Vol. I.

p. 160 l. -12 Specializing $\lambda_\gamma \mapsto 0$

l. -3, -1 $\sigma_\gamma(g_t)$ instead of $\rho_\gamma g_t$

p. 170

Corollary 2.3.12 requires a condition such as every left ideal contains a minimal left ideal. Clearly it is false when R has socle 0.

p. 308

2.1.10$'$ Given $L < R$ and $a \in R$ define $La^{-1} = \{r \in R : ra \in L\}$. If L is a maximal left ideal in R then so is La^{-1}, for any $a \neq 0$ in R, and core$(La^{-1}) \supseteq$ core (L).

p. 312

2.5.4$'$ A more explicit way of viewing Theorem 2.5.22. Say a ring R is *special* if there is a non-nilpotent element a such that every nonzero ideal of R contains a power of a. Prove that every prime special ring R satisfying the conditions of Theorem 2.5.22 is primitive. (Hint: The left ideal $\sum(l - a^i)$ is comaximal with every nonzero two-sided ideal.) Consequently any ring R satisfying the conditions of Theorem 2.5.22 and having no nonzero nil ideals is a subdirect product of special primitive rings. (Hint: Requires the proof of proposition 2.6.7.)

2.10.0 The following properties of a module M are equivalent: (i) M is injective; (ii) Any map $f : N \to E$ satisfies $f(N) \subseteq M$, where M is viewed as a submodule of its injective hull E; (iii) As in (ii), but for any essential extension E of M. Note that complications arise when we dualize condition (iii) to check projectivity, since M could be a non-projective module without any proper covers.

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2.5.39 Suppose that R is a domain such that for any a, b in R there is $n = n(a, b)$ such that $[a, [a, \dots, [a, b] \dots]] = 0$ (taken n times.) Then R is commutative.

2.5.40 Here is an interesting application of exercise 39, due to Avram Klein, extending an earlier result of Makar-Limanov: Suppose R is a noncommutative domain. Then the polynomial ring $R[\lambda, \mu]$ contains a free multiplicative semigroup. (Hint: Write $Z = Z(R)$, and ad a for $[a, \_]$. Take a, b such that $(\mathrm{ad}^n a)b \neq 0$ for all n. (Such a, b exist by exercise 39.) The claim is that $a' = a + \lambda$ and $b' = b + (a + \lambda)\mu$ generate a free semigroup. Indeed otherwise there are monomials $f \neq g$ in noncommuting indeterminates such that $f(a', b') = g(a', b')$; matching degrees first in μ and then in λ enables one to assume $\deg_Y f = \deg_Y g = d$ and $\deg_X f = \deg_X g$. One may assume f ends in Y and g ends in X. Write $f(X, Y - X\mu - g(X, Y - X\mu) = \sum h_i \mu^{d-i}$, where $h_i \in ZX, Y$. Then $h_1(a+\lambda, b) = 0$, and by induction on $\deg h_1$ one can show $(\mathrm{ad}\ ^n a)b = 0$. The trick is to rewrite $h_1 = \tilde{h}_1(X, [X, Y]) + \beta Y X^n$, using the equation $XY = YX + [X, Y]$, and note $\tilde{h}_1(a + \lambda, [x, b]) = 0$; clearly $\deg \tilde{h}_1 \leq \deg h - 1$, providing the inductive step.

p. 324

2.8.2$'$ Given R-modules M, N, say M is *N-projective* if every map $M \to N/L$ lifts to a map $M \to N$. (Thus "projective" means N-projective for every N.) Prove that if M is N-projective and $\pi : M' \to M$ is any cover (i.e. $K = \ker \pi$ is small in M' then for any map $f\colon M' \to N$ one has $fK = 0$. (Hint: f induces an epic $\bar{f}\colon M = M'/K \to N/f(K)$, so there is $g\colon M \to N$ such that $\eta g = f$, where $\eta\colon N \to fK$ is the canonical map. Let $h = g\pi$. For any x in M', $(f - h)x = fy$ for some y in K, so $(f-h)(x-y)=0$.

