# Game Theory

**References.**


(1) Berge and Ghoulia Houri, Programming, Games, and Transportation Networks (Nice treatment, but sometimes misleading, and some steps are skipped.)

(2) Berlekamp, Conway, and Guy, Winning Ways for your Mathematical Plays (2 volumes, consecutively paginated, on reserve in the library; we refer to it as BCG)

(3) Brams, Political Game Theory. (On reserve in Economics Library; excellent and gets to the point.)

(4) Nash, John F. Jr., Equilibrium points in n-person games, PNAS 36 (1950), 48–49. Remarkable 1-page paper which establishes the Nash equilibrium.

(5) Nash, John F. Jr., The bargaining problem, Econometria 18 (1950), 155–162. Another pearl, which reduces a deep mathematical result to almost a triviality.

(6) Nash, John F. Jr., Non-cooperative games, Annals of Math. 54 (1951), 286–295 the Nash equilibrium with a longer argument but easier math.

(7) C. van de Panne, Linear Programming and Related Techniques (Easy to understand, but slow-moving; gives a very thorough description of the simplex method, although I did not find proofs of the assertions)

(8) von Neumann and Morgenstern, The Theory of Games and Economic Behavior (1944. The classic)

(9) Myerson, Roger B., Game Theory: Analysis of Conflict, Harvard (1991)

(10) Robinson, J., An iterative method of solving a game, Annals of Math. 54 (1951), 296—301.


## I Introduction and basic concepts

Concerning the outcome, some games have a simple outcome (either win or lose, or possibly draw), and thus only one of two or three possible payoffs,. Other games involve a variable payoff, such as many card games where the payoff is tied to the number of points one accumulates. In a zero-sum game, a win (resp. loss) is counted as a positive (resp. negative) payoff.

Basic question: What is the definition of a game?

*Definition 1.1.* A game (of strategy) is a finite sequence of actions ("moves") taken in some discrete order by a finite number $n$ of players, according to a given set of rules which includes a given *starting position*, and in which the outcome consists of a certain payoff for each of them which depends on the position after each move. (Often the payoff is distributed only at the end of the game, but it convenient to consider payoffs as accumulated at each move). Thus, the payoff is a vector, $(p_1, \dots, p_n)$, where the $i$-th player receives payoff $p_i$. Actually, for various games such as those involving negotiations, each payoff $p_i$ might be a vector of several components, say in $\mathbb{R}^{(m)}$, so the payoff is in $\mathbb{R}^{(mn)}$.) In some games, the payoff might depend on some random event, such as the result of rolling dice.

Unless otherwise stipulated, we also assume "Local finiteness": At each turn, a player must choose his move from a finite number of choices.

Note that sports games, which often involve continuous action, are not games under this definition, unless somehow one separates the moves. It would not be difficult to define continuous games, but we shall not be concerned with that here.

Also at least one famous board game, *Monopoly*, does not satisfy the criterion that there are a finite number of moves until the outcome. Chess does because of the 50 move limit between taking pieces and/or advancing pawns.

Local finiteness excludes the following candidate for a game:

*Definition 1.2.* A non-locally finite game:

A picks a natural number;

B picks a natural number less than that of A;

A picks a natural number less than that of B; and so forth; The game ends when some player reaches 0 (and that player loses). Clearly A can win by starting with 1, but the point here is that in this example $A$ has infinitely many choices in the first move, so there could be arbitrarily long played games, although all have finite length.

*Definition 1.3.* The *length* of a played game is its number of moves.

Although in many games the players move simultaneously, it is convenient to stipulate that moves are made one at a time, and simply to separate a simultaneous move into two (or more) moves by adding the condition that each player is ignorant of the move made by the other players at this stage.

*Example 1.4.* Consider the game "paper, scissors, or stone," (PScSt) where two players choose "paper, scissors, or stone," and the winner is determined by the rule "Scissors cuts paper, stone breaks scissors, and paper covers stone." Even though both players make their move simultaneously, one could consider the first person as writing his move down secretly and then the second player choosing his move. Thus there are two moves made, leading to $3^2 = 9$ possible final positions.

One must distinguish between the abstract game $G$ and the actual game that is played out, which we shall call the *played game arising from* $G$. For example, a world championship match of chess might have 12 played games.

The mathematical theory of games is the attempt to determine the outcome of all possible played games. Since relatively easy games of strategy normally give rise to billions or trillions of played games, the object of the theory is to be able to control the payoff vector, presumably to maximize the payoff of one player (In a competitive game, this could involve minimizing the payoff to other players). A planned sequence of plays is called a *strategy*, and our main goal will be to find optimal strategies. The *value* of a game to a player is the best payoff which one side can obtain regardless of the strategy of the opponent, and an *optimal strategy* is one which will give this value. A player selecting the optimal strategy is *rational.* A game in which the rational strategy of each side leads to the same outcome is called a *strictly determined* game.

The question of rationality is more subtle than at first glance, especially in cooperative games of imperfect information. (cf. Chicken, cf. below) So perhaps a better definition of rational player would be one who knows the payoffs and would like to maximize his/her payoff over the course of time. We shall consider this point later.

Of course the rational strategy, as well as a proper determination of the payoff, might depend on knowledge, such as in quiz shows. Here is another example:

A picks a number $x > 0$; B then picks $y, z > 0$. $B$ wins if $x^2 + y^2 = z^2$; $A$ wins if it was impossible to pick such $y, z$; the game is a draw if there was a solution which

B missed. It is easy to see that A wins if he picks 1 or 2 and B can win otherwise. (Indeed $x^2 = (z+y)(z-y)$ with $x$ odd has the solution $z+y = x^2,\ z-y=1$, i.e. $z = \frac{x^2+1}{2}; y = \frac{x^2-1}{2}$. For $x = 2k$ one can use the Pythagorean triple: let $y = k^2 - 1$ and $z = k^2+1$.

But what if we used $x^n + y^n = z^n$? Until Wiles' solution of Fermat's Conjecture, even the judge would not know if A wins by choosing a number randomly. Thus, we shall call a player *wise* if he can determine his best strategy.

A game has *perfect information* when the moves alternate and, at the time of his move, the player knows all previous moves, and also knows what future moves are available to the other player (e.g. in chess). Otherwise the game has *imperfect information* (e.g. PScSt or poker). Note that in a game with perfect information, the players may not necessarily be wise. Thus the three concepts of "rational, wise, perfect information" all differ.

For any game, we designate an oriented graph whose edges correspond to the different moves, and whose vertices correspond to the position of the game before and after the move (i.e. edge) connecting them. For example the graph of PScSt is

$$\begin{tikzcd}
\text{starting position} & \text{move by First Player} & \text{move by Second Player} & \text{Payoff} \\
 & & (P,P) & (\tfrac{1}{2},\tfrac{1}{2}) \\
 & P \arrow[ur] \arrow[r] \arrow[dr] & (P,Sc) & (0,1) \\
 & & (P,St) & (1,0) \\
 & & (Sc,P) & (1,0) \\
0 \arrow[uuur] \arrow[r] \arrow[dddr] & Sc \arrow[ur] \arrow[r] \arrow[dr] & (Sc,Sc) & (\tfrac{1}{2},\tfrac{1}{2}) \\
 & & (Sc,St) & (0,1) \\
 & & (St,P) & (0,1) \\
 & St \arrow[ur] \arrow[r] \arrow[dr] & (St,Sc) & (1,0) \\
 & & (St,St) & (\tfrac{1}{2},\tfrac{1}{2})
\end{tikzcd}$$

The graph cannot have any cycles, since otherwise the game could give rise to an infinite played game, contrary to the definition. Thus the graph is a tree, called the "game tree," also called *extensive form*. Note that there might be payoffs at various stages, and also the branches of the tree might be made more complicated by the introduction of random events. Note that our definition of "position" must be broad enough to take into account the history of the game, since in chess if a position repeats four times there is a draw. (In fact, without this rule, chess would not be a game under our definition.) Since there are a finite number of vertices, the game tree has a path of maximal length, obviously starting at 0, which is called the "length" of the game. Note that any tree gives rise to a game (once we insert the payoff vectors), so in a sense game theory is equivalent to the theory of trees.

In a game with perfect information, each player knows at each move precisely where he is located in the tree. For example, PScSt is a game with imperfect information, and after the first turn the second player does not know whether he is located at point $P$, $Sc$, or $St$.

Note that even a trivial game such as PScSt has a rather complicated game graph, so the full game graph is not such an applicable tool unless the game is particularly simple. When possible, it is easier to describe the payoff in matrix form

Payoff Matrix for $PScSt$

| | | *Second Player* | | |
|---|---|---|---|---|
| | | Paper | Scissors | Stone |
| | Paper | $(\frac{1}{2},\frac{1}{2})$ | $(0,1)$ | $(1,0)$ |
| *First player* | Scissors | $(1,0)$ | $(\frac{1}{2},\frac{1}{2})$ | $(0,1)$ |
| | Stone | $(0,1)$ | $(1,0)$ | $(\frac{1}{2},\frac{1}{2})$ |

The matrix is also called the *normal form* of the game.

Sometimes one strategy always produces at least as good a payoff then another strategy. In this case we call the first strategy is *dominant* or *dominates*, the second strategy, and we can eliminate the second strategy from consideration.

For example if we changed the rules in PScSt so that St always beat P , we would have the normal form

| | | *Second Player* | | |
|---|---|---|---|---|
| | | Paper | Scissors | Stone |
| | Paper | $(\frac{1}{2},\frac{1}{2})$ | $(0,1)$ | $(0,1)$ |
| *First player* | Scissors | $(1,0)$ | $(\frac{1}{2},\frac{1}{2})$ | $(0,1)$ |
| | Stone | $(1,0)$ | $(1,0)$ | $(\frac{1}{2},\frac{1}{2})$ |

In this game, the dominant strategy for each side is to play stone, and thus the outcome is strictly determined (a draw of stone vs. stone).

**Games as sources of ideas.**

The beauty of game theory lies in the ability to encapsulate some fundamental ideas with a minimum of assumptions, thereby enabling one to formulate

Games come up in many possible forms, and, in addition to helping one to profit in Las Vegas, are aids in economics, finance, politics, negotiations, and enable one to reformulate various mathematical concepts, including number systems and information retrieval. Here is an example from a popular game show.

*Example 1.5.* The host displays 3 doors, and asks the contestant to choose one of them. Behind one of the doors is a choice prize (often an automobile), and behind the other two doors are prizes of lesser value. The contestant chooses one door, say number 1; then the host opens a different door (which does not yield the automobile), say number 2, and says "You are fortunate not to have chosen this door. Now would you like to switch your choice?" At this stage what should the contestant do? What is his chance of being correct?

The reader familiar with probability theory would apply the rule of restricted choice and switch doors. However, a much easier solution is obtained by picking an intuitive strategy for the game. Since one intends to switch doors, choose the door which does *not* contain the prize and then switch doors. If one has guessed correctly at the outset (obviously $\frac{2}{3}$ chance) then by switching one must land on the correct door (since one door is excluded by the guess, and one door is excluded by the host). Thus the strategy of switching doors gives a win $\frac{2}{3}$ of the time, so *not* switching wins only $\frac{1}{3}$ of the time.

Here is an interesting example, due to Nash, of a game that must have a winning strategy, although it is not known in general.

One says two squares on a checkerboard are *connected* if they are adjacent or if they touch on the diagonal going up to the right. Two players alternately choose squares; the object of the first player is to get a connected path from the right edge to the left edge, and the object of the second player is to get a connected path from top to bottom. (This can be formulated equivalently using hexagonal boxes; one says two hexagons are connected if they have a common edge, and the objective again is to obtain a connected path from one end of the board to the other.)

Clearly one player winning will block the other player, and comparing connected components shows that someone must win. One way of stating this formally is to take some maximal connected component of one player. If it does not win, then it is bounded on all sides by the other player, so that giving the component to the other player does not affect the outcome. But this has created a larger connected component, so iterating this procedure, one continues, and someone does not win, eventually the whole board would be given to one player, a contradiction.

PROPOSITION 1.6. *The first player must win at Nash's game with the proper strategy.*

*Proof.* If not then the second player would have a winning strategy. We show this leads to a contradiction by providing a winning strategy for the first player. The first player makes a random first move in square $T_1$, and pretends that he is the second player, ignoring the square $T_1$ unless his strategy calls for him to occupy it, in which case he makes another random move, and so forth. $\square$

THEOREM 1.7. *For any game $G$, there is a bound to the length of any played game (and thus, for $G$ locally finite, there are a finite number of played games arising from $G$*

*Proof.* By contradiction. Assume $G$ gives rise to played games of arbitrarily long length. For any choice of first move, we could define a new game $G_i$, each of whose starting positions is after this corresponding first move. (For example, if the game is chess, there are 20 possible opening moves for White, 16 pawn moves and 4 possible night moves, and each of this could define a new game starting with Black's first move.) By assumption, one of these, call it $G_{1_1}$ must give rise to played games of arbitrarily long length. Continue and get a played game which does not terminate.

One can also devise a proof by constructing a game for the occasion. We devise a game in which each player receives a shekel for each turn played. We shall call a game "promising" if it has a continuation of arbitrarily long length. Obviously at any stage, a player can choose a promising continuation, and this is in his interest. Thus the game goes on indefinitely. □

We have really proved a significant result in graph theory, called the Konigsburg Graph Theorem (Every graph either has paths of infinite length, or else there is a bound for the length of each path).

Theorem 1.7 has a startling corollary.

THEOREM 1.8. *(Perfect Information Theorem) Any game G of perfect information is strictly determined.*

*Proof.* Induction on the length $\ell$ of $G$. Suppose the first player has $n_1$ possible moves. This leads to $n_1$ games starting after the first player's move, each strictly determined, by induction, so the first player simply chooses that game which will produce the best payoff for him. □

In fact, as each player chooses his best move, there will be an *ideal game* For example, chess is a strictly determined game, although nobody knows whether White must win or not since so far we do not have players wise enough to determine the strategy for the ideal game. Nevertheless, given any strictly determined game, in theory at least, we can assign a value for the game, which is the payoff for the ideal game. Likewise. Nash's game is strictly determined.

**Classifying games.**

Different types of games to be discussed below: Geometrical games (including board games), combinatorical games (dice, cards, dominos, shesh-besh), deductive games, optimization games (zero-sum, nonzero-sum) As in many mathematical situations, our task in analyzing games is made easier by sorting out different kinds of games.

