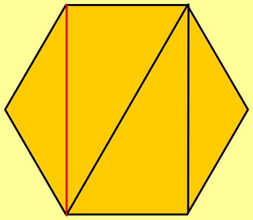
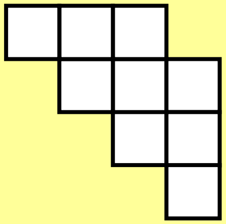
Flips, Arrangements and Tableaux





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Flips

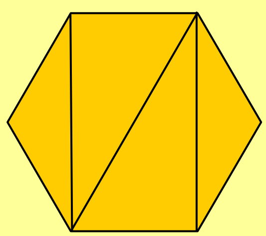
- Triangulations (TFT)
- Flips
- Flip Graph
- Diameter
- Stanley's Conjecture
- Main Result

Triangle-Free Triangulations

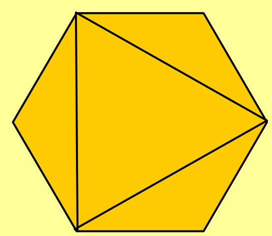
- **Definition:** A triangulation of a convex polygon is triangle-free (TFT) if it contains no triangle

Triangle-Free Triangulations

- **Definition:** A triangulation of a convex polygon is triangle-free (TFT) if it contains no “internal” triangle, i.e., a triangle whose 3 sides are *diagonals* of the polygon. The set of all TFT’s of an n-gon is denoted $TFT(n)$.



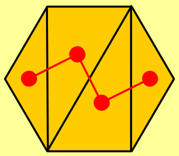
TFT

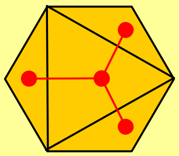


non-TFT

Colored TFT

- **<u>Note:</u>** A triangulation is triangle-free iff the dual tree is a path.





Colored TFT

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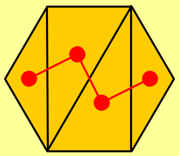
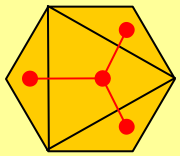
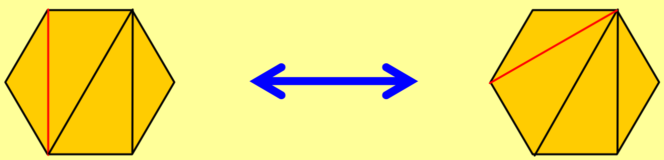




- The triangles of a TFT can be linearly ordered (colored) in two “directions”. Denote by $CTFT(n)$ the set of colored TFT’s.
- $\left|CTFT(n)\right| = n \cdot 2^{n-4}$