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2.8.33 In Schanuel's lemma, show that there is an isomporphism $P_1 \oplus K_2 \to K_1 \oplus P_2$ which lifts to an isomorphism $P_1 \oplus P_2 \to P_1 \oplus P_2$.

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Artinian modules are semi-LE. The following exercises sketch a proof of the Camps-Dicks Theorem that if M is an Artinian R-module then $T = \mathrm{End}_R(M)$ is a semilocal ring.

2.9.24 For M Artinian, iff $f\colon M \to M$ is monic then f is an isomorphism. (Hint: $f^iT = f^{i+1}T$ for some i.) **25** (Does not require M Artinian.) For any f, g in T show $\ker(f - fgf) = \ker f \oplus \ker(1-gf)$. (Hint: $x = (1-gf)x + gfx$.)

2.9.26 T is semilocal. (Extensive hint: Let $J = \mathrm{Jac}(T)$. One must show $\bar{T} = T/J$ is semisimple Artinian as a module over itself.

Step 1. Define a relation $<$ on $\mathcal{L}(M)$ by saying $K < N$ if K is a proper submodule of N. Looking at complements, show $\mathcal{L}(M)$ satisfies ACC with respect to this relation.

Step 2. Suppose $f \in T \setminus J$. For any $g \in T$ such that $1 - gf$ is not invertible, one has 1-gf not monic, and thus $\ker(f - fgf) > \ker f$.

Step 3. Let $\mathcal{S} = \{f \in T \setminus J : \bar{f}$ is idedmpotent (in $\bar{T}$) and $\bar{T}/(1-\bar{f})$ is semisimple Artinian. $1 \in \mathcal{S}$. Take f in $\mathcal{S}$ with $\ker f$ maximal with respect to $<$, and $g \in T$ with $gf \notin J$, such that $\ker gf$ is maximal possible with $\ker(gf - fhgf) > \ker gf$, so $gf - fhgf \in J$, i.e. $\overline{gf} = \overline{fhgf}$. Conclude that $\overline{fhgf}$ is idempotent in $\overline{fTf}$, so $\overline{f - fhgf}$ is idempotent. Furthermore $\overline{Tgf}$ is simple, since for any $a \in T$ for which $\overline{agf} \neq 0$ one has h satisfying $\overline{gf} = \overline{fhagf} \in \overline{Tgf}$. Conclude $\overline{T}\overline{(1-(f-fhgf))} = \overline{T}(1-\overline{f}) \oplus \overline{Thgf}$ is semisimple Artinian, but $f - fhgf \in \mathcal{S}$, so $f - fhgf \in J$. Hence $\overline{T} = \overline{T(1-(f-fhgf)}$ is semisimple Artinian.)

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Matlis' conjecture.

Suppose $M \oplus N$ is a direct sum of indecomposable injectives. Is M a direct sum of indecomposable injectives? This is known as Matlis' conjecture (and sometimes is asked more generally for LE-modules, in view of exercise 2.10.9). The next two exercises give some sample results along these lines, under the above hypothesis.

2.10.25. M is a sum (not necessarily direct) of indecomposable injectives. (Hint: Any x in M is contained in a finite direct sum of indecomposable injectives. But the kernel of the natural projection $E \to M$ intersects Rx trivially, so is 0, i.e. x is contained in a copy of E inside M.)

2.10.26. Prove Matlis' conjecture when R is left Noetherian. (Hint: M has a large submodule which is a direct sum of indecomposable injectives and hence is injective, and thus equals M.)

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New Exercise $\mathbf{0'}$ If L is a maximal left ideal of R and $a \in R \setminus L$ then La^{-1} is a maximal left ideal of R.

Volume II

p. 176 **0.** R has subexponential growth iff $\lim_{n\to\infty}(\log G_S(n))/n = 0$.