*Definition 1.9.* Two games are *equivalent* if they have identical optimal strategies. This would happen with a modification of payoffs by some linear transformation $f(x) = ax + b$ where $a > 0$.

As a special case, we have:

*Definition 1.10.* Two games are *isomorphic* if there is a 1:1 correspondence of their graphs which yield identical payoffs.

In a sense, an isomorphism is just a renaming of the "same" game.

Games are called *competitive* if the payoff of each person depends inversely on the outcome of the others. One case of a competitive game is when the sum of all payoffs are constant, whichever played game one chooses; such a game is called a *fixed-sum* game. For example, in tournaments, the sum of the payoff of a chess game is always 1 (winner gets 1, loser gets 0, or both side get $\frac{1}{2}$ in a draw. A fixed-sum game is *zero-sum* if this sum is zero; this is normally the case with games played for monetary payoff. In a zero-sum game between two players A and B, we usually consider the value in terms of the first player $A$, since the value for $B$ is the negative of the value for $A$.

Games which are not competitive are called *cooperative.*

*Remark 1.11.* Every fixed-sum game is equivalent to a zero-sum game. Indeed if the sum of the game is $c$, then we could invent a new game $G'$ where the payoffs are

$$(p_1 - \frac{c}{n}, \dots, p_n - \frac{c}{n})$$

instead of the payoff $(p_1, \dots, p_n)$ of $G$.

## II. GAMES OF PERFECT INFORMATION

We start with games based on positional play, all of which have perfect information. Their analysis often involves interesting mathematical principles.

**0. Dots.** One of the most famous games, familiar to all since childhood. The players alternate, and anyone filling a box gets an extra turn.

Conway is reputed to have challenged his class in game theory to a round of dots in a $3 \times 3$ board (the simplest nontrivial case). He claimed he could win in the first position and (at least) draw in the second position, and invariably did so!

Let us consider. If all players play until all available places are filled (before a box can be completed) then each box will have two lines. (Anyone who gives a box up at the beginning will not win, except in the exceptional case of a box and a margin.) On the other hand, a line from the center borders two boxes, so the number of lines is congruent $\pmod 2$ to the number of lines from the center. This would seem to imply that the second player wants an even number of lines from the center, except in the case that the first player fills one isolated box and then the second must give up 3 (the last position illustrated in the first row).

Thus the basic strategy of the first (resp. second) player must be to obtain an odd (resp. even) number of lines emanating from the center, with the exception mentioned earler. A secondary strategy of the first player is to form a corner on the side, and/or prevent a diameter from being drawn (since the only winning position for the second player must then be the "big box", cf. first position, which can be avoided later).

Suppose the first player starts with a line from the center. The second player continues the line to the other side, and can force a drawer by preventing the first player from drawing any other lines from the center (or else making them into diameters).

Strategy: The first player starts on the side. If the second player draws a line from the center, the first player draws a perpendicular through the center and wins. If the second player continues on the side, the first player should form a corner on the side. In the next turn he draws an appropriate line on the edge, and the following turn draws a line from the center (to prevent the "big box") and wins.

There are several other possibilities involving boxes given away in the midst of the play, cf. BCG p. 513, but this is in the same spirit. BCG also consider the $3 \times 3$ and $4 \times 4$ boxes, but not a complete description. One strategy which arises in the larger games is that of refusing to fill the last two boxes in a chain, in order to be the first to start the next chain.

(Philosophy: When I tried this out on my wife and daughter, they responded that the point of the game of dots is not to win, but merely to pass away time.)

**2. NIM-the ultimate fair game.**

We start with a class of games called "fair" or "impartial" games; this is a game in which switching players does not affect the payoffs, i.e. the game is independent of the identity of the player. For example, chess is fair. Nevertheless, the first player seems to have an advantage in chess, so to balance this, tournaments have the players alternate playing white and black. We can formalize this by simultaneously playing 2 games the first of which A plays first, and the second of which B plays first. We call this a *paired game*. (In general, given $n$ players one would need $n!$ games.) Since the two games in the pair are independent of each other, the strategy of the paired game is simply the combination of the strategies for each game, so its solution includes the solution of the original game. But the paired game has no advantage for either side, and in particular the value must be 0.

We focus on NIM: You have $t$ piles of objects. Each person can remove any number that he wants from any pile. The one to remove the last object wins; in other words the person who cannot move loses. Clearly NIM is impartial.

1 pile. Strategy: Take all of them.

2 piles. Strategy: Make the piles even. (Then match the second player's move each time) Thus $(n, n)$ is a win for B for all $n$, but $(m, n)$ is a win for A for all $m \neq n$. One funny quirk is that if the one to remove the last object loses, the outcome of the game is the same (provided each pile has at least 2)! The point is that $(1, 0)$ and $(0, 1)$ are wins for B, so $(1, n)$ and $(n, 1)$ are wins for A for all $n > 1$. Now $(2, 2)$ is a win for B, since A takes 1 and B takes all, so any $(2, n)$ is a win for A, and inductively we see $(n, n)$ is a win for B for all $n$, and $(m, n)$ is a win for A for all $m \neq n$.

This can be displayed graphically, by making a grid, starting with $(0, 1)$ and $(1, 0)$ marked B; anything that can be reached (i.e. from above or from the right) is a win for A. This leaves $(1, 1)$ as a win for B. The idea is that if A starts on $(1, 1)$ he *must* move to a square previously marked A, which thus is a win for B. But now A can get to $(1, 1)$ from $(n, 1)$ and $(1, n)$ for any $n$, so these are all marked A. This "backwards analysis" works as follows: One marks all the lattice points A that give A an immediate win, and then marks the lattice points B that force A to move onto a point previously marked A (since this now means a win for B). Now one marks the lattice points A that enable A to move onto a point newly marked B, and so forth.

3 piles. Strategy: Write each number $m_1, m_2, m_3$ to the base 2. (For example if $m_1, m_2, m_3$ are 58, 51, and 30 resp., write them as

111010

110011

011110

$* \ *{*}*$

Any column which has an odd number of 1's you can mark with a *. (This is like taking the "xor" bit-sum; a way of doing it mentally is canceling duplication in the representations base 2, and then adding.) Then remove the correct number to eradicate all *'s. This means 010101 from the bottom, for example. The next reduction must change some columns, and thereby create columns with *; then the same strategy can be used, and by induction, one can finish.

Note this strategy can always be attained: Find a number, say $m_i$, which has a 1 in the highest order column in which $*$ appears, and take the xor sum of $m_i$ with the $*$; this will by removing an appropriate amount from $m_i$. (Note that the largest number need not suffice as the $m_i$; for example the only winning strategy given the position 2,2,1 is to remove the 1.

This proves that any game of NIM is determined, giving a win to $B$ iff the "xor" sum is 0. Furthermore this argument shows that the same strategy works for any number of columns, so finishes NIM, at least for the next few seconds.

**Adding NIMbers.**

We write $\mathbf{n}$ to denote the pile of $n$ counters. Note that as games by themselves all $\mathbf{n}$ are equivalent, namely the first person wins by taking them all. However, insofar as adding games is concerned, they yield different results since $\mathbf{1} + \mathbf{2}$ differs from $\mathbf{1} + \mathbf{1}$. Then we note $\mathbf{n} + \mathbf{n} = 0$ for all $n$, and thus $\mathbf{m}+ \mathbf{n} = \mathbf{k}$ if $(m, n, k)$ is a winning combination for $B$, or, equivalently, as we saw above, iff there are an even number of 1's appearing in the three numbers in any given columnn, if the numbers are written in binary notation. (Clearly this definition is well-defined). It is easy to see that this operation is associative, so we have an Abelian group in which any substitution provides a different winning position for B. This enables one to calculate more quickly; BCG calls these "NIMbers".

*Remark.* $\mathbf{m} + \mathbf{n} = \mathbf{k}$, where $k$ is the smallest number for which the equation $\mathbf{u} + \mathbf{v} = \mathbf{k}$ does not hold for any $(u, v) < (m, n)$ (where $u = m$ or $v = n$).

(Proof: Whatever player A reduces from one pile, player B can reduce from the other.)

**Chinese NIM.** If one wants to make the 2-pile NIM less trivial, one could permit a player to remove an equal number from both piles. Now $(m, n)$ is a win for A whenever $1 \leq m, n \leq 1$, so $(1, 2)$ and $(2, 1)$ are wins for B. Hence $(1, n+1)$,, $(2, n)$, $(n, 2)$, $(n+1, 1)$, $(n, n+1)$, and $(n+1, n)$, are wins for A for all $n \geq 2$, so $(3, 5)$ and $(5, 3)$ are wins for B. Hence $(n, n+2)$ $(3, n+3)$, $(5, n-1)$, and others are wins for A for all $n > 3$, implying $(4, 7)$ and $(7, 4)$ are wins for B. But now comes a twist – since we already have $(5, 3)$ a win for B, we have to go to row 6 to get $(6, 10)$ a win for B, and the next win for B after that is $(8, 13)$. (After that is $(9, 15)$.) This game is analyzed in an article by Gardiner.

A more complicated version is that $A$ can remove any combination which inductively has been shown to be a win for $B$.

If one permits a player to take an equal number from 2 piles, this does not affect NIM for an odd number of piles, but NIM for an even number of piles is unsolved under these rules.

**Other NIMs.**

BCG, p. 53, deals with other variants of NIM. There is “poker-NIM”, in which one may add chips which he has taken before. However, any adding move can be negated by deleting the same number.

Northcott’s game involves starting with the following position of checkers on a checkerboard:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $B$ | | | $W$ | | | |
| | | | $W$ | | | | $B$ |
| | | $W$ | $B$ | | | | |
| $B$ | | | | | $W$ | | |
| | | $W$ | | | | $B$ | |
| $B$ | | | | | | | $W$ |
| | $W$ | | | | | | $B$ |
| | $W$ | $B$ | | | | | |

The idea is to move along a row, and see if you can trap your opponent. This is really NIM in disguise. Why? The differences are just the number of counters in each column, so this reduces to poker NIM.

Another variant of $2 \times 2$ NIM is to view it as a chessboard in which either side is permitted to make a rook move towards the origin; the first one who cannot move loses. One could try this with other chess-pieces, say knights, and then one could try a game in which player A must reduce the $x$-coordinate by 2 (i.e., move from $(m, n)$ to $(m - 2, n \pm 1)$) whereas player B must reduce the $y$-coordinate by 2 (from $(m, n) to (m \pm 1, n - 2)$. This game is quite simple, the reverse game is more interesting (where the last person to move loses). Combinations of this game (playing with several knights simultaneously) are studied in BCE, pp. 260 ff.

General NIM: One starts with a vector $\mathbf{v} = (v_1, \dots, v_n)$, and is allowed to remove from a choice of vectors $\mathbf{s}_1(\mathbf{v}), \dots, \mathbf{s}_m(\mathbf{v})$, where this choice depends on $\mathbf{v}$ but not on the player.

Note one could get an equivalent game by replacing each $\mathbf{v}$ by

$$\hat{\mathbf{v}} = k\mathbf{v} = (kv_1, \dots, kv_n),$$

where $\mathbf{s}_i(\hat{\mathbf{v}}) = \mathbf{s}_i(\mathbf{v})$. Also one can “translate” a game of NIM by adding the same number to each component of each vector. (The difference vectors $\mathbf{s}_i$ would remain the same except the ones which reach 0.)

Sprague-Grundy Theorem. Every impartial, locally finite game $G$ of perfect information can be reduced to General NIM in one column. Proof is by induction. We recall that $G$ is bounded, so we can proceed by induction on the bound on the number of turns. The first turn yields say $n$ new games $G_1, \dots, G_n$, each with a smaller bound, so by induction each $G_i$ is equivalent to some General NIM, starting with say $v_i$ units. We apply the translations of the previous paragraph to make all the numbers in the different games distinct, and then put $\tilde{v} = 1 + \max v_i$. Now define the $s_i(\tilde{v}) = \tilde{v} - \mathbf{v}_i$ and one has the required General NIM game for $G$.

**Projects.** Dots for up to 4 boxes square, Fancy NIM for 4 piles, a believable traffic game.

## 2. Unfair games: Hackenbush.

Oan make more elaborate games of NIM, by having counters which only $A$ can touch, or which only $B$ can touch (and all counters above such a counter would also be removed). Writing $A$ for a counter that only A can touch, $B$ for a counter that only B can touch, and $C$ for a counter that either could touch, a typical game might look like

$$\begin{array}{ccc} C & C & \\ A & B & A \\ C & C & A \end{array}$$

A has a large advantage in this game, since by removing $C$ at the bottom of the second column produces a game in which B's best move is to remove the first column, and then A has 2 extra remaining moves, so the value of this game is 2 for A, and similarly if B moves first (since A wins with 2 extra moves).

Since the existence of $C$ counters complicates the analysis, we first consider only $A$ and $B$ counters. We say the game of Hackenbush terminates when some player cannot move (since he has removed all of his counters), in which case the other player gets the number of his counters remaining on the table. At this stage, the payoff for $A$ is the number of $A$ counters (or minus the number of $B$ counters). Thus, intrinsically, any Hackenbush game of nonzero value must be unfair.

Given a Hackenbush game $G$ we define $\nu_A(G)$ to be the value (for $A$) when A moves first; $\nu_B(G)$ to be the value (for $A$) when B moves first. Note that these might not be the same.

To get examples where $\nu_A(G) \neq \nu_B(G)$ we consider mixed columns. Intuitively it is better to remove a counter which lies beneath counters of the opponent, but it is bad to remove a counter lying beneath another of your counters. For example, consider the game

$$\begin{array}{c} B \\ A. \end{array}$$

Since A's first move removes B, the game would have approximate value 0, but intuitively A has a stronger position than B, since $A$ can topple $B$'s piece, so we would like to give the game some positive value. (This is reflected in the game

$$\begin{array}{ccc} B & & \\ A & A & B. \end{array}$$

$\nu_A(G) = 1$ and $\nu_B(G) = 0$.