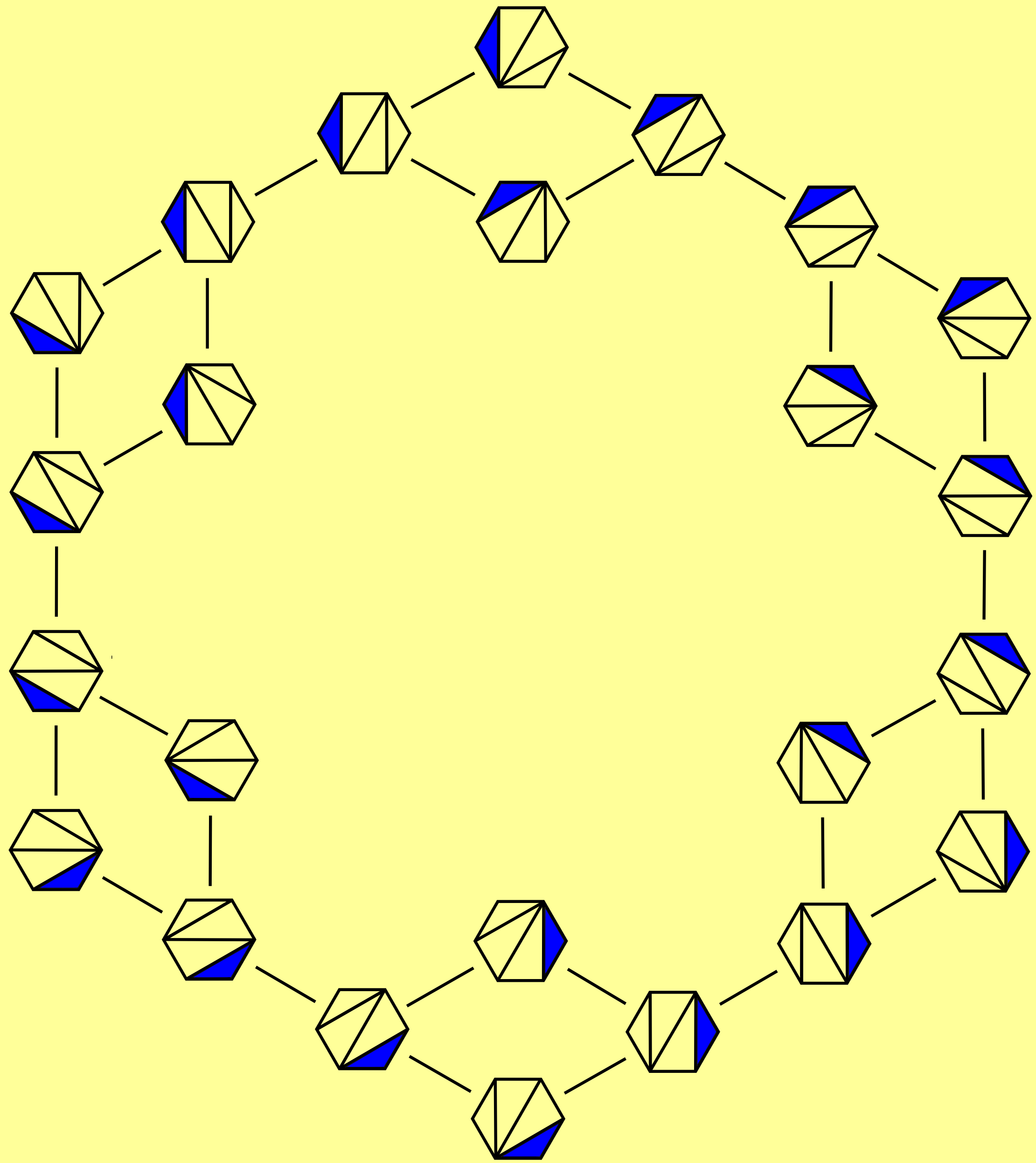
Flip Graph

- Flip = replacing a diagonal by the other diagonal of the same quadrangle.

