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Non-vanishing of periods of automorphic functions

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Abstract. We study a period defined for three automorphic functions on GL(2). We give a lower bound on the average size of such a period which is essentially sharp. We deduce from this bound a non-vanishing result for this period. As an application we obtain non-vanishing of certain L-functions at $\frac{1}{2}$.

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0 Introduction

0.1. Periods and *L*-functions. Relations between periods of automorphic functions, automorphic L-functions, and their special values, is one of the central themes in number theory. Non-vanishing of L-functions is even a more classical subject. There are many examples of an interplay between these topics.

In this note we will discuss non-vanishing of certain periods of automorphic functions on GL(2). We will provide applications of these results to non-vanishing of L-functions.

There are many different techniques for showing non-vanishing of L-functions (see [MM], for example). Most notable are the celebrated positivity argument of Hadamard–de la Vallee Poussin, Selberg's theory of Eisenstein series (see [JS]), as well as more recent works, especially of Iwaniec and Sarnak ([IS]). Those methods usually use deep arithmetic information (like Euler products of L-functions, estimates of exponential sums etc.).

We would like to present a simple example of non-vanishing of certain periods of automorphic functions which does not require any arithmetical input (except of a lattice). We study a period which is defined for a three automorphic representations of GL(2) considered as the automorphic representation of the group $\mathrm{GL}(2)^3$. The period is defined then as the integral over the diagonal imbedding of GL(2). This period was extensively studied in recent years in a connection with a conjecture of H. Jacquet (see [HK]).

We give a *lower* bound on the average size of such a period. This bound is essentially sharp. We deduce from this that for infinitely many representations such a period is nonzero.

For arithmetic lattices (or more precisely for congruence subgroups) this could be translated into non-vanishing of certain L-functions at $\frac{1}{2}$ due to a known relation of these periods to L-functions.

Our method is that of an analysis in automorphic representations which we introduced in [BR]. The method is *elementary* in a sense that it does not use any arithmetical information. The construction we use was suggested by the work [S] of Peter Sarnak.

0.2. Triple products. Let $G = \mathrm{SL}(2, \mathbb{R})$ and $\Gamma \subset G$ be a lattice. For simplicity of the exposition we will mostly discuss only co-compact lattices (for example arising from quaternion algebras). The general case could be treated analogously (see 2.4 where we discuss how to extend our results to this case).

Let $X = \Gamma \backslash G$. It is well known that $L^2(X)$ decomposes into a discrete sum of irreducible unitary representations of $\mathrm{SL}(2, \mathbb{R})$ (since X is compact). All unitary representations of $\mathrm{SL}(2, \mathbb{R})$ are classified: these are representations of principal and complementary series and representations of discrete series. For simplicity we will discuss only class 1 representations (i.e. those which have K-fixed vectors w.r.t the maximal compact subgroup $K = \mathrm{SO}(2)$). These are representations of principal and complementary series. Such representations π_λ are parameterized by a number $\lambda \in \mathbb{C}$ (see 1.1) and belong to the principal series ($\lambda \in i\mathbb{R}$) or to the complementary series ($\lambda \in (0, 1]$).

Let $\{\pi_i\}$ be those representations of class 1 which appear in the decomposition of $L^2(X)$ into irreducible representations. Each π_i has a unique K-fixed vector ϕ_i of norm one. Hence $\{\phi_i\}$ is a basis for $L^2(X)^K$. These ϕ_i's are known classically as *Maass forms*. Namely, let $\pi_i = \pi_{\lambda_i}$ be an automorphic representation of class 1 and ϕ_i the (unique) K-fixed vector in it. Such an automorphic function could be viewed as a function on $G/K = \mathfrak{H}$ where it turns out to be an eigenfunction of the hyperbolic Laplacian Δ with the eigenvalue $\mu_i = \frac{1}{4}(1 - \lambda_i^2)$ (see [B]).

To state the problem, we fix one automorphic function as above, ϕ, and consider the function ϕ^2 on X. Since $\phi^2 \in L^2(X)$, we may consider its spectral decomposition with respect to the basis $\{\phi_i\}$. Since ϕ^2 is K-invariant it will have non-zero projections only to representations which have a non-zero K-fixed vector.