However we can modify it into a stable game. The Hackenbush game $G$ is it stable with value $n$ if

We shall say that a Hackenbush game is "doubled" if its position is placed beside itself; we call this game $2G$. By "doubling" the previous $G$ we get

$$\begin{array}{cc} B & B \\ A & A, \end{array}$$

which has value 1 for each side. (Namely A removes one of the counters, and B has only one counter remaining, so that leaves the final A counter standing.) The Hackenbush game $G$ is it stable with value $n$ if $\nu_A(2G) = \nu_B(2G) = 2\nu_A(G)$. Thus $G$ should have value $\frac{1}{2}$.

Now we can define the *value* $\nu(G)$ of a Hackenbush game to be $n$ if $G$ is stable of value $n$, and inductively $\nu(G) = \frac{n}{2}$ if the doubled game has value $n$. Thus, when $G$ is stable after $t$ doublings, we have

$$\nu(G) = \frac{\nu(2^t G)}{2^t}.$$

(At this stage it is not clear that every game of Hackenbush has a value.)

We proved for $G = \begin{matrix} B \\ A \end{matrix}$ that $\nu(G) = \frac{1}{2}$.

Similarly

$$G = \begin{matrix} B \\ B \\ A \end{matrix}$$

has value $\frac{1}{4}$ for A, since

$$\begin{matrix} B & B \\ B & B \\ A & A \end{matrix}$$

is played follows: A removes one of its counters and then B the top remaining counter (or vica versa), in each case yielding the game

$$\begin{matrix} B \\ A, \end{matrix}$$

which inductively has value $\frac{1}{2}$. Thus $\nu(G) = \frac{1}{4}$.

**Lemma.** *The Hackenbush game* $G_n = \begin{matrix} B \\ \vdots \\ B \\ A \end{matrix}$ *has value* $2^{-n}$ *where* $n$ *is the height of the column.*

*Proof.* Double the game. The play by A is to remove a counter and thus a whole column, and then B removes his uppermost counter, thereby yielding $G_{n-1}$ which, inductively has value $2^{n-1}$. $\square$

What about $\begin{smallmatrix} B \\ A \\ A \end{smallmatrix}$ ?

At first glance the induction might seem to be going in the wrong direction, but one needs to prove that doubling a game a finite number of times will in fact lead to a stable game. We shall see this shortly.

More generally one could "add" games by putting one alongside another, one would like the value of the sum to be the sum of the values of the individual games; for example

$$\begin{matrix} A & B & & A & B & \\ A & B & A & A & B & A \\ A & B & A & A & B & A \end{matrix}$$

has value 4.

Here A can remove his counter in the first column so the game has approximate value 1 for A, and 0 for B, but according to our calculations the game should have true value $\frac{3}{4}$.

One might note that by definition the value of all these Hackenbush positions are of the form $\frac{m}{2^t}$. This raises the question of whether any number $\frac{m}{2^t}$ is the value of a suitable (finite) Hackenbush game. The best way is to write the various positions horizontally. We have seen:

$AAA \dots A$ has value $n$; $B \dots BA$ has value $\frac{1}{2^n}$.

Let us check some other games, seen easily by doubling them:

$ABA$ has value .75;

$BAA$ has value 1.5.

Intuitively we see that if a typical word $\ldots BAA \ldots A$ ends at the bottom with a string of $n$ $A$'s, then we can give that string *guessed value* $n$, and the $B$ before it the guessed value of $-\frac{1}{2}$, the letter preceding it the value of $\pm\frac{1}{4}$, etc. (Analogously if the bottom is a string of $B's$.) It is easy to show that every string provides a unique value of the form $\frac{m}{2^t}$. The guessed value of a column is the sum of all the guessed values in the column.

THEOREM 2.1. *Any game given by a string of A's and B's has value $\nu(G)$ equal to the guessed value described in the previous paragraph.*

*Proof.* This theorem is done in two passes. Write $\mu(G)$ for the guessed value of $G$ for $A$. First we prove that $\nu_A(G) \geq \mu(G)$, and the best strategy for A is to optimize the guessed value.

Note if $A$ is repeated on the bottom then by taking the top of these counters $A$ still removes all the $B$, so the extra $A$ counters on the bottom remain and clearly provide value 1 each. Thus in proving the theorem we may assume $A$ is *not* repeated on the bottom, and likewise $B$ is not repeated on the bottom.

Any column will be rewritten as a string $W_t W_{t-1} \ldots W_1 W_0$, where each $W_i$ is $A$ or $B$, and the string is written from top to bottom, i.e. $W_t$ denotes the top counter and $W_0$ denotes the bottom counter.

The theorem will be proved by induction on the maximum height $t$ of the columns. If every column has height at most 1 the theorem is obvious, so we assume the theorem is true for all games for which all columns have height $< t$.

We claim that given any column of height $t$, and move by $A$ decreases the guessed value for $A$, and $A$ has precisely one move to decrease his guessed value by the smallest possible amount $2^{-t}$. Indeed, if $W_t = A$ then $A$ removes $W_t$ (and no other move is as good); if $W_t = B$ then taking the largest $j$ such that $W_j = A$, player $A$ removes $W_j$, thereby losing guessed value $2^{-j}$, but he also topples $W_{j+1}, \ldots, W_t$, all of which are $B$ with guessed values $2^{-(j+1)} + \ldots 2^{-t}$, so the net loss for A is $2^{-t}$.

So A can remove the appropriate piece from a column of height $t$, which now has smaller height. We assume B now makes his best possible move. B's move now will increase A's guessed value, so he has to find another column of height $t$ and make the analogous move. (If there are no other columns of height $t$ then B is out of luck, and in fact A's guessed value increases.) The increased in guessed value is $2^{-j} \geq 2^{-t}$. Thus after some number of moves, we have a new game $H$, and $\mu(H) \leq \nu_A(H)$ by induction, when no columns of height $t$ remain. On the other hand, $\nu_A(H) \leq \nu_A(G)$ since $B$ has by assumption made his/her best possible moves. (After we prove $A$ has made the best possible move we shall see $\nu_A(H) = \nu_A(G)$. Putting everything together, we have

$$\mu_A(G) \leq \mu(H) \leq \nu_A(H) \leq \nu_A(G),$$

as desired.

Now we claim in the stable case that $\mu(G) = \nu(G)$. Having seen that removing the appropriate piece from a column of length $t$ is the optimal strategy for maintaining the guessed value, we now see that if there are an even number of columns, then both players can choose the optimal strategy and reach a game $H$ with the same guessed value $\mu(G)$, which by induction is $\nu(H)$, since $H$ has height $< t$. Furthermore, there is a strategy of each side which will produce the outcome $\mu_A(G)$ in the stable case and any deviation from this strategy will act to the detriment of the player deviating (since it will adversely affect $\mu_A(H)$, which we may assume is $\nu_A(H)$ by induction). But we can obtain such a game by doubling $G$, and subsequent doublings required to solve $H$ only leave the number of columns of $G$ even. □

Note that this proof also gives the best strategy for the game, and shows that a game of height $t$ becomes stable after $t$ doublings (since one doubling at each height suffices).

Reversing the discussion, given any number, write it in binary notation, say of the form $m.d_1d_2d_3\dots d_n$ where each $d_j = 1$. Then write a string of $m$ $A$'s on the right, preceded by $BA$ (for the decimal point), and, always tacking onto the left, write $A$ when $d_i = 1$ and $B$ when $d_i = 0$, ignoring $d_n$. For example the game of value $\frac{5}{16} = .0101$ must be $BABBA$. Thus every number of this form is represented as the value of a (unique) Hackenbush string.

COROLLARY 2.2. *The value of the true payoffs of two Hackenbush positions is the sum of the true payoffs.*

*Proof.* The value of the sum is obviously the sum of the values, so the same inductive proof works.

Modification: Consider vertical trees emanating from the bottom, and any move cuts off a branch (whose vertex is marked by A or B). I THINK the values are still binary.

(Another modification: Give different payoffs to different counters depending on their branches. I think this permits other values.)

One should note that any real number can be written as an infinite string in binary notation, and thus this value could be obtained by an infinite game; for example $\frac{2}{3} = .010101010101\cdots$, so its game is $\cdots BABABABA$.

As BCG point out, one could get any ordinal number in this way, by permitting ordinal numbers of counters; these are technically not games, but each played game is finite, although not bounded. What about $BB\cdots BA$ ? But now one can describe square roots of ordinal numbers, etc!

**Mixed Hackenbush.**

As mentioned above, the presence of $C$ counters complicates the picture, since for example $\begin{smallmatrix} C \\ C \end{smallmatrix}$ is a win for the first player, and thus does not have a value (as described above). Such a position is called “fuzzy” in BCG; a game where the first person (no matter who it is) to move should win is called $*$ in BCG. Thus most NIM positions are fuzzy. A fuzzy game can favor one player, in the following sense: $\begin{smallmatrix} A \\ C \end{smallmatrix}$ is fuzzy, whereas

$$\begin{matrix} A & A \\ C & C \end{matrix}$$

is a win for A, no matter what. Another interesting facet is that instinctively the best strategy for either side is to remove $C$ counters first in order to prevent the opponent access to such a counter. In fact, one wants to be the last player to be able to remove a C counter, and thus this stage of the game is played according to the strategy of ordinary NIM. BUT if a player sees that he has a losing NIM position, he will remove a counter of his lying above a C counter in order to rectify the position.

Some examples:
the game

$$\begin{array}{cc} B & \\ B & \\ B & B \\ A & C \end{array}$$

is clearly a win for A. (In general any player would want to take the $C$ first, to prevent the other from taking it.) Thus the values of fuzzy games have very peculiar properties as numbers.

Consider

$$\begin{array}{ccc} C & & A \\ A & & A \\ & C & \end{array}$$

Clearly this game is fuzzy, since the first person to move takes away the bottom C and wins. BCG's version of Hackenbush has the more general set-up of a vertical graph whose edges are colored blue, red, or green where A can remove blue, B can remove red, and anyone can remove green. Once an edge is removed, one also removes any part of the graph which is not connected to the ground.

Another variant is "COL" [BCG, pp. 39ff] where players alternate coloring maps; the value of this game is determined in [BCG, p. 49].

**3.** ***n* in a row.** Obvious advantage for the first person. $n$ in a row (Scientific American, September 1993.) Threatening position: $n-1$ in a row or two $n-2$ in a rows. Two open $n-2$ in a row force an open $n-1$ in a row and thus a win. Thus $n = 3$ is won on the first move.

$n = 4$ is a win for a $4 \times 30$ board

$n = 5$ is a win on $15 \times 15$ board

$n = 6$, $n = 7$ open

$n \geq 8$ is drawn.

$n \geq 9$ the draw can be demonstrated by pairing off the squares of the board in such a way so that every line of 9 contains at least one pair (so that if one player occupies half the pairs, the game is drawn). This is called a Hales-Jewett pairing.

If you use a square then there is an obvious Hales-Jewett pairing, namely by a domino-type layout

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| | 5 | 5 | 6 | 6 | 7 | 7 | |
| 9 | 9 | 10 | 10 | 11 | 11 | 12 | 12 |
| | 13 | 13 | 14 | 14 | 15 | 15 | |

This game can be played using fixed planar shapes, i.e. the object is to form one of these shapes. The following shapes are known wins:

$$X \qquad XX \qquad XXX \qquad XXXX$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $XX$ | $XXX$ | $XXXX$ | $X$ | $X$ | $XX$ | $XXX$ |
| $X$ | $X$ | $X$ | $XXX$ | $XXXX$ | $XX$ | $XX$ |

$$\begin{array}{l} XX \\ \quad XXXX \end{array}$$

is thought to be a win, but is unknown. All other shapes are proven draws. (One can reduce to 12 minimal shapes, which are all draws.)

**4. Pente.** This is a very interesting variant, which was given out almost free as a promotion in the US about 4 years ago. The game is played on an (almost) infinite board and the first to form 5 in a row (also permitting diagonals) wins. The catch is that in a position $ABBA$ the two $B$ counters are removed immediately and given to player A, and a player defaults by losing 10 counters. Obviously A has an advantage, but the question is whether this game is finite, i.e. if A can force a win?

**5. Four in a row, with a vertical board.**

This game involves 4 in a row either vertical or horizontal or diagonal, but such that any given column must be filled in order (since the checker falls to the bottom). In other words, position i,j must be filled before (i+1,j). Suggested strategy: Color the squares red and black, and the first player only choose red squares (since any reasonable strategy will prevent horizontal and vertical rows of 4 from forming). This game is sold in the US with an $8 \times 8$ board. There is an obvious strategy here, based on even-odd parity, since it is easy to block vertical and horizontal victories, and the diagonal must all be of the same parity; I did not work this out completely, but expect that the game could be solved fairly easily.

## III KNOWLEDGE

We just studied games having perfect information. When one lacks perfect information, the first natural question is how best to utilize the information that one has. This raises the question of the nature of knowledge. Several ancient paradoxes involve deductions from partial knowledge.

The most famous example is that three women are in a room, and each of them has smudged rouge. Each sees that the other has, but is too polite to mention it. Someone walks into the room and says, "Someone has smudged rouge." After a while, all three blush. Why? (An older version is that a king is looking for a wise counsellor, and is down to a short list of 3. He puts a red spot on each person's forehead, and tells them all that they have either a red or blue spot, and at least one is red. The winner realizes that the other would have won if her spot were blue, and thus it must be red. .)

Also there is the tale of the unfaithful wives in a small town with a river. A long time ago, before the age of pc, a village had the rule that any husband who knows his wife is unfaithful would throw her off the bridge at precisely midnight following when he learns she is unfaithful. Each husband knew the status of all wives but his. However, nothing happened until a social worker came and said in horror, "There exist unfaithful wives in this town!" What happens? (Hint: Induction on the number of unfaithful wives.)

2. The executioner. (One prisoner of three is to be executed; the executioner is not allowed to tell the prisoner who is to be executed, and is not allowed to tell a prisoner his own status. After much pleading from prisoner A, the executioner tells prisoner A, "Prisoner C is not going to be executed." How does this affect prisoner A? On the one hand, since there are only two prisoners left, we might expect A's chances of execution to have increased. On the other hand, it does not make sense that the chances have changed at all, since the fact that the executioner said something should be irrelevant. This can be understood better when compared the game of the doors (Example 1.5). In fact there is a strict analogy between the plight of prisoner A and the contestant who does *not* change doors. (Poor prisoner A cannot change his identity, after all.) Accordingly, his chances for execution are indeed $\frac{1}{3}$. Ironically, the chances of prisoner B for execution have risen to $\frac{2}{3}$.