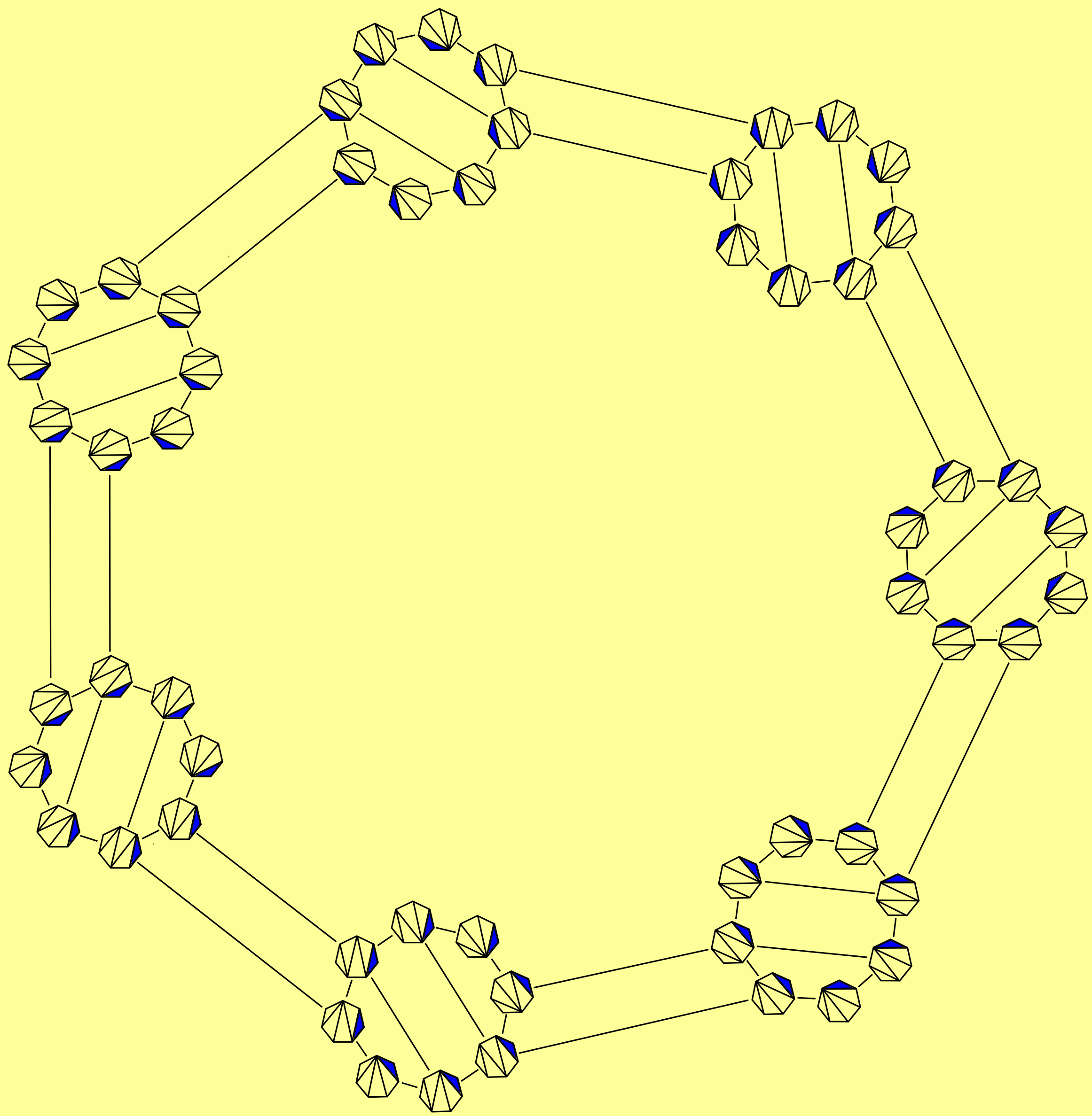


- The colored flip graph Γ_n has vertex set $CTFT(n)$ with edges corresponding to flips.

Γ_6



Γ_7



Diameter of Colored Flip Graph

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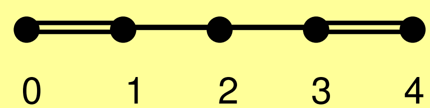
- **Theorem:** *[A-Firer-Roichman, '09]*
  For $n > 4$:
  (a) The diameter of Γ_n is $n(n-3)/2$.