We have then:

$$\phi^2 = \sum c_i \phi_i.$$

Here coefficients are the triple scalar products

$$c_i = \langle \phi^2, \phi_i \rangle = \int_X \phi \cdot \phi \cdot \bar{\phi}_i \, dx.$$

Clearly, we can view these as a period of $\phi \cdot \phi \cdot \bar{\phi}_i$, defined on X^3, over the diagonal copy of X.

Later we will explain why these triple products are of interest and how they are related to the theory of L-functions (see also [S], which was our starting point). First we remark that these c_i are small. Namely we showed in [BR] that the coefficients c_i decay exponentially as $\exp\left(-\frac{\pi}{4}|\lambda_i|\right)$.

More precisely, let us introduce (normalized) coefficients $b_i = |c_i|^2 \exp\left(\frac{\pi}{2}|\lambda_i|\right)$. The main result of the paper [BR] was the proof of the following theorem which settled a conjecture of P. Sarnak (see [S]):

Theorem 1. *There exists a constant* $C > 0$ *such that*

$$\sum_{\lambda_i} b_i e^{-|\lambda_i|\varepsilon} \le C \cdot |\ln \varepsilon|^3 \quad \text{as } \varepsilon \to 0.$$

The aim of the present note is to prove that infinitely many of these triple products are nonzero. We establish a lower bound on the average size of the coefficients c_i. This bound is essentially sharp i.e. it differs by a power of logarithm from the upper bound above.

Theorem 2. *There exists a constant* $c > 0$ *such that*

$$\sum_{\lambda_i} b_i e^{-|\lambda_i|\varepsilon} \ge c \cdot |\ln \varepsilon| \quad \text{as } \varepsilon \to 0.$$

Corollary 1. *For any given* ϕ *as above there are infinitely many automorphic functions* ϕ_i *such that* $\int_X \phi \cdot \phi \cdot \bar{\phi}_i \, dx \neq 0$.

Remark. Constants c and C above are effectively computable.

0.3. Non-vanishing of *L*-functions. One of main interests in triple products and their bounds stems from their relation to the theory of automorphic L-functions. This relation exists of course only for arithmetic lattices and cuspidal functions which are eigenfunctions of Hecke operators. In that case we can translate the above Theorems into the language of automorphic forms on adele groups (see [B]).

Let G be $\mathrm{GL}(2)$ or the group of units of a quaternian algebra both defined over $\mathbb{Q}$ and $G(\mathbb{A})$ the corresponding adele group. Recall that any automorphic π has the form $\pi \simeq \pi^\infty \otimes_p \pi_p$, where π^∞ is the archimedean and π_p are non-archimedean components.

We consider the period defined in 0.2. Namely, let π_i, $i = 1, 2, 3$, be infinite dimensional irreducible cuspidal representations of $G(\mathbb{A})$ with the trivial central character. Define the trilinear form

$$I = I_{\pi_1, \pi_2, \pi_3} : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$$

by

$$f_1 \otimes f_2 \otimes f_3 \mapsto \int_{\mathbb{A}^\times G(\mathbb{Q})\backslash G(\mathbb{A})} f_1(g) f_2(g) f_3(g)\, dg.$$

Here $f_i \in \pi_i$.

Taking into account that $\pi \simeq \check{\pi}$ for representations discussed in 0.2 we have $c_i = I_{\pi,\pi,\pi_i}(\phi^0 \otimes \phi^0 \otimes \phi_i^0)$, where ϕ^0 means the $K(\mathbb{A})$-fixed vector in π and ϕ_i^0 in π_i.

The period I is non-zero if it is non-zero for some choice of f_i's.

We will use now the following result of B. Roberts. We formulate only a part of it.