In questions 1, the existence of common knowledge changes the circumstances. In question 2, the extra knowledge affects B but not A. (B is now 2/3 likely to be executed, and A 1/3).

The modern explanation is that there is common knowledge, i.e. knowledge which everybody knows and which everybody knows that everybody knows, etc. The problem with this is that the statement that there exists a women with smudged rouge has always been common knowledge, so one must formulate the explanation carefully.

Before defining common knowledge, we want to describe knowledge. Define $\Omega$ to be the set of states. Since we cannot measure a state in its entirety, we define $P_i(\omega)$ to be the set of states which are possible according to the $i$-knowledge if a given state $\omega$ is thought to have occurred, which we shall call the states *consistent* with $\omega$. Then we have

(P1) $\omega \in P_i(\omega)$;

(P2) If $\omega' \in P_i(\omega)$ then $P_i(\omega') \subseteq P_i(\omega)$;

(P2) is quite clear, since if our knowledge permits us to conclude $\omega'$ then it certainly enables us to conlude anything consistent with $\omega'$. Note that (P2) is a consequence of a broader assumption:

(P3) The $P_i(\omega) : \omega \in \Omega$ comprise a partition of $\Omega$.

In other words, if we say that two events $\omega, \omega'$ are *related* if $\omega'$ is consistent with $\omega$, then (P1), (P2) say this relation is reflexive and transitive, whereas (P3) says it is symmetric (and thus an equivalence).

Here is an argument that consistency is symmetric. If $\omega$ is consistent with $\omega'$ and $\omega'$ is not consistent with $\omega$, that means that $\omega$ cannot hold, which would yield a contradiction.

One could take the $P_i$ to be cosets of a vector space, or more generally, algebraic curves. This would enable us to utilize linear algebra and perhaps more generally algebraic geometry.

Note that consistency is dependent on a person's knowledge. Thus if one does not know anything, then all events are consistent! There are many examples in which consistency is symmetric, but here is an example in which it is not. For example, if I can measure the lower bound of an interval, then the interval $[1, 2]$ is consistent with $[0, 2]$, but not vica versa. On the other hand, if all the indivual events are mutually exclusive (which is an implicit assumption in the exposition

I read) and one cannot measure the discrepancies between two consistent states, then consistency is clearly symmetric.

Once we assume consistency is an equivalence relation $\sim$, we can take $\Omega/\sim$, which is the set of observed states.

If $P_i(\omega)$ is a singleton then player $i$ knows that $\omega$ occurred. If everyone knows a certain state is excluded from $\Omega$, then this is called *common knowledge*.

For example, consider the space

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ (WWW) & (WWB) & (WBW) & (WBB) \\ \\ 5 & 6 & 7 & 8 \\ (BWW) & (BWB) & (BBW) & (BBB) \end{array}$$

Then $P_1$ is the partition $\{15\},\{26\},\{37\},\{48\}$; $P_2$ is the partition $\{13\},\{24\},\{57\},\{68\}$; $P_3$ is the partition $\{12\},\{34\},\{56\},\{78\}$.

After the announcement that there is a spot, $\neg\{1\}$ becomes common knowledge, and we have new partitions

$$\{1\},\{5\},\{26\},\{37\},\{48\};$$
$$\{1\},\{3\},\{24\},\{57\},\{68\};$$
$$\{1\},\{2\},\{34\},\{56\},\{78\}.$$

Once player 1 communicates that he has excluded $\{5\}$, we see that $\neg\{5\}$ becomes common knowledge, etc.

Of course each $i$ has its own corresponding partition $P_i$. The communications between the players is the key to utilizing the knowledge.

On the other hand, we can define the knowledge function $K_i$ (of the $i$-th player) on $\Omega$. Suppose we are interested in a certain property, but are able only to determine other properties in our states. Thus we want to know when our observation implies a state has a certain property.

An *event* $E$ is a subset of $\Omega$. Intuitively $E$ is the set of states which that we have observed. $E$ implies another event $D$ iff $D \subset E$, in other words any state holding for $D$ also holds in $E$. Given $P_i$ one defines $K_iE$ to be the set $\{\omega : P_i(\omega) \subseteq E.\}$ These are the states which we know occurred, since all the possible states according to a given measurement fall within $E$. Thus $K_i$ is a function from the power set of $\Omega$ to itself.

Then (P1), (P2) imply (K0) thru (K3), and (P3) then implies (K4). Thus we see the theory runs more smoothly with the $P_i$.

So $K_iE$ is the largest subset $E' \subset E$ for which if $E'$ implies a state then we know $E'$ implies this state. $K_i$ satisfies the following axioms:

(K0) $K_i\Omega = \Omega$;

(K1) $V \subseteq W$ implies $K_iV \subseteq K_iW$;

(K2) $K_iV \subseteq V$ (axiom of knowledge)

(K3) $K_iV = K_i^2V$ (transparency)

(K4) $(\neg K_i)^2V \subseteq K_i^V$ (wisdom)

(K4) is most problematic: It says if you don't know that you don't know something then you know it; one has to be wise indeed for this. Note that $\neg$ means the set complement.

(K2) says if you know something then it is true; (K3) says that if you know you know something then you know it.

PROPOSITION 2. $K_i(V \cap W) = K_iV \cap K_iW$.

*Proof.* $\subseteq$ from (K1) and $\supseteq$ from (K3). $\square$

An *i-truism* is an event such that $K_iE = E$. which cannot occur without the person knowing it. Thus $K_iE$ is a truism. Thus $P_i(\omega)$ is the intersection of truisms containing $\omega$. In other words, taking $E$ to be the set of states in which this event occurs, $K_iE = E$.

*Remark 3.* Applying set complements to each side of (4) yields

$$K_i(\neg K_i)V \supseteq K_i^V,$$

which means that $\neg K_iV$ is a truism.

Using (K3) it is clear that any state implied by $E$ is implied by the $i$-truism $K_iE$. Also the intersection of truisms is a truism.

Define $KV$ by $\cap_i K_i(V)$. This satisfies all the (Ki) except (K3). Nevertheless, since

$$V \supseteq KV \supseteq K^2V \dots$$

we see that there is some $n$ such that $K^nV = K^{n+1}V \dots$; we call this $K^\infty V$.

## IV GAMES OF IMPERFECT INFORMATION

**Combinatorial games.** Although this is perhaps the most common kind of game, we will not be discussing these, because they really belong more to a probability course. I don't want to be guilty of teaching you poker or shesh-besh. However, the theory of probability is said to have been discovered by Pascal in response to a request from a gambler acquaintance.

### 2-PERSON SINGLE-MOVE GAMES

2-person games are the most well-studied. Even the simplest situation, where each person has one move with $n$ possibilities, has many subtleties. In this case one can describe the game best in normal form, via a $n \times n$ matrix, and we shall spend considerable time with this.

**Zero-sum single-move games: saddle points.**

*Example 4.1.* We start with the following game in normal form:

BATTLE OF THE BISMARK SEA

| | | *Japanese* | |
|---|---|---|---|
| | | Sail north | Sail south |
| *Adm. Kenney* | Search north | $(-2,2)$ | $(-2,2)$ |
| | Search south | $(-1,1)$ | $(-3,3)$ |

The payoff here is the number of days it takes the Americans to find the Japanese, viewed negatively for the Americans and positively for the Japanese. For convenience in notation, it is customary in a zero-sum game between two players to write just the payoff for the first player.

BATTLE OF THE BISMARK SEA

| | | *Japanese* Sail north | Sail south | **Minima** (for rows) |
|---|---|---|---|---|
| | Search north | $-2$ | $-2$ | $\underline{-2}$ |
| *Adm.Kenney* | Search south | $-1$ | $-3$ | $-3$ |
| | **Maxima (col)** | $-1$ | $\underline{-2}$ | |

In a zero-sum game, we wrote the payoff (number of days needed to find Japanese) matrix from the Americans' point of view. The first player's most conservative strategy is to assume his opponent will pick the worst result for him, so the first player pessimistically expects the worst of each row. In such a case, his strategy would be to maximize the minimum along each row. This is called the *minimax* since historically it was done with zero-sum games, and one wanted to minimize the other person's gain. Thus Kenney's minimax strategy is the minimum of losses of 2 and 3, which is 2. (So his payoff by minimax is -2). Since the Japanese payoff is the negative of Adm. Kenney's, their minimax strategy is to minimize the maximum of each column. Both these strategies agree to go north, with payoff $-2$ for Admiral Kenney this is called a *saddle point.* Note that if one person chooses the saddle point then the other must choose the saddle point or else will suffer, so choosing the saddle point gives an *equilibrium*.

Here is a simpler example.

*Example 4.2.* Reuven and Shimeon play the following game: Each shows simultaneously the number of fingers on a hand. If both are congruent $\pmod 2$ then Reuven gives Shimon 2 shekels; if Reuven is even and Shimon is odd then Shimon pays Reuven 3 shekels; if Shimon is odd and Reuven is even then Reuven pays Shimon 4 shekels. The matrix describing the normal form of this game (from the point of view of Reuven ) is called the *payoff matrix*, and in this case is

| $_{Reuven}\backslash^{Shimon}$ | Even | Odd |
|---|---|---|
| Even | $-2$ | $+3$ |
| Odd | $-4$ | $-2$ |

Strictly speaking, the payoff matrix here is $A = \begin{pmatrix} -2 & +3 \\ -4 & -2 \end{pmatrix}$

*Remark 2.* If Reuven and Shimon changed sides then the payoff matrix $A$ would become the negative transpose $-A^t$; we get the negation because we consider the payoff to Shimon instead of Reuven (since Shimon is now in Reuven 's previous position).

Clearly Reuven should choose even, since in either case he does better, and thus Shimon should choose even, and Reuven will lose 2 each time. This is called a "saddle point", or "point of equilibrium". Also we say that Reuven 's strategy of choosing even "dominates" the strategy of choosing odd. This example is pretty trivial, but there can be more subtle points; for example consider the game:

| $_{Reuven}\backslash^{Shimon}$ | Even | Odd |
|---|---|---|
| Even | $-3$ | $+8$ |
| Odd | $-2$ | $-1$ |

Although it might appear that neither row is superior, if Reuven is pessimistic then Reuven will pick the second row to minimize his loss, and thus Reuven should pick the first column to maximize his gain. Formally if the payoff matrix is $(a_{ij})$, we define a *maximin* is $(i_0, j_0)$ for which $\max_i a_{i_0 j} = \max_i \min_j a_{ij}$.

Dually, *minimax* is $(i_0, j_0)$ for which $\max_i a_{ij_0} = \min_j \max_i a_{ij}$; this would be the maximin solution if we reversed the game, i.e. have each player determine the strategy of his opponent.

LEMMA 4.3. $\max_i \min_j a_{ij} \leq \max_j \min_i a_{ij}$.

*Proof.* Clearly for any $i$, $\min_j a_{ij} \leq \max_j \min_i a_{ij}$. $\square$

Consider the matrix

$$\begin{array}{ccc} 1 & 8 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{array}$$

A *saddle point* or *point of equilibrium* (if it exists) is a maximin from the point of view of each player, i.e. it satisfies $\min_j \max_i a_{ij} = \max_i \min_j a_{ij}$. In our first game, $a_{21} = -2$ is the saddle point. This is the mutually agreed solution, since either side would worsen its payoff by unilaterally changing strategy. In the second game, $\min_j \max_i a_{ij} = \min\{8, 6, 9\} = 6$ whereas $\max_i \min_j a_{ij} = \max\{1, 4, 2\} = 4$. But neither is a point of equilibrium.

The strategy at a point of equilibrium is called a *pure strategy.*

Exercise: analyze the game given by the matrix

| $_{Reuven}\backslash^{Shimon}$ | option 1 | option 2 | option 3 |
|---|---|---|---|
| option 1 | $-2$ | $+3$ | $+5$ |
| option 2 | $-1$ | $0$ | $+8$ |
| option 3 | $-2$ | $-3$ | $+8$ |

Mins on row are -2,-1,-3, so maximin is -1, which occurs in the (2,1) position. Maxs on columns are -1,3,5, so minimax is -1. Equilibrium point at (2,1)

| $_{Reuven}\backslash^{Shimon}$ | option 1 | option 2 | option 3 |
|---|---|---|---|
| option 1 | $-2$ | $+3$ | $+5$ |
| option 2 | $-1$ | $-2$ | $+8$ |
| option 3 | $-3$ | $-2$ | $+8$ |

Min's on row are -2,-2,-3, so maximin is -2. Max's on columns are -1,3,5, so minimax is -1. No equilibrium point. This is the usual case.

When considering the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of a two-person game with two options for each side, one may apply row permutations or column permutations (by rearranging the options if necessary) and thus assume that $a_{11} \geq a_{ij}$ for all $i, j$. If

$a_{12} \geq a_{22}$ then choosing the first row is a dominating strategy for the first person, and the game is solved (and has a saddle point). Similarly, if $a_{21} \geq a_{22}$ then the first column dominates for the second player and there is a saddle point. Thus the only remaining case is when $a_{12} < a_{22}$ and $a_{21} < a_{22}$. In this case we have no saddle point. (These are called *elusive* games, to be discussed below.

Lemma: The maximin is the negative of the minimax of $-G^t$, so switching sides does not alter this analysis. Also, if one row (say the first row) dominates the others then the minimum $a_{1j}$ of the first row is greater than the corresponding entry $a_{ij}$ on each other row, and thus greater than all other minimums, so the maximin is $a_{1j}$. On the other hand, the maximum of each column is on the first row, so we get the same minimax $a_{1j}$. This proves that a dominant strategy provides a maximin-minimax, although in a 3 by 3 game the converse is not necessarily true. (However for a 2 by 2 game the maximin-minimax can only come from a dominant row or column, as seen via the last paragraph.)

An example of a game without a saddle point is $PScSt$. We will return to 0-sum games later. Of course, in a competitive game, it is in each player's interest to keep his strategy a secret. If the competitor's strategy is known, then one can easily find one's own strategy to maximize against it.