(after corollary 6.2.25′) Surprisingly a PI-ring R can have a nilpotent ideal N for which GK-dim $R/N >$GK-dim R, as described in exercises 6.3.20ff .

Digression 6.3.28′ For use in invariant theory one would like to adjoin the characteristic coefficients of *all* the elements of R (not just "enough"); let us call this $\tilde{C}$, and let $\tilde{T}(R) = R\tilde{C} \subseteq Q$. We shall see now when C is a field that also $\tilde{C}$ is affine over C, and thus Noetherian; since $\tilde{T}(R)$ is f.g. over $\tilde{C}$ (as in (iv)), it follows that both $\tilde{T}(R)$ and its center are affine over C. In fact, modulo a result from commutative algebra to be quoted in the proof, we have

Proposition. *(Notation as above) $\tilde{C}$ is f.g. as C'-module, and thus is affine over C.*

Proof. We view $R = C\{r_1, \dots, r_t\}$ in $Q \otimes_C K \approx M_n(K)$, where K is the algebraic closure of $Z(Q)$. Then each r_k can be identified with the matrix $(\xi_{ij}^{(k)}$, and the characteristic polynomial of r_k is $\det(\lambda \cdot 1 - (\xi_{ij}^{(k)})$, whose coefficients are certainly in $C[\xi_{ij}^{(k)} : 1 \leq i,j \leq n,\ 1 \leq k \leq t]$. Now take any $\tilde{c}$ in $\tilde{C}$. Writing $c = \sum c_u \alpha_1^{u_1} \dots \alpha_t^{u_t}$, where each α_i is a characteristic coefficient of a suitable element a_i of R, we see each a_i is integral over C' (by Shirshov's Theorem), so its conjugates are also integral over C', and thus each α_i is integral over C', i.e. each α_i is contained in the integral closure $\bar{C}'$ of C' in $C[\xi_{ij}^{(k)}]$. Hence $c \in \bar{C}'$, thereby proving $\tilde{C} \subseteq \bar{C}'$. But $\bar{C}'$ is f.g. as a C'-module, by a standard result of commutative algebra, cf. Zariski-Samuel, vol. 1, p. 267 or Matsamura [ ], p. 240. Hence $\bar{C}'$ is a Noetherian C'-module, so its submodule $\tilde{C}$ is Noetherian and in particular f.g. $\square$.

The Gelfand-Kirillov dimension modulo nilpotent ideals.

The next few exercises describe how the GK dimension passes modulo a nilpotent ideal.

6.3.20 Suppose M is an $R-T$ bimodule, where R, T are F-algebras. Let $W = \begin{pmatrix} R & M \\ 0 & T \end{pmatrix}$, cf. Example 1.9. Then GK-dim $W \leq$GK-dim R+GK-dim T. (Hint: Any finite dimensional subspace is contained in a suitable space $V = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A, B, C are respective finite dimensional subspaces of R, M, T. Then $V^n \subseteq \begin{pmatrix} A^n & A^nBC^n \\ 0 & C^n \end{pmatrix}$; take logarithms mod n and then let $n \to \infty$.

6.3.21 If $I, J \lhd R$ with $IJ = 0$ then GK-dim $R \leq$GK-dim R/I+GK-dim R/J. (Hint: Apply Proposition 6.3.14 to exercise 20.) **6.3.22** If $N \lhd R$ and $N^t = 0$ then GK-dim $R/N \leq t\cdot$GK-dim R.

6.3.23 Let $R = F\{X_1, X_2\}/ < X_2 >^m$. Then $R/\mathrm{Nil}(R) \approx FX_1 = F[X_1]$ has GK-dimension 1, but GK-dim $(R) = m$. (Hint: By exercise 22; also there are $\binom{n}{m-1}$ monomials of degree $m-1$ in X_2.

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40. If k is relatively prime to index(R) then $R^{\otimes k} \approx R$. (Hint: Reduce to symbols via Merkurjev-Suslin.)