Theorem ([R]). *If for some π the period $I_{\pi,\check{\pi},\sigma}$ is non-zero then $L(\sigma, 1/2) \neq 0$.*

This together with Corollary 1 gives the following (see 2.4 for the proof):

Corollary 2. *For infinitely many cuspidal representations π_i with π_i^∞ of class one the value $L(\pi_i, \frac{1}{2}) \neq 0$.*

The next application is to the non-vanishing of a more complicated L-function. Let $L(\pi_1 \otimes \pi_2 \otimes \pi_3, s)$ be the Garrett triple L-function (see [B] for example). It is also connected with the triple period. Namely, D. Jiang proved the following:

Theorem ([J]). *Let $G \simeq \mathrm{GL}(2)$. The nonvanishing of I_{π_1,π_2,π_3} implies that the value at $s = 1/2$ of $L(\pi_1 \otimes \pi_2 \otimes \pi_3, s)$ is nonzero.*

This together with the Corollary 1 gives the following (see 2.4 for the proof)

Corollary 3. *For a fixed cuspidal representation π there are infinitely many cuspidal π_i with π_i^∞ of class one such that the value $L(\pi \otimes \check{\pi} \otimes \pi_i, \frac{1}{2}) \neq 0$.*

Remarks. 1. Non-vanishing of $L(\pi_i, 1/2)$ for infinitely many i was proven earlier by Y. Motohashi ([M]) using Kuznetsov's formula. His method possibly gives a quantitative result as well. Non-vanishing for $L(\pi_i, 1/2)$ also can be proven by other means such as those in [IS].

2. The main interest in the non-vanishing of triple periods stems from the conjecture of H. Jacquet which suggests that $L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2}) \neq 0$ if and only if (an appropriate) triple period is nonzero. It is proven in many cases (see [HK] and [J]).

3. Recently Iwaniec and Sarnak proved a remarkable result on non-vanishing of $L(\pi_k, \frac{1}{2})$ for holomorphic cusp forms π_k of weight k (see [IS]). Namely, they showed that for at least 50% of π_k holds the bound $|L(\pi_k, \frac{1}{2})| \gg |\ln(k)|^{-2}$ as $k \to \infty$. Their proof uses sophisticated number theory.

We emphasize that we do not use any arithmetic information and in particular Corollary 1 is true for ϕ's which are not Hecke eigenfunctions.

4. Our method fails in an attempt to prove non-vanishing of a similar L-function $L(\pi_1 \otimes \pi_2 \otimes \pi_i, \frac{1}{2})$ for the case $\pi_1 \not\simeq \check{\pi}_2$. In particular our method does not naturally

apply to non-vanishing of $L(\pi, \frac{1}{2})$ for holomorphic cuspidal representations. In that sense we also use positivity in a crucial way.

5. Finally, we would like to remark on the *upper* bound following from the Theorem 1. This bound is implicitly contained in [BR], but it is not emphasized enough since the main objective there was to give bounds on Fourier coefficients of cuspidal functions. Theorem 1 gives a bound for the second moment of the usual Rankin-Selberg L-functions $L(\pi_1 \otimes \pi_2, s)$ on the critical line: $\int_0^T |L(\pi_1 \otimes \pi_2, 1/2 + it)|^2 \, dt \ll T^{2+\varepsilon}$.

Similar result can be obtained from the approximate functional equation. We note, however, that our proof does not use the explicit form of the functional equation for the involved Eisenstein series (i.e. that the constant term of these Eisenstein series is expressed in terms of the Riemann zeta function). In particular our proof gives non-trivial bounds in non-arithmetic cases as well.

1 Analytic continuation

1.1. Analytic vectors and their analytic continuation. Proof of the Theorem 2 is based on the principle of analytic continuation of vectors in a representation of G. We discussed it in [BR] in more details and here will only remind the idea.

Let $G = \mathrm{SL}(2, \mathbb{R})$ and (π, G, V) be a continuous representation of G in a topological vector space V. A vector $v \in V$ is called *analytic* if the function $\xi_v : g \mapsto \pi(g)v$ is a real analytic function on G with values in V. This means that there exists a neighborhood U of G in its complexification $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ such that ξ_v extends to a holomorphic function on U. In other words, for the elements $g \in U$ we can unambiguously define the vector $\pi(g)v$ as $\xi_v(g)$, i.e., we can extend the action of G to a somewhat larger set.

We consider a typical representation of G—a representation $(\pi_\lambda, G, \mathfrak{D}_\lambda)$ of the principal series. Namely, fix $\lambda \in \mathbb{C}$ and consider the space $\mathfrak{D}_\lambda$ of smooth homogeneous functions of degree $\lambda - 1$ on $\mathbb{R}^2 \backslash 0$, i.e., $\mathfrak{D}_\lambda = \{\phi \in C^\infty(\mathbb{R}^2 \backslash 0) : \phi(ax, ay) = |a|^{\lambda-1} \phi(x, y)\}$; we denote by π_λ the natural representation of G in the space $\mathfrak{D}_\lambda$.