Diameter of Colored Flip Graph

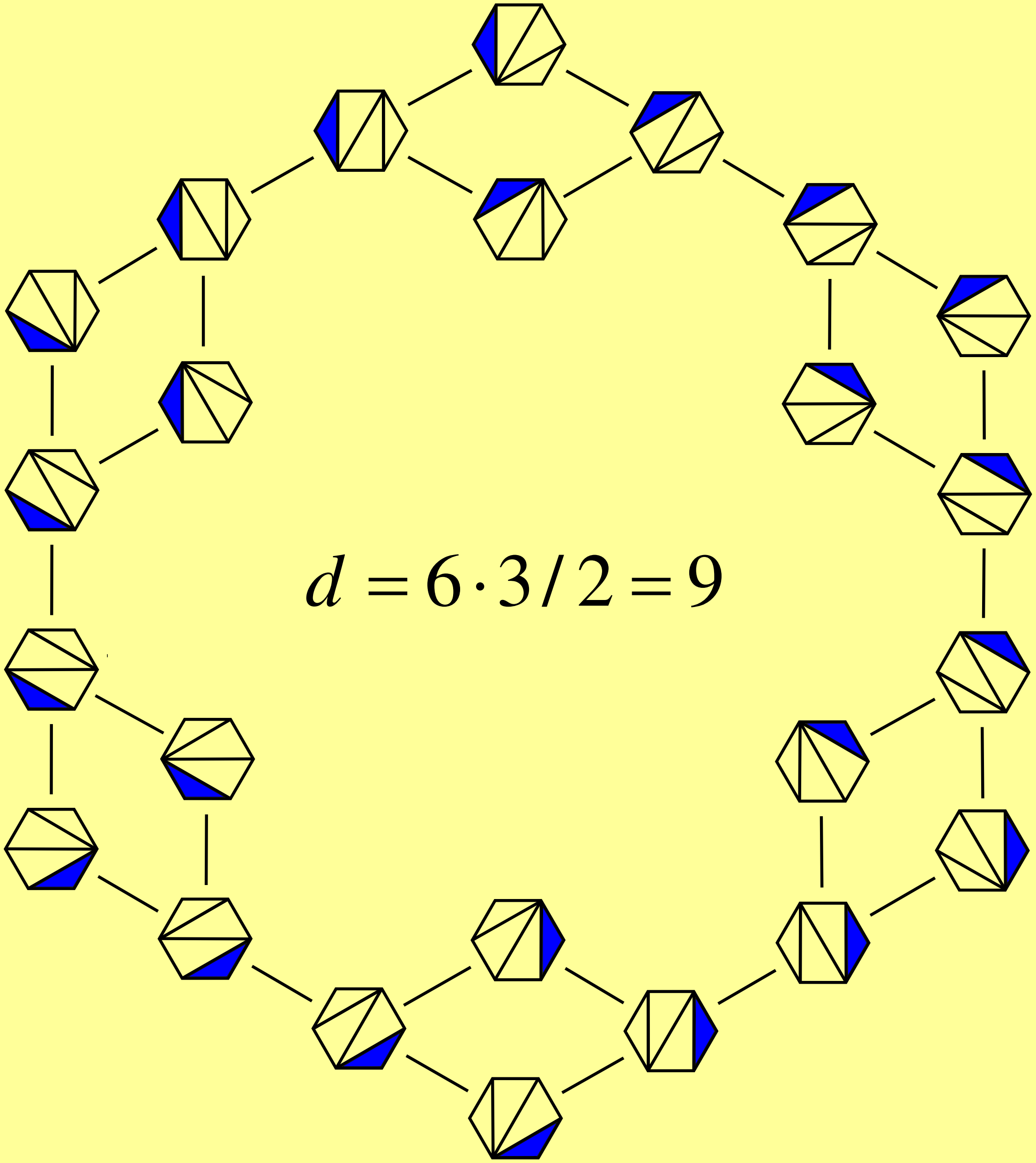
- **<u>Theorem:</u>** *[A-Firer-Roichman, '09]*
  For $n > 4$:
  (a) The diameter of Γ_n is $n(n-3)/2$.

  The proof involves an action of an affine Weyl group of type $\tilde{C}$.

