel groups.

It is quite natural to consider restricted versions of Problem 7.8 as is considered for the Burnside problem. Let Q_n be the quasi-Engel variety defined by the identity $q_n \equiv 1$. Let $F = F_{k,n}$ be the free group with k generators in the variety Q_n.

Problem 7.10. Is there a solvable group with k generators $F^0_{n,k}$ in Q_n such that every solvable group $G \in Q_n$ with k generators is a homomorphic image of $F^0_{n,k}$?

In fact, one should prove one of the following three statements equivalent to Problem 7.10:

Problem 7.11.
- Is the intersection of all co-solvable normal subgroups H_α in F also co-solvable?
- Do all locally solvable groups from Q_n form a variety?
- Is every residually finite, quasi-Engel group locally solvable?

Remark 7.12. For Engel groups, the restricted Burnside problem has a positive solution [Wi1], [Plo6].

Consider the class of profinite groups. J. S. Wilson and Zelmanov [WZ] proved that every profinite Engel group is locally nilpotent. So a relevant question is

Problem 7.13. Is every profinite quasi-Engel group locally solvable?

Remark 7.14. In the case of an affirmative solution of Problem 7.2, the corresponding sequence $q_n(x,y)$ should be chosen for the definition of quasi-Engel groups and for the problems listed above.

There is also a bunch of Burnside-type problems related to Thompson-type properties and weak Engel-type properties. In fact these are the problems related to two- and one-and-a-half generation for infinite groups and infinite-dimensional Lie algebras. See [GPS] for details.

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