*Cooperative 2-person games.*

Cooperative games have more subtleties. An example of a cooperative game is the "Battle of the Sexes;" perhaps a better although less colorful name is "partnership". In this game, if one side makes the decisions and the other is passive then the active side gets 2 and the passive side gets 1; if both sides try to make the decisions or both sides are passive then each side loses 1. The normal form is

PARTNERSHIP

| | | *Second* | |
|---|---|---|---|
| | | Complain | Passive |
| *First* | Complain | $(-1,-1)$ | $(2,1)$ |
| | Passive | $(1,2)$ | $(-1,-1)$ |

In this game, there are two partners. If one is active and the other is passive, it is good for the partnership, whereas if both are active they may go at crosspurposes, and if both are passive the enterprise lacks leadership. (A more colorful name for this game is "Battle of the Sexes.") The maximin solutions (from each side) both are $(-1,-1)$ which is the worst possible result for everyone! However, this is no longer a point of equilibrium. On the other hand, (2,1) and (1,2) are both points of equilibrium. Interestingly, in this game is in the interest in one side to announce his intention to complain, since then the other side would choose to be passive, and the total payoff would be 3. However, the fairest solution would be for each side to agree to complain alternatively (with the other side being passive).

A related cooperative game is "Chicken," in which two sides drive towards each other in automobiles. The side that chickens off gets 0, whereas the one who bulls on gets 4. However, if neither side backs off, they all land in a hospital and lose 10. Thus the normal form is

CHICKEN

| | | *Second* | |
|---|---|---|---|
| | | Bull on | Chicken off |
| | Bull on | $(-10,-10)$ | $(4,0)$ |
| *First* | | | |
| | Chicken off | $(0,4)$ | $(3,3)$ |

Here the minimax strategy is to back off, so presumably each side should back off, but this is not an equilibrium, since if A knows B will back off, then he will go on. The two equilibria are where one side goes on and the other chickens off, although the "best" overall result is where both back off. Again, in first analysis, it is in A's interest to announce in advance he will go on. This game is also called the "Cuban missile crisis."

An even stranger solution comes from the "Prisoner's Dilemma." In this, two suspects are arrested. If they each keep quiet they will get 1 year in prison. If one turns State's witness and testifies against the other he goes free and the other gets 10 years. However, if both turn State's witness then they both get 6 years. The normal form is

PRISONER'S DILEMMA

| | | *Second* | |
|---|---|---|---|
| | | Rat | Keep quiet |
| | Rat | $(-6,-6)$ | $(0,-10)$ |
| *First* | | | |
| | Keep quiet | $(-10,0)$ | $(-1,-1)$ |

Here the equilibrium is (-6,-6), i.e. for both to rat although it produces a result not in the interest of either prisoner. The best overall solution, for both to keep quiet, is an anti-equilibrium! In this game, it would be in the interest of someone to announce that he will keep quiet, but then to rat. On the other hand, it is in the strong interest of the first player to convince the second player not to rat.

**metagames.**

Our object in what follows is how to find a formal justification of the obvious case-by-case analysis:

1. In Chicken it is in one's interest for the strategy to be known, since it scares off the opponent; nevertheless, the opponent could feel manipulated in this way.

2. In prisoner's dilemma one wants to convince the opponent to stay quiet.

We start with prisoner's dilemma. This game becomes more interesting if there were enough communication with the opponent (or if the game is played often enough) for one could base one's strategy on prior experience of the opponent. In other words, one can form a strategy based on one's idea of the opponent's strategy (perhaps based on repeated instances of the game), to wit:

*Rat*: Rat no matter what

$Q$ : Keep quiet no matter what

$Tit$ : Tit for tat: Rat iff the opponent is expected to rat

$Tat$ : Tat for tit: Keep quiet iff the opponent is expected to rat.

These strategies are called "meta-strategies". Clearly if A chooses Tit then B should keep quiet, since this provides (-1,-1) instead of (-6,-6). But now B, knowing that A is choosing a meta-strategy, now must respond to any possible meta-strategy, and thus defines a new game, called a *metagame*, in which he provides a move in response to your planned meta-strategy. Thus any move by the opponent consists of a vector $(v_1, v_2, v_3, v_4)$ where $v_1, v_2, v_3, v_4 \in \{Rat, Q\}$, and $v_1$ denotes the planned response to $Rat$, $v_2$ the response to $Q$, $v_3$ the response to $Tit$, and $v_4$ the response to $Tat$. For example, if B chooses $(Rat, Rat, Q, Rat)$ to A's $Tit$ then he chooses $Q$ and A chooses $Q$ and the payoff is $(-1, -1)$.

Note that $A$ has $2^2 = 4$ possible strategies in the metagame, giving B $2^4 = 16$ meta-strategies. Although $Tit$ is an attractive meta-strategy for $A$, it is not dominant, since $Rat$ does better against $Q$. On the other hand B does have a dominant strategy in this metagame, namely $(Rat, Rat, Q, Rat)$ (It is correct to rat against any meta-strategy except $Tit$.) The payoff vector is

$$((-6, -6), (-10, 0), (-1, -1), (-10, 0))$$

and B maximizes his payoff by choosing (-1,-1), the payoff for the equilibrium, $(Rat, Rat, Q, Rat)$ against $Tit$. (If A moves from $Tit$ he does worse, and we noted B's best choice against $TIt$ is $Q$.

In view of the equalities among many payoffs, this metagame actually has 3 points of equilibrium, the others being $(Rat, Rat, Rat, Rat)$ against $Rat$, which is the saddle point of the original game, at $(-6, -6)$, and $(Rat, Q, Q, Rat)$ against $Tit$. Here the payoff vector is

$$((-6, -6), (-1, -1), (-1, -1), (-10, 0))$$

So the advantage here is that there is a dominant strategy for B, and this determines the game.

The metagame for Chicken is even more subtle. Again, there are the four analogous strategies:

$G$: Go on no matter what

$C$: (Chicken): Back off no matter what;

Tit: Tit-for-Tat

Tat: Tat-for-Tit

There is an obvious dominant strategy here for B: It is correct to bull against $C$ or $Tat$, and to back off against $G$ or $Tit$. Thus B has a dominant meta-strategy $(C, G, C, G)$. It is in the interest of A to go on and get 4, yielding an equilibrium in this metagame, producing the outcome (4,0) which is not in the interest of B! (Since the game is symmetric, one would intuitively expect the fair result to be (1,1).) In order to force this outcome the first player must be willing to take on a non-dominant meta-strategy: $(G, C, C, G)$ or the more aggresive $(B, B, C, B)$. This forces B to avoid $B$ and encourages him to choose $Tit$, producing the (1,1) payoff. There also is an equilibrium when B plays the very aggressive $(G, G, G, G)$, The corresponding payoff vector is now

$$((-10, -10), (0, 4), (-10, -10), (0, 4)),$$

so A plays $C$ or $Tat$, and A gets (0,4).

Conjecture: Any dominant strategy remains dominant in the metagame when taken on the diagonal (i.e. repeating it in the vector). (There are parallels in diplomacy, e.g. the Cuban missile crisis resulted in the correct resolution although the negotiations were on the brink of disaster.) How could the dominant strategy fail to be the best strategy? Because one also has to take into account the payoff of the other side. Maybe define “absolutely dominant” if the first component of the payoff is at least as high and the other components are not higher. Thus (0,4) does not absolutely dominate (-10,-10) in the first player’s strategy since he might choose (-10,-10) in order to discourage the second player from following this option. Or, perhaps B requires a payment from A for his cooperation with A.

But B of course gets the best results with the greedy strategy $(B, B, B, B)$, thereby forcing A to take $C$ or $Tat$. It might be that in the repeated game, A will eventually switch to $Tit$ just to try to force the first player to abandon his strategy. If A does not, then we have disaster.

Second level: A’s response to B’s meta-strategies. These would be $2^16$ choices.

## IV Mixed strategies in (zero sum) games

We return to analyze the 2-person single-turn game algebraically.

**Elusive 2 by 2 games.** 1. Reuven and Shimeon now play the following game: Each shows simultaneously the number of fingers on a hand. If both are congruent (mod 2) then Reuven gives Shimon 2 shekels; if Reuven is even and Shimon is odd then Shimon pays Reuven 3 shekels; if Shimon is odd and Reuven is even then Shimon pays Reuven 1 shekel.

| $_{Reuven}\backslash^{Shimon}$ | Even | Odd |
|---|---|---|
| Even | $-2$ | $+3$ |
| Odd | $+1$ | $-2$ |

Who wins? This is simplest example of a classic situation. Either side can gain a positive result by guessing properly, and gets a negative result by guessing improperly. If Reuven can guess what Shimeon is planning then he will base his strategy accordingly, and it could be either choice (odd or even). Suppose in this matrix that Reuven chooses a mixed strategy he will give an odd number $p_1$ of the time and an even number $p_2 = 1 - p_1$ of the time, and likewise Shimon will give an odd number $q_1$ of the time and an even number $q_2 = 1 - q_1$ of the time. Then if Shimon always chooses even, Reuven ’s payoff is $-2p_1 + p_2$, whereas if Shimon always chooses odd, Reuven ’s payoff is $3p_1 - 2p_2$. Thus Reuven can guarantee a good payoff by setting these equal, i.e.

$$-2p_1 + p_2 = 3p_1 - 2p_2,$$

so $3p_2 = 5p_1$, and thus $p_1 = \frac{3}{8}$ and $p_2 = \frac{5}{8}$. Note that Reuven ’s payoff is now $-\frac{1}{8}$ in either case, so Shimon wins the game.

Let us look at the game from Shimon ’s point of view. Reuven ’s payoff is $-2q_1 + 3q_2$ or $q_1 - 2q_2$, depending on whether Reuven chooses even or odd, so equating these yields $3q_1 = 5q_2$, so $q_1 = \frac{5}{8}$ and $q_2 = \frac{3}{8}$; Reuven ’s expectation is still $-\frac{1}{8}$ Thus each player is led to a strategy which will provide the same result.

In general, by choosing the first row $p_1$ times and the second row $p_2$ times, player A gets $p_1a_{11} + p_2a_{21}$, or $p_1a_{12} + p_2a_{22}$, and since $p_1 + p_2 = 1$, we get two lines, one connecting $a_{11}$ and $a_{21}$, and the other connecting $a_{12}$ and $a_{22}$. The dominant situation is that one line lies above the other; the non-dominant is that the two lines cross. At all points other than the crossing point, B can pick the lower of the two lines, and thus hurt the payoff to A, so it is in A's interest to find where the two lines cross, i.e. $p_1a_{11}+p_2a_{21} = p_1a_{12}+p_2a_{22}$, and thus $p_2(a_{21}-a_{22}) = p_1(a_{12}-a_{11})$, i.e., $\frac{p_2}{p_1} = \frac{a_{12}-a_{11}}{a_{21}-a_{22}}$, so

$$p_2 = \frac{a_{12}-a_{11}}{a_{12}-a_{11}+a_{21}-a_{22}}; \qquad p_1 = \frac{a_{21}-a_{22}}{a_{12}-a_{11}+a_{21}-a_{22}}.$$

We need a positive solution for $p_1$ This happens when $a_{12}-a_{11}$ and $a_{21}-a_{22}$ have the same signs, i.e. precisely when the game is elusive. Note the intersection of the two lines is

$$\begin{aligned} a_{11}&\frac{a_{21}-a_{22}}{a_{12}-a_{11}+a_{21}-a_{22}} + a_{21}\frac{a_{12}-a_{11}}{a_{12}-a_{11}+a_{21}-a_{22}} \\ &= \frac{a_{12}a_{21}-a_{11}a_{22}}{a_{12}-a_{11}+a_{21}-a_{22}} = -\frac{|G|}{a_{12}-a_{11}+a_{21}-a_{22}}. \end{aligned}$$

This is symmetric with respect to the reversal of A and B, as should be expected.

## Solutions of 0-sum games

In general, given an $n \times n$ payoff matrix $(a_{ij})$, and assuming Reuven chooses the $i$-th strategy with probability $p_i$ and Shimon chooses the $j$-th strategy with probability $q_j$, Reuven 's payoff is $f(p,q) = \sum_{i,j} p_iq_ja_{ij}$.

Von Neumann proved that there exists at least one point $\hat{p}$ and $\hat{q}$ such that $f(p,\hat{q}) \le f(\hat{p},\hat{q}) \le f(\hat{p},q)$ for all vectors $p,q$. These would then provide optimal strategies for both sides, and $f(\hat{p},\hat{q})$ is the value of the game for Reuven .

The classical way to obtain such a solution is by the method of linear programming, which we shall see later. First we want to analyze this game algebraically.

**Algebraic solution of an $n \times n$ game.**

Let us try to reach the optimal mixed solution algebraically. As a preliminary observation, let us note that the constraint $q_1 + \cdots + q_n = 1$ is irrelevant, since we could always divide all the $q_j$ by $\sum q_j$ at the end (similarly with the $p_i$, so our solution vectors could be obtained in projective space. Although we usually do not utilize this observation, it is useful at times. Remember that Reuven 's task is to maximize his payoff, whereas Shimon 's task is to minimize Reuven 's payoff.

Perhaps one mixed strategy (say for Shimeon) $\hat{q}' = (q_1', \dots, q_n')$ might *dominate* another $\hat{q}'' = (q_1'', \dots, q_n'')$ in the sense that, regardless of Reuven 's strategy, $\hat{q}'$ provides a lower payoff than $\hat{q}''$, i.e. $\sum_j a_{ij}q_j' \le \sum_j a_{ij}q_j''$ for all $i$. We call such a game "partially dominated". We define the *dominated components* as those $j$ for which $q_j' < q_j''$. Intuitively, dominated components $j$ are "bad", in the sense that any strategy $\hat{q} = (q_1, \dots, q_n)$ is dominated by

$$\hat{q} + \alpha(\hat{q}' - \hat{q}'') = (q_1 + \alpha(q_1' - q_1''), \dots q_n + \alpha(q_n' - q_n''))$$

for any $\alpha > 0$, and thus $q$ is dominated by some strategy for which some $q_j = 0$. Thus we could discard any dominated component. A game without dominated strategies will be called "totally elusive".

For example, any $2 \times 2$ game with a dominated strategy obviously is determined, since one component dominates the other. Now let us turn to the algebraic approach.