The restriction on S^1 gives an isomorphism $\mathfrak{D}_\lambda \simeq C^\infty_{even}(S^1)$ and for basis vectors of $\mathfrak{D}_\lambda$ one can take the vectors $e_k = \exp(2ik\theta)$. If $\lambda = it$, then $(\pi_\lambda, \mathfrak{D}_\lambda)$ is a unitary representation of G with the invariant norm $\|\phi\|^2 = \dfrac{1}{2\pi} \int_{S^1} |\phi|^2 \, d\theta$.

Consider the vector $v = e_0 \in \mathfrak{D}_\lambda$. We claim that v is an analytic vector and we want to exhibit a large set of elements $g \in G_{\mathbb{C}}$ for which the expression $\pi(g)v$ makes sense. The vector v is represented by the function $(x^2 + y^2)^{(\lambda-1)/2} \in \mathfrak{D}_\lambda$. For any $a > 0$ consider the diagonal matrix $g_a = \mathrm{diag}(a, a^{-1})$. Then

$$\xi_v(g_a) = \pi_\lambda(g_a)v = (a^2 x^2 + a^{-2} y^2)^{(\lambda-1)/2}.$$

The last expression makes sense as a vector in $\mathfrak{D}_\lambda$ for any *complex* a such that

$|\arg(a)| < \frac{\pi}{4}$ (since in this case $\Re(a^2x^2 + a^{-2}y^2) > 0$). Hence, we see that the function ξ_v analytically extends to the subset $I = \left\{g_a : |\arg(a)| < \frac{\pi}{4}\right\} \subset \mathrm{SL}(2, \mathbb{C})$.

As g approaches the boundary of I, the vector $\pi(g)v \in \mathfrak{D}_\lambda$ has very specific asymptotic behavior that we will use in order to get an information about this vector.

1.2. The method. We describe here the idea behind the proof of Theorem 2.

Let $L_i \subset L^2(X)$ be the space corresponding to the automorphic function ϕ_i as above (see 0.2). Let $pr_i : L^2(X) \to L_i$ be the orthogonal projection. Since the function ϕ^2 is K-invariant and there is at most one K-fixed vector in each irreducible representation of $\mathrm{SL}(2, \mathbb{R})$, we have $pr_i(\phi^2) = c_i\phi_i$.

Since the G-action commutes with the multiplication of functions on X,

$$pr_i((\pi(g)\phi)^2) = pr_i(\pi(g)(\phi^2)) = c_i\pi_i(g)\phi_i.$$

By the principle of analytic continuation, the same identity holds for the complex points $g \in I$ (see 1.1). Since all the spaces L_i are orthogonal, we get the following basic relation for the complex points g:

$$\|(\pi(g)\phi)^2\|^2 = \sum_i |c_i|^2 \|\pi_i(g)\phi_i\|^2. \tag{1.1}$$

Here $\|\cdot\| = \|\cdot\|_{L^2}$ denotes the L^2-norm in $L^2(X)$.

It is important that in (1.1) we deal with complex points g and for such g the operators $\pi(g)$ are *non-unitary*. As a result, relation (1.1) gives a non-trivial information.

Now, consider the behavior of the function $(\pi(g)\phi)^2$ near the boundary of I. Take $\varepsilon > 0$ and an element $g_\varepsilon \in I$ which is approximately at the distance ε from the boundary of I. For example, set $g_\varepsilon = \mathrm{diag}(a_\varepsilon^{-1}, a_\varepsilon)$ for $a_\varepsilon = \exp\left(\left(\frac{\pi}{4} - \varepsilon\right)i\right)$.