41. If D is a division algebra of degree p^t, p prime, then there is a field $L \supseteq F = Z(D)$ such that $[L : F]$ is prime to p and $D \otimes L$ has a maximal subfield E_0 with a chain $E_0 \supseteq E_1 \dots \supseteq E_t = L$ in which each $[E_i : E_{i+1}] = p$. (Hint: In proposition 7.2.11 take $E_0 = KL$ in E, noting Gal(E/L) is solvable.)

42. If p divides $m = $ index(R) then index($R^{\otimes p}$) divides m/p. (Hint: Assume R is a division ring and compute index($R_1^{\otimes p}$) where $R_1 = C_{D\otimes L}(E_1)$, notation as in exercise 41.)

Group Algebras satisfying a PI. Passman [89] has found a much shorter proof of the Isaacs-Passman Theorem, that the group algebra $F[G]$ satisfies a PI (where $\mathrm{char}(F) = 0$) iff G has an Abelian subgroup of finite index; we present the proof here. By Theorem 8.1.52, one may assume $G = \Delta(G)$.

Lemma 8.1.53. *Suppose $G = \triangle(G)$, and W is a finite central subgroup of G. If $0 \neq \alpha \in F[W]$ and $\alpha f(X_1, \dots, X_n)$ is a multilinear identity of $F[G]$ (over $F[W]$) then G has subgroups $A \supseteq B$ such that B is finite, A has finite index in G, $B \subseteq Z(A)$, and A/B is commutative.*

Proof. One is done unless $G' \not\subseteq W$. Take x, y in G such that $z = xyx^{-1}y^{-1} \notin W$. Let $C = C_G(x) \cap C_G(y)$, which has finite index in G.

We proceed by induction on the number of monomials $m = m(f)$ of f. Clearly $m \geq 2$, so one may assume that X_1 occurs before X_2 in some but not all monomials of f. Let g be the sum of those monomials in which X_1 occurs before X_2, and $h = f - g$. Then

$$0 = yx\alpha f(c_1, \dots, c_n) = yx\alpha(g(c_1, \dots, c_n) + h(c_1, \dots, c_n))$$

and

$$0 = \alpha f(xc_1, yc_2, \dots, c_n) = xy\alpha g(c_1, \dots, c_n) + yx\alpha h(c_1, \dots, c_n)$$

Subtracting these two equations yields

$$0 = \alpha(xy - yx)g(c_1, \dots, c_n) = \alpha yx(z - 1)g(c_1, \dots, c_n).$$

If $z \notin C$ then the coefficient of 1 yields $\alpha yxg(c_1, \dots, c_n) = 0$, so $g(c_1, \dots, c_n) = 0$ for all c_i in C; replacing G by C and f by g, one is done by induction.

If $z \in C$ then $z \in Z(C)$ has finite period, by corollary 8.1.33, so $W' = W < z >$ is a finite subgroup of C, and we replace G by C, W by W', and f by g, and again are done by induction.

Proof of the Isaacs-Passman Theorem. Take A, B, G as in Exercise 27. Replacing G by A, we may assume G' is finite and central; take HG of finite index, with H' minimal possible. Then $K' = H'$ for any subgroup K of H having finite index. For any prime ideal P of $F[H]$ let S be the central localization of $F[H]/P$. S is generated over its center by the image of a finite number of elements of H, whose common centralizer C thus has finite index in H. Thus the image of C in S is central. Hence $H' = C' \subseteq 1 + P$. Since $F[H]$ is semiprime, we conclude $H' = 1$, i.e. H is Abelian.

The original Isaacs-Passman proofs have explicit bounds on the index of the Abelian subgroup in terms of the PI-degree; part of this can be gleaned from Exercise 27.

Exercise 8.1.27 Using an ultraproduct argument, show that the index of the Abelian subgroup of G can be bounded by a function of the PI-degree of $F[G]$.