We start from Shimon 's point of view, and look for a mixed strategy which will yield the same payoff, no matter what Reuven plays. At this stage we are not requiring this payoff to be best for Shimon , only that it be independent of Reuven 's strategy. Let $E$ denote this unified payoff for Shimon . Then

$$\sum_j a_{ij} q_j = E \quad \text{for each} \quad i = 1, \dots, n. \tag{1}$$

Also $\sum q_j = 1$. Thus the mixed strategy which will yield the same result regardless of Reuven 's strategy is the solution to the $n$ equations (1) along with

$$q_1 + \cdots + q_n - 1 = 0, \tag{2}$$

which yield $n+1$ equations in the $n+1$ unknowns $q_1, \dots, q_n, E$. Put in matrix terms this is

$$\begin{pmatrix} a_{11} & \dots & a_{1n} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

The solution (from Shimon 's point of view) can be found easily by means of Cramer's rule, which in particular says

$$E = \frac{\det(a_{ij})}{\det \begin{pmatrix} a_{11} & \dots & a_{1n} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix}}, \tag{4}$$

and this solution is unique (provided the denominator is nonzero, which we assume for the time being.) Now let us do this from Reuven 's point of view. This could be accomplished by making the same analysis with $A^t = (a_{ji})$, as noted in Remark 1, so our solution is

$$E = \frac{\det(a_{ji})}{\det \begin{pmatrix} a_{11} & \dots & a_{n1} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{1n} & \dots & a_{nn} & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix}}. \tag{5}$$

But $\det A^t = \det A$, and likewise

$$\det\begin{pmatrix} a_{11} & \dots & a_{n1} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{1n} & \dots & a_{nn} & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix} = (-1)^2 \det\begin{pmatrix} a_{11} & \dots & a_{n1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{1n} & \dots & a_{nn} & 1 \\ -1 & \dots & -1 & 0 \end{pmatrix}$$
$$= \det\left(\begin{pmatrix} a_{11} & \dots & a_{1n} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix}^t\right)$$
$$= \det\begin{pmatrix} a_{11} & \dots & a_{1n} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix},$$

so the values of $E$ in (4) and (5) are the same, i.e. Reuven 's unified strategy yields the same payoff as Shimon 's unified strategy!

Intuitively, we would like $E$ (determined by (4) and (5)) to be the optimal guaranteed payoff for both Reuven and Shimon . Let us consider Shimon . Actually, the solution for $E$ could be viewed more generally as an intersection of the hyperplanes determined by the hyperplanes

$$E_i = \sum_{j=1}^{n-1} a_{ij} q_j + a_{nj}(1 - \sum_{j=1}^{n-1} q_j); \tag{6}$$

We view this geometrically in $n$-space, where the first $n-1$ axes correspond to $q_1, \dots, q_{n-1}$, and the last axis (height) corresponds to the payoff. Their intersection will be optimal for Shimon unless there is some other point $(q'_1, \dots, q'_{n-1})$ at which the values $E'_i \le E_i$ for each $i$. (Because if $E'_i > E_i$ then Reuven could pick strategy $i$ and Shimon would lose.) In other words, if we go in the direction of the vector $v = (q'_1, \dots, q'_{n-1}) - (q_1, \dots, q_{n-1})$, we see that each of our hyperplanes is tilted upward, so we could continue in this direction until hitting a "boundary" point, where some $q_j = 0$ or 1. Thus, our algebraic solution will produce an optimal solution Reuven iff there are no "redundant" choices for Reuven (i.e. strategies which he would never do). For example in the game

| Reuven\Shimon | Even | Odd |
|---|---|---|
| Even | 4 | $-3$ |
| Odd | 4 | $-1$ |

$E = 4$, which is attained by $q_1 = 1$ and $q_2 = 0$, but it is by no means optimal for Shimon ; in fact this is his worst strategy. On the other hand, Reuven cannot find a strategy to attain $E = 4$, since his solution is $p_1 = -\frac{5}{2}$ and $p_2 = \frac{7}{2}$, which is absurd! (He must pick moves with probabilities lying between 0 and 1).

We have uncovered the hidden difficulty that some of the $q_i$ might turn out to be negative in the algebraic solution. This happens in the $2 \times 2$ game precisely when $a_{11} - a_{12}$ and $a_{21} - a_{22}$ have the same sign, which means that one strategy for

Reuven dominates the other. Thus we conclude that in the $2 \times 2$ case, a game has an algebraic solution which can be attained for either side, iff the game is elusive.

In general, let

$$J = \{j : q_j \geq 0\} \quad \text{and} \quad J' = \{1, \ldots, n\} \setminus \{j : q_j \geq 0\} = \{j : q_j < 0\}.$$

Then

$$\sum_{j \in J} a_{ij} q_j = E + \sum_{j \in J'} a_{ij}(-q_j)$$

We can use this equation to modify strategies. Explicitly, let $s' = \sum_{j \in J} q_j$, and $\hat{q}' = (q_1', \ldots, q_n')$ be defined as $q_j' = 0$ if $q_j \leq 0$ and $q_j' = \frac{q_j}{s'}$ otherwise. Likewise let $s'' = \sum_{j \in J'} q_j = 1 - s'$, and $\hat{q}'' = (q_1'', \ldots, q_n'')$ be defined as $q_j'' = 0$ if $q_j \leq 0$ and $q_j' = \frac{q_j}{s''}$ otherwise. Note that $s' > 1$ so $s'' < 0$ Also let $E_i'$ be the $i$-th payoff for $\hat{q}'$ and $E_i''$ be the $i$-th payoff for $\hat{q}''$.Then

$$s' E_i' = E_i - s'' E_i'',$$

so

$$E_i' - E_i'' = \frac{1}{s'} E - \frac{s' + s''}{s'} E_i'' = \frac{1}{s'} E - \frac{1}{s'} E_i''.$$

This means that given a mixed strategy, we can "improve" it by looking at the $i$ with the worst payoff and then replacing $\alpha \hat{q}'$ by $\alpha \hat{q}$ (or visa versa) for the largest $\alpha$ which is appropriate (either until we reach the next strategy or until one of the strategies becomes 0). I don't know how one would proceed from here, but this approach might shorten the time for the method to operate.

Of course if the denominator in (4) is 0 then we have a difficulty, but this can only happen if the rows are linearly dependent, i.e. if some linear combination of the rows of the payoff matrix yield a vector $(m, m, \ldots, m)$. For example the payoff matrix could be $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. This happens iff the columns of the payoff matrix satisfy this same property, i.e. $\sum a_{ij} s_j = m$, for $1 \leq i \leq n$ and $\sum s_j = 0$. But then some $s_j$ are negative and we have the situation described in the previous paragraph. Let us summarize our results.

Theorem 3. *In a square matrix game, there is an algebraic solution for which Reuven and Shimon have the same payoff, but this can be obtained in reality iff the probabilities $p_i$ and $q_j$ are non-negative. In this case, one can show easily that this is the best strategy for each.*

In case some $q_j$ are negative we still saw that the algebraic approach yields information, and it would be interesting to incorporate this into the linear programming solution. Incidentally, we shall see below that every game came be reduced to a square game.

Another point: Any mixed strategy could be substituted for one of the strategies involved in it, and then yields a new payoff matrix. Such a transformation yields an equivalent game iff the transformation matrix $B$ has the property that both its entries and the entries of its inverse are non-negative. (For example, a diagonal matrix with positive entries would be an example of such a matrix.) Interesting question: What is the class of matrices having this property?

If the strategy involves the same payoff $E$ for each player, independent of the choice of the other player, then we may assume the payoff matrix satisfies $a_{1i} = a_{i1}'$ for each $i$.

**Farkas' Theorem.**

**Farkas' Theorem.** *Suppose $A = (\alpha_{ij})$ is an arbitrary $\ell \times m$ matrix over the field $\mathbb{R}$. Write $\mathbf{a}_i = (\alpha_{i1}, \ldots, \alpha_{im})$ for $1 \le i \le \ell$, the rows of $A$. The system $\sum_j \alpha_{ij}\lambda_j > 0$ of linear inequalities, for $1 \le i \le \ell$, has a simultaneous solution over $\mathbb{R}$, iff every non-negative, nontrivial linear combination of the $\mathbf{a}_i$ is nonzero.*

Proof: ($\Rightarrow$) If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is a solution then, for any $\beta_i \in \mathbb{R}$,

$$(\sum_i \beta_i \mathbf{a}_i) \cdot \mathbf{x} = \sum_{ij} \beta_i \alpha_{ij} x_j = \sum_i \beta_i (\sum_j \alpha_{ij} x_j) > 0.$$

($\Leftarrow$) Consider the cone $\mathcal{C}$ of vectors of the form $A\mathbf{v}^t$, where $\mathbf{v} = (v_1, \ldots, v_m)$ is non-negative and $\sum v_i = 1$. $\mathcal{C}$ is convex, i.e. if $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ then $t\mathbf{x} + (1-t)\mathbf{y} \in \mathcal{C}$, for all $0 \le t \le 1$. $\mathbf{0} \in \mathcal{C}$. By a compactness argument there is some point $\mathbf{p} \in \mathcal{C}$ at a minimum distance from $\mathbf{0}$.

Take any $\mathbf{q} \ne \mathbf{p}$ in $\mathcal{C}$. By hypothesis on $\mathbf{p}$, the angle $\mathbf{0pq}$ cannot be acute, so

$$0 \ge (\mathbf{q} - \mathbf{p}) \cdot (\mathbf{0} - \mathbf{p}) = \mathbf{p} \cdot \mathbf{p} - \mathbf{q} \cdot \mathbf{p}.$$

Hence $\mathbf{q} \cdot \mathbf{p} \ge \mathbf{p} \cdot \mathbf{p} > 0$ for each $\mathbf{q} \in \mathcal{C}$, so conclude taking

$$\mathbf{q} = \mathbf{a}_i = e_i A \in \mathcal{C}$$

for each $i$. (The solution is $\mathbf{p}$.

This enables us to prove that any fair game has an optimal strategy, by means of:

**Corollary (Fundamental Theorem of Game Theory).** *If there is no $\mathbf{x} > 0$ in $\mathbb{R}^{(m)}$ (written as a column) with $A\mathbf{x} < 0$, then there exists $\mathbf{w} \ge 0$ (written as a row) in $\mathbb{R}^{(m)}$ with $\mathbf{w}A \ge 0$.*

*proof.* Define $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$. By hypothesis there is no simultaneous solution to the inequalities $-\mathbf{a}_i \cdot \mathbf{x} > 0$, $x_j > 0$. In other words, defining $\tilde{A}$ to be the $\ell + m \times m$ matrix $\begin{pmatrix} -A \\ I \end{pmatrix}$, there is no solution to $\tilde{A} \cdot \mathbf{x} = 0$. Hence by Farkas' Theorem, the rows of $\tilde{A}$ are dependent with non-negative coefficients. $\square$

Indeed, subtract the largest constant possible such that the hypothesis still holds, and the corollary says that the first player can force this result to be obtained. We shall generalize this result soon.

**Summary.** *Algebraic solution of a 1-move game*

First discard all dominated strategies via linear programming. This will yield a square matrix (prove), and this can be solved by the solution of the simultaneous equations the payoffs to be equal.

**Julia Robinson's solution of a 1-move game by successive approximations.**

Recall that in a fair game, reversing the players yields the negative of the transpose of the payoff matrix (since we also have to reverse the person receiving the payoff), so a game is symmetric iff the payoff matrix is skew-symmetric (and in particular, 0 on the diagonal). By pairing the game, we shall assume it is symmetric. (Explicitly, the matrix for this is an $mn \times mn$ matrix, whose payoff for Reuven choosing the pair $(i,j)$ and Shimon choosing the pair $(i',j')$ is $a_{ij'} - a_{j'i}$, which is antisymmetric. Thus the matrix is symmetric.

Since we assume the opponent is picking the best possible strategy, the best we can hope for in a fair symmetric game is a draw, and on the other hand, the best strategy should result in a draw. Thus we look for a mixed strategy with outcome 0.

The idea here is for Reuven to learn from Shimon since Shimon is playing as well as possible. Thus, Reuven always bases his move on the history of what Shimon has done until then.

We start by solving the mixed strategy for the symmetric game. The algorithm is very easy: Let $i_0$ be arbitrary, and inductively, let $j_k$ be Shimon 's best strategy against Reuven 's mixed strategy up to this point. Define

$$p_{i,t} = \frac{\#\text{ of } j_k \text{ equal to } i, \text{ with } k \leq t}{t+1},$$

and $p_i = \lim_{t\to\infty} p_{i,t}$.

For example, suppose

$$A = \begin{pmatrix} 0 & 2 & -1 & 3 \\ -2 & 0 & 2 & 1 \\ 1 & -2 & 0 & 1 \\ -3 & -1 & -1 & 0 \end{pmatrix}.$$

Note that Shimon will never choose the last column, so this becomes irrelevant to our consideration.

Here is an iterated strategy:

| $k$ | $(p_1, p_2, p_3, p_4)$ | Shimon 's response | payoff |
|---|---|---|---|
| 1 | $(1,0,0,0)$ | 3 | $-1$ |
| 2 | $(1,0,1,0)$ | 3 | $-1$ |
| 3 | $(1,0,2,0)$ | 2 | $-\frac{2}{3}$ |
| 4 | $(1,1,2,0)$ | 2 | $-\frac{2}{4}$ |
| 5 | $(1,2,2,0)$ | 1 | $-\frac{2}{5}$ |
| 6 | $(2,2,2,0)$ | 1 | $-\frac{1}{3}$ |
| 7 | $(3,2,2,0)$ | 1 | $-\frac{2}{7}$ |
| 8 | $(4,2,2,0)$ | 1 | $-\frac{1}{4}$ |
| 9 | $(5,2,2,0)$ | 1 | $-\frac{2}{9}$ |
| 10 | $(6,2,2,0)$ | 1 | $-\frac{1}{5}$ |
| 11 | $(7,2,2,0)$ | 3 | $-\frac{3}{11}$ |
| 12 | $(7,2,3,0)$ | 3 | $-\frac{1}{4}$ |
| 13 | $(7,2,4,0)$ | 3 | $-\frac{3}{13}$ |
| 14 | $(7,2,5,0)$ | 3 | $-\frac{3}{14}$ |
| 15 | $(7,2,6,0)$ | 3 | $-\frac{3}{15}$ |
| 16 | $(7,2,7,0)$ | 3 | $-\frac{3}{16}$ |
| 17 | $(7,2,8,0)$ | 3 | $-\frac{3}{17}$ |
| 18 | $(7,2,9,0)$ | 2 | $-\frac{4}{18}$ |
| 19 | $(7,3,9,0)$ | 2 | $-\frac{4}{19}$ |
| 20 | $(7,4,9,0)$ | 2 | $-\frac{4}{20}$ |
| ... | ... | ... | |

The convergence is painfully slow, but we are not so far from the algebraic solution, which is $p_1 = p_3 = .4$, $p_2 = .2$ (since $p_4$ is dominated by $p_2$). This would have been the vector (8,4,8,0).