$\tilde{C}_4$



Γ_6

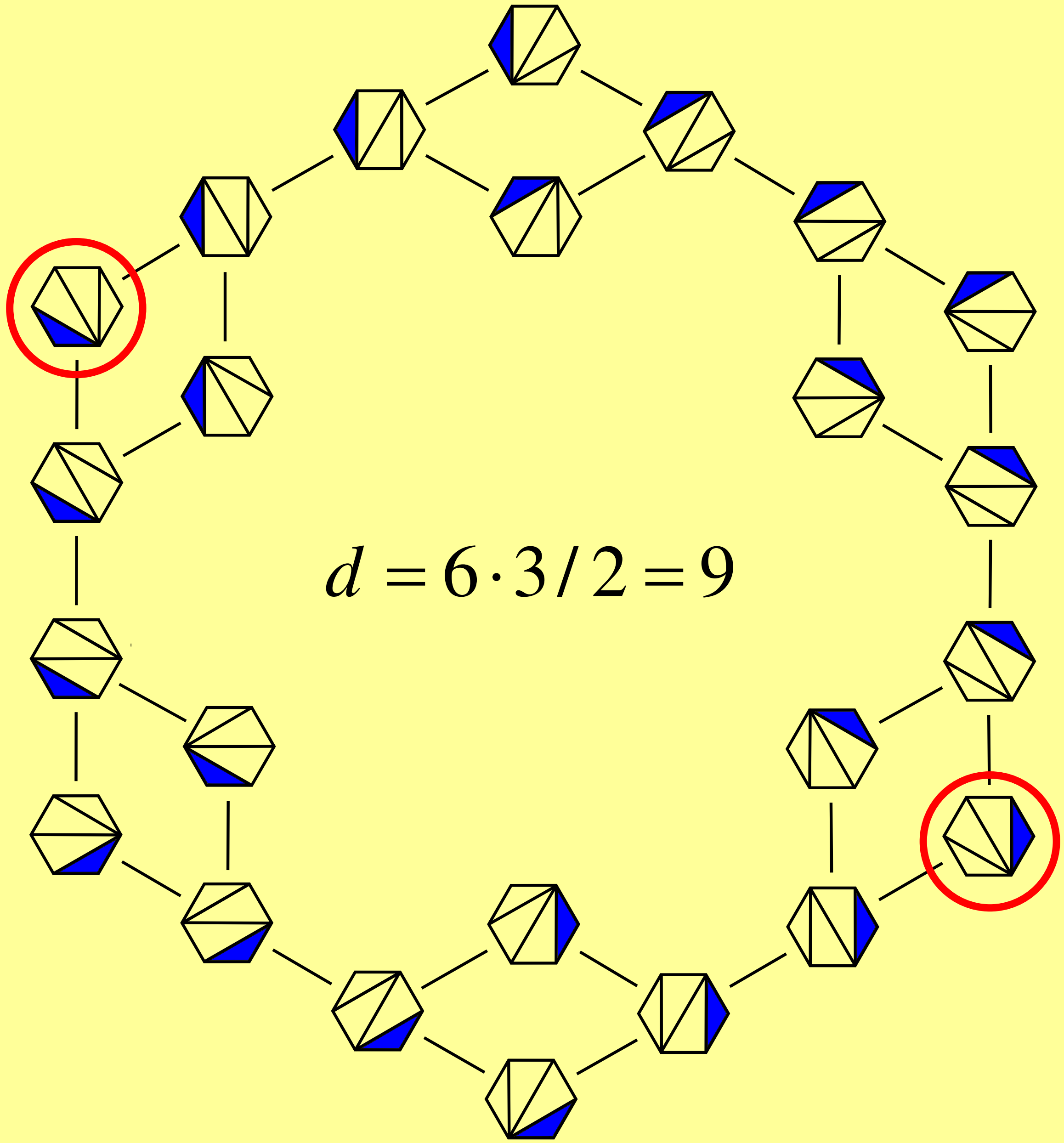


$$d = 6 \cdot 3/2 = 9$$

Diameter of Colored Flip Graph

- **<u>Theorem:</u>** *[A-Firer-Roichman, '09]*
  For $n > 4$:
  (a) The diameter of Γ_n is $n(n-3)/2.$
  (b) Any colored TFT and its reverse are antipodal (distance = diameter).
  (reverse = same triangulation, opposite direction)

Γ_6



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Stanley's Conjecture

- **Observation:** The diameter $n(n-3)/2$ of Γ_n is also the number of diagonals in the n-gon!