With shorthand notations $v_\varepsilon = \pi(g_\varepsilon)e_0$ and $\phi_\varepsilon = v(v_\varepsilon)$ formula (1.1) becomes

$$\|\phi_\varepsilon^2\|^2 = \sum |c_i|^2 \|\phi_{i,\varepsilon}\|^2. \tag{1.2}$$

Our goal is to give an *lower* bound on the left hand side of (1.2) and a *upper* bound of each of the $\|\phi_{i,\varepsilon}\|^2$ as $i \to \infty$ and $\varepsilon \to 0$. The later problem is invariantly defined in terms of representation theory and so can be computed in any model of the representation π_i (e.g., in $\mathfrak{D}_{\lambda_i}$). A direct computation gives

$$\|\phi_{i,\varepsilon}\|^2 \leq C \cdot |\ln \varepsilon| \cdot \exp\left(\left(\frac{\pi}{2} - \varepsilon\right)|\lambda_i|\right) \quad \text{for some } C > 0. \tag{1.3}$$

On the other hand, $\mathrm{vol}(\Gamma\backslash G) \cdot \|\phi_\varepsilon^2\|^2 \geq \|\phi_\varepsilon\|^4$, by Cauchy-Schwartz, and a simple computation shows that

(1.4) $\|\phi_\varepsilon\|^2 \geq c|\ln \varepsilon|,$

with some $c > 0$.

Bounds 1.3 and 1.4 imply Theorem 2. (see 2.2)

2 Proofs

2.1. We restate bounds claimed in 1.3 and 1.4.

Proposition. *Let* (π, G, L) *be an irreducible unitary representation of* $\mathrm{SL}(2, \mathbb{R})$ *and* $v \in L$ *a unit K-fixed vector. Consider* g_ε *and* $v_\varepsilon = \pi(g_\varepsilon)v$ *as above. Then*

(1) *There exists* $c > 0$ *such that* $\|v_\varepsilon\|^2 \geq c|\ln(\varepsilon)|$ *as* $\varepsilon \to 0$.

(2) *There exists* $C > 0$ *such that if* $\pi \simeq \pi_\lambda$ *is a representation of the principal series, then* $\|v_\varepsilon\|^2 < C|\ln \varepsilon| \exp\left(\left(\frac{\pi}{2} - \varepsilon\right)|\lambda|\right)$ *for any* λ *and* $\varepsilon < 0.1$.

2.2. *Proof of Theorem 2.* We have the following basic relation (1.2)

$$\|\phi_\varepsilon^2\|^2 = \sum |c_i|^2 \|\phi_{i,\varepsilon}\|^2.$$

Using (1) in the Proposition we have the lower bound:

$$\|\phi_\varepsilon^2\|^2 \geq \mathrm{vol}(\Gamma\backslash G)^{-1} \cdot \|\phi_\varepsilon\|^4 \geq \mathrm{vol}(\Gamma\backslash G)^{-1} \cdot c(\ln(\varepsilon))^2.$$

Using (2) in the Proposition we arrive at

$$C \cdot \sum |c_i|^2 \cdot |\ln \varepsilon| \exp\left(\left(\frac{\pi}{2} - \varepsilon\right)|\lambda_i|\right) \geq \mathrm{vol}(\Gamma\backslash G)^{-1} \cdot c(\ln(\varepsilon))^2$$

which is equivalent to Theorem 2.

2.3. *Proof of Proposition.* What we claim in the Proposition could be computed in any model of the representation π. We use the model $\mathfrak{D}_\lambda$ as in 1.1. Hence the vector v_ε is given by

$$v_\varepsilon(x, y) = (a^2x^2 + a^{-2}y^2)^{(\lambda-1)/2},$$

with $\arg(a) = \frac{\pi}{4} - \varepsilon$. Let $Q_\varepsilon(x, y) = a^2x^2 + a^{-2}y^2$. Then $\|v_\varepsilon\|^2 = \int_{S^1} |Q_\varepsilon^{(\lambda-1)/2}|^2\, dt$. Let $(x, y) = t \in S^1$ and denote $m_\varepsilon(t) = |Q_\varepsilon(t)|$ and $a_\varepsilon(t) = \arg(Q_\varepsilon(t))$. We have $\|v_\varepsilon\|^2 = \int_{S^1} e^{|\lambda|a(t)} m_\varepsilon^{-1}(t)\, dt$. It is easy to see that for $|t| < 0.1$ we have the following $c_1|t - i\varepsilon| \geq$

$m_\varepsilon(t) \geq c_2|t - i\varepsilon|$ and for $|t| \geq 0.1$ $m_\varepsilon(t) \leq c_3$. Hence $\int_{S^1} m_\varepsilon^{-1}(t)\,dt = c|\ln(\varepsilon)|$, which the claim in (1). Claim in (2) follows from this and Cauchy-Schwartz since $a_\varepsilon(t) \leq \pi/2 - 2\varepsilon$.