THEOREM 4. $\mathbf{p} = (p_1, \ldots, p_n)$ *(as defined above) exists, and is the optimal strategy for Reuven (and thus for Shimon , since the game is symmetric).*

*Proof.* We need to show the value of the game is 0. We show that the payoff of the mixed strategy $\mathbf{p}$, $\lim_{t\to\infty} \frac{u_t}{t+1} = 0$, where $u_t$ is the payoff of the sequence $p_{1,t}, \ldots, p_{n,t}$.

First note that the payoff for the $j$th strategy against $i_k$ is $a_{i_k,j} = -a_{j,i_k}$. Let

$$\mathbf{c}_{t+1} = (\sum_{k=0}^{t} a_{1,i_k}, \cdots \sum_{k=0}^{t} a_{n,i_k}),$$

i.e., $t\mathbf{c}$ is the payoff vector for the strategy at row $t$. Then Shimon 's payoff is $\frac{\max_j c_{t+1,j}}{t+1}$, which we want $\leq \epsilon$, for arbitrarily small $\epsilon$, so it is enough to prove $\max_j c_{t+1,j} \leq (t+1)\epsilon$ for all $j$. It is convenient to prove the following more general result

LEMMA 5. *(Main Lemma) Suppose* $\mathbf{c}_1$,$\mathbf{c}_2, \ldots,$ *is a sequence of vectors for which* $\mathbf{c}_0 \leq \mathbf{0}$ *(in each component) and each* $\mathbf{c}_{k+1} = \mathbf{c}_k + \mathbf{a}_{i_k}$*, where* $\mathbf{a}_{i_k} = (a_{1,i_k}, \ldots, a_{n,i_k})$ *and* $a_{ij} = -a_{ji}$*. Write* $c_{k,j}$ *for the* $j$*-th component of* $\mathbf{c}_k$*. Then for large enough* $t$*,*

$$\max_j c_{t,j} \leq t\epsilon$$

*for all $j$.*

*Proof.* We will find a function $f = f(n; a, \epsilon)$, where $n \in \mathbb{N}$, $a \in \mathbb{R}^+$, and $\epsilon \in \mathbb{R}^+$, such that for any antisymmetric matrix $(a_{ij})$ with each $a_{ij} \le a$ and each $t > f(n; a, \epsilon)$, that each $\max_j c_{tj} \le t\epsilon$. If $n = 1$ then $a = 0$ so the assertion is obvious (since $c_{t1} = c_{01} \le 0$ is non-positive). Thus we assume $n > 1$.

STEP 1. For any $t$, there exists $j \le n$ such that the entry $c_{t,j} \le 0$. (Indeed take the sum of the $j_k$- components from $1 \le k \le t$, counting repetitions of the subscript. Then for any give $i_k$, $c_{t,i_k} = \sum_\ell a_{i_\ell, i_k}$, so

$$\sum_k c_{t,i_k} = \sum_{\ell,k} a_{i_\ell,i_k}.$$

But $\sum_{\ell,k} a_{i_\ell,i_k} = 0$ since the subscripts are symmetric in the components but $(a_{ij})$ is skew-symmetric. Thus the left-hand side $\le 0$, so one of its summands $\le 0$.

STEP II Given $\eta > 0$ (to be determined later), define $t_1 = f(n-1, a, \eta)$. Also, write $j_k$ for that $j$ for which $\max_j c_{k,j} = c_{k,j_k}$. We consider $t_1$ consecutive rows starting from a given $t$.

First we claim that if there is an index $i \ne j_{t_1}, \dots, j_{t+t_1}$ then $\max c_{t+t_1,j} \le \max c_{t,j} + t_1\eta$. (Indeed, take $b = \max_{j \ne i} c_{t_1,j}$ and define the vector of length $n-1$, $\mathbf{c}'_k$, by

$$\mathbf{c}'_{k,j} = \begin{cases} \mathbf{c}_{k+t_1,j} - b & \text{for} \quad j < i; \\ \mathbf{c}_{k+t_1,j-1} - b & \text{for} \quad j > i; \end{cases}$$

thus, we have eliminated the $i$-th column and normalized to make $\mathbf{c}' < \mathbf{0}$. (Note that neither the $i$-th row (which is the $i$th column of the previous step) nor the $i$-th column enters into the computation. Thus one can strike out the $i$-th row and column from the antisymmetric matrix $(a_{ij})$, so is still left with an antisymmetric $n-1 \times n-1$ matrix. Of course one has to compress the indices, which are $1, 2, \dots, i-1, i+1, \dots, n$. Then by induction

$$t\eta \ge \max_j \mathbf{c}'_{t_1,j} = \max_j(\mathbf{c}_{t+t_1,j}) - \max_{j \ne i} c_{t_1,j} \ge \max_j(\mathbf{c}_{t+t_1,j}) - \max_j c_{t_1,j},$$

as desired.)

On the other hand, we claim that if $j_t, \dots, j_{t+t_1}$ run over all the indices, then we claim $\max_j c_{t+t_1,j} \le t_1 a$. Indeed take $j'$ for which $c_{t,j'} \le 0$ (by step I), and by assumption we have some $k$ between $t$ and $t+t_1$ for which $j_k = j'$. This means that $c_{k,j'}$ is the maximum of the $c_{k,j}$.

Thus for any $j$,

$$c_{k,j} \le c_{k,j'} \le c_{t,j'} + (k-t)a \le (k-t)a,$$

so

$$c_{t+t_1,j} \le c_{k,j} + (t + t_1 - k)a \le (k-t)a + (t+t_1-k)a \le t_1 a,$$

as desired.

STEP III (Conclusion of proof of main lemma) Take $\eta > 0$ to be determined below, compute $t_1 = f(n-1, a, \eta)$ and given arbitrary large $t$, write $t = qt_1 + r$, by means of the Euclidean algorithm. By Step II, for any $u < q$ we see that

$$\max_j c_{(u+1)t_1+r,j} \leq \begin{cases} \max_j c_{ut_1+r,j} + t_1\eta & \text{or} \\ t_1 a, \end{cases}$$

so iterating over the $q$ values of $u$ we cannot add $t_1\eta$ more than $q+1$ times, and noting that $t_1 \leq \frac{t}{q}$ we get

$$\max_j c_{t,j} \leq t_1 a + (q+1)t_1\eta \leq t(\frac{a}{q} + \eta + \frac{\eta}{q}),$$

Since we may assume $\eta < 1$, we are done whenever $\frac{a+1}{q} + \eta \leq \epsilon$; thus we put for example $q = \frac{2(a+1)}{\epsilon}$, and take $\eta = \frac{\epsilon}{2}$ and $F(n; a, \epsilon) = qt_1$. $\square$

Given an arbitrary (fair) game, suppose (Reuven , Shimon ) starts with a strategy $(i_0, j_0)$, and inductively take $i_{k+1}$ (resp. $j_{k+1}$) to be Reuven 's (resp Shimon 's) best response to the sequence $j_0, \dots, j_k$ (resp. $i_0, \dots, i_k$. Then both $i_t$ and $j_t$ tend to the optimal strategies for Reuven and Shimon , since one sees easily that $((i_t, j_t))$ tends towards the best strategy in the symmetrized game. Thus we have proved that every game can be solved iteratively.

A corollary to this is that the iterative solution is the algebraic solution we reached earlier (when it exists!), since the algebraic solution is unique.

**Nash's theoretical solution.**

There is an extremely elegant solution of John Nash, which proves that any competitive fair 1-move game with $n$ players has an equilibrium. Indeed, consider $n$-tuples of payoff vectors $(b_1, \dots, b_n)$ where the $i$-th player gets $b_i$ based on a certain mixed strategy. At each stage, the $i$ player looks at the current strategies and picks a strategy to improve his payoff (if possible); otherwise he sticks with his strategy. In fact, he can choose several possible strategies, but assuming the payoff is a linear function, his choice of improved payoff is convex, in the sense that combining two improved payoffs still gives an improved payoff. Also, if there is a convergent sequence of improved payoffs, then the limit is also an improved playoff. Each player does this, leading to a new vector $(b'_1, \dots, b'_n)$. There is a theorem in mathematical analysis called the Kakutani Fixed Point Theorem, which says that any convex mapping closed under limits has a fixed point, and this is the Nash equilibrium!

**Theory of linear programming applied to game theory.**

Clearly Shimon 's task is to minimize $w$ such that $-\sum a_{ij}q_j + w \geq 0$ for $i = 1, \dots, n$ and $\sum q_j = 1$. Analogously, Reuven 's task is to solve the "dual problem", to maximize $u$ such that $-\sum a_{ij}p_i + u \leq 0$ for $i = 1, \dots, m$, with $\sum p_i = 1$. Not worrying about symmetric games, we note that the strategies are unaffected by adding the same constant quantity $a$ to each $a_{ij}$, so we may assume each $a_{ij} > 0$. Thus we may assume $u, w > 0$. Writing $y_j = \frac{q_j}{w}$, we have the new problem of finding non-negative $y_j$ such that

$$\sum a_{ij}y_j \leq 1 \quad \text{for each } i, \quad \text{with} \quad \sum y_j \quad \text{maximal.} \tag{8}$$

Then taking $w = \sum y_j$ we simply put $q_j = \frac{y_j}{w}$. The dual analysis holds for the $p_i$.

This leads us to solve a problem in linear programming, which can be solved by the simplex method.

**Topological background.** Let $V$ be a vector space over $\mathbb{R}$.

By *multilinear form* $f$ we mean a multilinear function $f(x_1, \dots, x_n) = \alpha_1 x_1 + \dots + \alpha_n x_n$.

*Definition 8.* A *half-space* $H^f_\alpha$ (for a number $\alpha$ and a multilinear form $f$) is the set of points $(x_1, \dots, x_n)$ given by $f(x_1, \dots, x_n) \geq \alpha$. A *polytope* is an intersection of a finite number of half-spaces. The *profile* øS of a convex set $S$ is its set of vertices.

A "line" is a subspace of dimension 1, whereas a "hyperplane" is a coset of a subspace $V_1$ of dimension $n-1$.

*Remark 12.* Every hyperplane is defined as $\{(v_1, \dots, v_n) : \alpha_i v_i = \alpha_0\}$.

A *supporting hyperplane* of a set $S$ is a hyperplane which touches $S$, but such that all of the points of $S$ are on the same side of its half-space.

THEOREM 13. *Every supporting plane of a non-empty compact convex set $S$ contains a vertex of $S$.*

*Proof.* Induction on $n = \dim S$. If $n = 0$ then $S$ is a point and there is nothing to prove; in general any supporting hyperplane $P$ which does *not* contain $S$ is also a supporting hyperplane for $P \cap S$, which has lower dimension, and thus contains a vertex $v$, which is also a vertex of $S$. □

THEOREM 14. *A point $v$ of a polytope is a vertex iff $v$ is the intersection of the generating planes through $v$.*

COROLLARY 15. *The profile of a polytope is finite.*

COROLLARY 16. *Every bounded nonempty polytope is a convex polyhedron.*

**The simplex method for solving the Fundamental Problem of Linear Programming, for the case of a hyperplane.**

Here we suppose $f$ is multilinear. Then the constraints are given by a finite set of half-spaces, so our possible solution set is a polytope $T$. Take a vertex $v$ and all $T_v$ to be the intersection of all generating hyperplanes of $T$ which pass through $v$. Then $T'$ also is a polytope. By an easy transformation we may put the origin at $v$, i.e. assume $v = 0$.

CASE I. $f(x) \leq f(0)$ for any $x$ on an edge of $T'$. Then$f(x) \leq f(0)$ for all $x$ in $T'$ (and thus for all $x$ in $T$), so $v$ is the desired point.

CASE II. There is a point $x_0$ on an edge such that $f(x_0) > f(0)$. Then for any $\alpha \in \mathbb{R}$. we have

$$f(\alpha x_0) - f(0) = \alpha(f(x_0) - f(0).$$

Take $\alpha$ maximal such that $\alpha x_0 \in T$. Then $\alpha x_0$ is the next vertex, and since $T$ has a finite number of vertices, we conclude by iteration.

Note that the problem here is to determine the polytope in question, since the solution lies on one of its vertices. For $m = n = 2$ this can be done quite easily on graph paper. For example using (8) we have four inequalities whose intersection is (in the nondegenerate case) a quadrilateral formed from the $X$ and $Y$ axes and two other lines $L_1, L_2$. The point $L_1 \cap L_2$ provides the solution when the game is elusive. But we have seen already that the $2 \times 2$ game can be solved quickly via algebra.

In general Dantzig provided the "simplex method" to solve the basic problem in linear programming. We start by converting (8) into the equalities

$$\sum_j a_{ij} y_j + z_j = 1 \tag{9}$$

and want

$$f = \sum y_j \tag{10}$$

to be maximal.

We start with the *basic feasible solution* $z_j = 1, q_j = 0$, and $f = 0$. At this stage the $z_j$ are called the *basic variables* and the $y_j$ are the *nonbasic variables.* We now want to replace as many $z_j$ as needed by $y_j$, at each stage increasing $f$. Clearly increasing $y_1$ would improve $f$, and, justified by the theory above, we increase $y_1$ as much as possible to find the next vertex, i.e. $y_1 = \min_i \frac{1}{a_{i_0 1}}$. Suppose this minimum occurs at $i_1$. Then we *make* $y_1$ a basic variable instead of $z_{i_1}$, using $y_1 = \frac{1}{a_{i_0 1}}(1 - z_{i_1} - \sum_{j>1} a_{i_0 j} y_j)$ to eliminate all other appearances of $y_1$ in (9) and (10). Thus (10) has become

$$f = \frac{1}{a_{i_0 1}} - z_{i_1} + \sum_{j>1} (1 - \frac{1}{a_{i_0 1}} a_{i_0 j}) y_j \tag{11}$$

If all of the coefficients $(1 - \frac{1}{a_{i_0 1}} a_{i_0 j}) \leq 0$ then we are done; any nonpositive coefficient corresponds to a dominated move. If some coefficient is positive then for that value of $j$ we make $y_j$ a basic variable, finding the suitable $z_i$ to replace by the modified version of (9). We continue doing this until we are done; since we have followed the general procedure described above, we must finish after a finite number of steps.