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- **Conjecture:** *[Stanley]*

  Each diagonal is flipped (once) in any geodesic between antipodes.

Stanley's Conjecture

- **Main Theorem:** *[A-Roichman, '10]*
  Each diagonal is flipped (once) in any geodesic between a colored triangulation and its reverse.

Arrangements

- A certain hyperplane arrangement
- Arc permutations
- Flip graph and chamber graph

Hyperplane Arrangements

- The hyperplane arrangement of type A_{n-1} :
$$x_i = x_j \qquad (1 \le i < j \le n)$$
corresponds to the complete graph K_n.
- Remove from K_n the edges
$$(1,2), (2,3), \ldots, (n-1, n), (n,1)$$
to get a slightly smaller arrangement H.

Arc Permutations

- **Definition:** A permutation on $1, \ldots, n$ is an arc permutation if each prefix of it forms, as a set, an interval modulo n (with $n = 0$).
- **Example:**

  $\pi = 12543 \quad (n = 5)$ is an arc permutation:

  $$1 \quad 12 \quad 125 = 120 \quad 1254 \quad 12543$$
- $\pi = 125436 \quad (n = 6)$ is not:

  $$125 \neq 120$$

Flip Graph and Chamber Graph

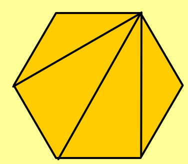
- **Theorem:** The colored flip graph Γ_n is isomorphic to the graph whose vertices are (equivalence classes of) arc permutations, and whose edges connect permutations separated by a unique hyperplane in H (i.e., are in adjacent chambers).

Tableaux

- Counting geodesics
- Truncated Shifted Shape
- Standard Young tableaux
- Geodesics and tableaux

Counting Geodesics

- Let T_0 be a (colored) star triangulation.

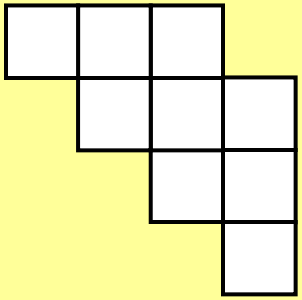


What is the number of geodesics from T_0 to its reverse?

Truncated Shifted Shape

- The truncated shifted staircase shape

$(3,3,2,1):$



Truncated Shifted Tableaux

- The standard Young tableaux of truncated shifted staircase shape $(3,3,2,1)$:

$$\begin{array}{cccc} 1 & 2 & 3 & \\ & 4 & 5 & 6 \\ & & 7 & 8 \\ & & & 9 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 4 & \\ & 3 & 5 & 6 \\ & & 7 & 8 \\ & & & 9 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & \\ & 4 & 5 & 7 \\ & & 6 & 8 \\ & & & 9 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 4 & \\ & 3 & 5 & 7 \\ & & 6 & 8 \\ & & & 9 \end{array}$$

Geodesics and Tableaux

- **Theorem:** The number of geodesics in Γ_n from T_0 to its reverse is twice the number of standard Young tableaux of truncated shifted shape $(n-3, n-3, n-4, \ldots, 1)$.

$$\begin{array}{c|cccc} & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 2 & 4 & \\ 2 & & 3 & 5 & 6 \\ 3 & & & 7 & 8 \\ 4 & & & & 9 \end{array} \quad \longleftrightarrow \quad 13 \quad 14 \quad 24 \quad 15 \quad 25 \quad \ldots \quad 36 \quad 46$$

sequence of flipped diagonals

Fine della lezione.

Grazie per l’attenzione!