2.4. *Proof of Corollaries 2 and 3.* If Γ is co-compact this is simply a reformulation of the Theorem 2. However if Γ is not co-compact we have an extra term arising from the Eisenstein series. We claim that this term is negligible.

We assume that $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and hence there is one cusp and we denote $E(s)$ the corresponding Eisenstein series. We denote $b(t) = \left|\left\langle \phi^2, E\left(\frac{1}{2} + it\right)\right\rangle\right|^2 \exp\left(\frac{\pi}{2}|t|\right)$. We have then as in 1.2:

$$\sum_{\lambda_i} b_i |\ln\varepsilon| e^{-|\lambda_i|\varepsilon} + \int_{-\infty}^{\infty} b(t)|\ln\varepsilon| e^{-|t|\varepsilon}\,dt \geq \|\phi_\varepsilon^2\|^2 \geq c|\ln\varepsilon|^2.$$

We claim that

$$(^*) \qquad \int b(t) e^{-|t|\varepsilon}\,dt = O(1), \quad \text{as } \varepsilon \to 0.$$

The bound $(^*)$ immediately implies claims in Corollaries.

The bound $(^*)$ follows from a bound on the usual Rankin-Selberg L-function $L(s) = L(\phi \otimes \phi, s)$. Namely, $b(t) = g(t)|L(\phi \otimes \phi, s)/\zeta(2s)|^2$. Here ζ is the Riemann zeta function and the function $g(t)$ has an explicit expression in terms on Γ-functions (see [B]) from which we infer that $g(t) = 1/t^2 \cdot (1 + o(1))$ as $t \to \infty$. Hence $(^*)$ would follow from a bound

$$(^{**}) \qquad \int_T^{2T} |L(1/2 + it)|^2\,dt \ll T^A$$

with any $A < 2$. This could be shown by noting that $L(s) = L_{sym}(\phi, s)\zeta(s)$, where $L_{sym}(\phi, s)$ is the symmetric square L-function. $L_{sym}(\phi, s)$ is of order three L-function which has three Γ-factors at infinity. Hence the integral $\int_T^{2T} |L_{sym}(\phi, 1/2 + it)|^2\,dt$ could be bounded, by the use of approximate functional equation, by $T^{3/2}$ as in [Iw]. Using the classical Weyl bound $|\zeta(1/2 + it)| \ll t^{1/6}$ and taking into account that $|1/\zeta(1 + it)| \ll \ln t$ we obtain $A = 11/6$ in $(^{**})$.

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References

[B] Bump, D.: Automorphic forms and representations. Cambridge University Press, 1997

[BR] Bernstein, J., Reznikov, A.: Analytic continuation of representations. Ann. of Math. **150** (1999), 329–350

[G6] Gelfand, I., Graev, M., Piatetski-Shapiro, I.: Generalized Functions, 6. Saunders, 1967

[HK] Harris, M., Kudla, S.: The central critical value. Ann. of Math. **133** (1991), 605–672

[Iw] Iwaniec, H.: On the order of vanishing of modular *L*-functions. Sem. Theorie Nombres, Bordeaux **2** (1990), 365–376

[IS] Iwaniec, H., Sarnak, P.: The non-vanishing of central values. Preprint, 1998

[J] Jiang, D.: Nonvanishing of the central critical value. Internat. Math. Res. Notices **2** (1998), 73–84

[JS] Jaquet, H., Shalika, J.: A non-vanishing theorem for zeta functions of GL_n. Invent. Math. **38** (1976), 1–16

[M] Motohashi, Y.: Spectral mean values of *L*-functions. J. Number Theory **42** (1992), 258–284

[MM] Murty, R., Murty, K.: Non-vanishing of *L*-functions and applications. Prog. in Math. 157. Birkhäuser, 1997

[R] Roberts, B.: Nonvanishing of GL(2) *L*-functions at 1/2. Math. Ann. **312** (1998), 575–597

[S] Sarnak, P.: Integrals of products of eigenfunctions. Internat. Math. Res. Notices **6** (1994), 251–260

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