NOTE that in the continuation, when we create a new basic variable $y_j$ we need not replace some $z_{i_j}$, but unfortunately might replace some other $y_j$ previously found. Thus, although we eventually finish this procedure, it need not be after $n$ stages. (Important question, how many steps do we need; van den Panne conjectures it is between $m$ and $3m$.)

**Convex functions with linear constraints.**

Let us view all this in a more general framework. We say a function $f\colon V \to \mathbb{R}$ is *convex* if for any nonnegative numbers $\alpha_1, \alpha_2$ and any given $v_1, v_2$ in $V$ we have $f(\alpha_1 v_1 + \alpha_2 v_2) \leq \alpha_1 f(v_1) + \alpha_2 f(v_2)$. (We say $f$ is *concave* if $-f$ is convex.) Note that any linear function is both concave and convex.

*The Fundamental problem of linear programming* Given $(a_{ij})$ and $b_i$, find $y_1, \dots, y_n$ with $\sum a_{ij} y_j \geq b_i$ for $1 \leq i \leq r$ and $\sum a_{ij} y_j = b_i$ for $i = r+1, \dots, m$, such that the convex function $f(y_1, \dots, y_n)$ is minimized.

Lemma 17. *If $C$ is a closed convex nonempty set in $\mathbb{R}^{(n)}$ not containing the origin, then there is a linear form $f$ and $\alpha > 0$ such that if $x \in C$ then $f(x) > \alpha$.*

*Proof.* Take a closed ball $B$ with center at the origin, which intersects $C$. Then $C \cap B$ is compact and thus has some point $v$ with minimum norm. Take $\alpha = ||v||^2$; then taking $f(x) = \langle v, x \rangle$ (the scalar product) we shall see $f(x) \geq \alpha$ for all $x \in C$.

Indeed, suppose on the contrary there is $x \in C$ with $\epsilon = ||v||^2 - \langle v, x\rangle > 0$, Then for any $\beta$ we get

$$|(1-\beta)v + \beta x||^2 \leq ||v||^2 - 2\beta||v||^2 + \beta^2||v||^2 + 2\beta(1-\beta)(||v||^2 - \epsilon) + \beta^2||x||^2$$
$$= ||v||^2 - 2\beta(1-\beta)\epsilon + \beta^2(||x||^2 - ||v||^2).$$

Choosing $0 < \beta < 1$ such that $\frac{\beta}{1-\beta}(||x||^2 - ||v||^2) < 2\epsilon$ shows we have found a new point in $C$ with smaller norm, contradiction. $\square$

Lemma 18. *If $C$ is a nonempty convex set in $\mathbb{R}^{(n)}$ not containing the origin, then there is a linear form $f$ such that if $x \in C$ then $f(x) \geq 0$.*

*Proof.* For any $x \in C$ write $A_x = \{v : ||v|| = 1 \text{ and } \langle v, x\rangle \geq 0\}$. Any finite intersection of the $A_x$ is 0 (seen by applying lemma 17 to the polyhedron formed by these points), so, since the unit sphere is compact we have $\cap_{x\in C} A_x \neq \emptyset$. $\square$

Theorem 19. *( Separation Theorems) (i) Any two nonempty disjoint convex sets $C$ and $C'$ are separated by a plane.*

*(ii) Any two nonempty disjoint convex sets $C$ and $C'$ are strictly separated by a plane, provided $C$ is compact and $C'$ is closed.*

*Proof.* In each case, we make a translation so that both $C$ and $C'$ miss the origin.

(i) $C + (-C')$ is convex and misses the origin, so there is a linear form $f$ such that $f(c - c') \geq 0$ for all $c \in C$, $c' \in C'$. Thus $\inf f(c) \geq \sup f(c')$.

(ii) Now $C + (-C')$ is also closed, so in (i) one has $f(c - c') > \alpha$ for some $\alpha > 0$; the rest is easy. $\square$

The separation in this theorem need not be strict, i.e. the plane might have to touch one or the other (but not both), even when both sets are closed, for example the lower half-plane and the upper half of the hyperbola $xy = 1$.

Corollary 19. *The closure $C$ of the convex hull of a set equals the intersection of the closed half-hyperplanes which contain it.*

*Proof.* For every point not in this intersection, there is a hyperplane which separates it from $C$. $\square$

Theorem 20. *(Intersection Theorem) Suppose $C_1, \dots C_m$ are closed convex sets in $\mathbb{R}^{(n)}$ whose union is convex. If the intersection of every $m-1$ of these is nonempty then $\cap C_i \neq \emptyset$.*

*Proof.* . Take $a_i \in \cap_{j\neq i} C_j$, for each $i$; replacing $C_i$ by its intersection with the convex hull of $\{a_1, \dots, a_n\}$, we may assume each $C_i$ is convex???. The assertion is vacuous for $n = 1$ and easy for $n = 2$, by theorem 19, since any hyperplane separating $C_1$ and $C_2$ would intersect their union, by convexity of $C_1 \cup C_2$, which is absurd.

So assume inductively that the theorem is true for $m-1$, but assume $\hat{C} = \cap_{i=1}^{m-1} C_i$ is disjoint from $C_m$. Take a separating hyperplane $P$ and let $C_i' = P \cap C_i$. Then

$$\cup_{i=1}^{m} Ci' = P \cap (\cup_{i=1}^{m-1} C_i) = P \cap (\cup_{i=1}^{m} C_i),$$

which is convex. The intersection of any $m-2$ of the $C_i$ contains $\hat{C}$ and intersects $C_m$, so therefore intersects $P$. Thus the corresponding $m-2$ of the $C_i'$ intersect; by induction we have

$$\emptyset \neq \cap_{i=1}^{m-1} C_i' = \hat{C} \cap P,$$

which is absurd. $\square$

Corollary 21. *Suppose a convex set $C$ is disjoint from $\cap_{i=1}^m C_i$ but is intersects the intersection of any $m-1$ of the $C_i$. Then $C \not\subseteq \cup C_i$.*

*Proof.* Induction on $m$, the case $m = 1$ being trivial. Take $a_i \in \cap_{j\neq i} C_j$, as before, and $C_i'$ be the intersection of $C_i$ with the convex hull $A$ of $a_1, \dots, a_n$. Then $\cup C_i$ is not convex, so misses some point $a$ of $A$. Then $a \in C \setminus \cup C_i$. $\square$

Corollary 22. *(Helley's Theorem) Suppose $C_1, \dots, C_m$ are convex sets in $R^{(n)}$, with $m > n+1$. If the intersection of any $n+1$ of the $C_i$ is nonempty then $\cap_{i=1}^m C_i \neq \emptyset$.*

*Proof.* By induction; we take $a_i \in \cap_{j\neq i} C_i$ and by iteration we see that the intersection of the polyhedra determined by all $a_j,\ j \neq i$ is nonempty. $\square$

**Von Neumann's theorem.** We now provide Von Neumann's proof that any matrix game has a solution.

Theorem 23. *(Fundamental Theorem) Suppose $C \subset \mathbb{R}^{(n)}$ is a convex set and $f_1, \dots, f_m$ convex functions on $C$. If $f_k(x) < 0$ has no solution in $C$, for $1 \le k \le m$, then there exists a function of the form $f(x) = \sum_{i=1}^{n+1} p_i f_{k_i}(x)$ with $p_i \ge 0$, $\sum p_i = 1$, and $\inf_{x\in C} f(x) \ge 0$.*

*Proof.* Let $S = \{v = (v_1, \dots v_m) \in \mathbb{R}^{(m)} :$ There exists $x \in C$ satisfying $f_i(x) < v_i$ for each $i\}$. Then $0 \neq S$, and $S$ is convex. By First Sep Thm there is a nonzero form $\sum p_i x_i$ which takes on positive values for all points in $S$. Thus for any positive numbers $\alpha_1, \dots, \alpha_m$ we have $(f_i(x) + t_i) \in S$, and thus

$$\sum p_i f_i(x) + \sum \alpha_i p_i \ge 0$$

for all $\alpha_i > 0$. Thus $\sum p_i f_i(x) \ge 0$ for all $x$ in $C$, but also each $p_i \ge 0$. Dividing through by $\sum p_i$ enables us to assume $p_i = 1$.

The theorem is proved for $m \le n+1$; for $m > n+1$ we need Helley's Theorem applied to reduce the problem to $n+1$, in which case we are already done. $\square$

*Definition 25.* A function $f\colon A \to \mathbb{R}$ is *lower semicontinuous* if for any $\alpha > 0$, the set $\{a \in A : f(a) > \alpha\}$ is open. $f$ is *upper semicontinuous* if $-f$ is upper semicontinuous.

Corollary 26. *Suppose $C \subseteq \mathbb{R}^{(n)}$ is compact convex and let $f_u\colon C \to \mathbb{R}$ be a (possibly infinite) family of convex functions which are lower semicontinuous. If the system $f_u(c) \le 0$ has no solution $c$ in $C$ then there are $p_1, \dots, p_n \ge 0$ with $\sum p_i = 1$ and $u_1, \dots u_n$ satisfying*

$$\inf_{c\in C} f(c) > 0.$$

*Proof.* For each $\epsilon > 0$ there is no $c$ satisfying all $f_u(c) \le \epsilon$; thus $C_{u,\epsilon} = \{c \in C : f_u(c) \le \epsilon\}$ are closed subsets with empty intersection, so an intersection of a finite number of subsets $C_{u_1,\epsilon_1} \cap \dots \cap C_{u_m,\epsilon_m}$ is empty, i.e. $f_{u_i} - \epsilon_i$ has no simultaneous nonpositive solution. One concludes with Theorem 23 to get

$$\inf_{c\in C} \sum_{j=1}^{n+1} p_j f_{u_{ij}}(c) \ge \sum p_j \epsilon_{ij} > 0.$$

□

THEOREM 27. *(Von Neumann's minimax theorem) Suppose $A \subset \mathbb{R}^{(m)}$ and $B \subset \mathbb{R}^{(n)}$ are nonempty compact convex sets and let $f(x,y) : \mathbb{R}^{(m)} \times \mathbb{R}^{(n)} \to \mathbb{R}$ be upper semicontinuous concave in the first component and lower semicontinuous convex in the second. Then $A \times B$ has a saddle point, i.e., there is an ordered pair $(a_0, b_0) \in A \times B$ with $f(a,b) \le f(a_0,b_0) \le f(a_0,b)$ for all $a \in A$ and $b \in B$.*

*Proof.* By upper semicontinuity, for any $b$, $f(a,b) : a \in A$ attains its upper bound $\max_{a\in A} f(a,b)$. Let

$$\alpha = \min_{b\in B} \max_{a\in A} f(a,b)$$

and

$$\beta = \max_{a\in A} \min_{b\in B} f(a,b);$$

we shall prove $\alpha = \beta$. Clearly $\alpha \ge \beta$, so we need to show $\beta > \alpha - \epsilon$ for every $\epsilon > 0$. Given $a$ in $A$ define

$$g_a(b) = f(a,b) - \alpha + \epsilon.$$

There is no $b$ for which all $g_a(b) \le 0$, so Corollary 25 yields $a_i$ and $p_i$ with $\sum p_i = 1$ such that $g = \sum p_i g_{a_i}$ is positive-valued. Then

$$f(\sum p_i a_i, b) \ge \sum p_i f(a_i, b) > \alpha - \epsilon$$

for every $b$, implying

$$\beta \ge \min f(\sum p_i a_i, b) > \alpha - \epsilon,$$

as desired. □

Sion (BG, p. 68) has generalized Von Neumann's theorem to quasi-convex instead of convex functions. (This means $\{c \in C : f(c) < \alpha\}$ is convex.)

## V OPTIMIZATION (NON-ZERO SUM) GAMES

**1. The traffic jam.**

**2. The stock market.** This would be a 0-sum game, except for two problems:
(1) There are other investments, so call it the investment game
(2) if the money supply were constant

**3. Grades.** Is this a 0-sum game or not?

**4. Strikes.**

## VI GAMES OF NEGOTIATION

We start with an amazingly simple analysis by Nash of distributing assets. Each person assigns a utility value for each object.

For example, if one has the utilities

| object : | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $U_1$ : | 1 | 3 | 10 | 5 | 4 | 2 |
| $U_2$ : | 6 | 4 | 2 | 3 | 5 | 5 |

Should we give $\{2,3,4\}$ to player 1 and $\{1,5,6\}$ to player 2, or $\{3,4,5\}$ to player 1 and $\{1,2,6\}$ to player 2, or $\{3,4\}$ to player 1 and $\{1,2,5,6\}$ to player 2?

We assume utility is linear, in the sense that one adds all the utilities together. Given a convex set $S$ of possible distributions, we write $c(S)$ for the optimal distribution. Also, we make the following assumptions:

(1) If $U_1(a) < U_1(b)$ and if $U_2(a) < U_2(b)$, then $a \neq c(S)$.
(2) If $T \supseteq S$ and $c(T) \in S$, then $c(S) = c(T)$.
(3) If $S$ is symmetric with respect to the players, then $c(S)$ is on the diagonal.

Nash's solution is to distribute the assets such that the product of the utilities is maximal. His argument: normalizing, we may assume that $U_1U_2$ takes on some maximal value on some point $a_0$ in $S$. Normalizing, (replacing $U_1(a)$ by $\frac{U_1(a)}{U_1(a_0)}$ and $U_2(a_0)$ by $\frac{U_2(a)}{U_2(p)}$ we may assume that $U_1(a_0) = U_2(a_0) = 1$. The line $L$ given by $U_1 + U_2 = 2$ is tangent to the hyperbola $H$ given by $U_1U_2 = 1$, so for any point $p \in S$ above $L$, the straight line connecting $p$ and $a_0$ crosses the $H$, and thus would yield a point of greater value for $U_1U_2$. Thus $S$ is bounded by $L$, and we enlarge $S$ to the square $T$ bounded by $L$ and centered around the line $U_1 = U_2$. Since $T$ is symmetric, $c(T)$ lies on the diagonal and thus is $a_0$. Hence $c(S) = a_0$, as desired.

The three values of the product of utility functions described above are as follows:

$$18 \cdot 16 = 288; \qquad 19 \cdot 15 = 285; \qquad 15 \cdot 20 = 300.$$