**Vol. I.**

p. 308 **2.1.10′** Given $L < R$ and $a \in R$ define $La^{-1} = \{r \in R : ra \in L\}$. If $L$ is a maximal left ideal in $R$ then so is $La^{-1}$, for any $a \neq 0$ in $R$, and core$(La^{-1}) \supseteq$ core $(L)$.

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**2.5.4′** A more explicit way of viewing theorem 2.5.22. Say a ring $R$ is *special* if there is a non-nilpotent element $a$ such that every nonzero ideal of $R$ contains a power of $a$. Prove that every prime special ring $R$ satisfying the conditions of theorem 2.5.22 is primitive. (Hint: The left ideal $\sum(l - a^i)$ is comaximal with every nonzero two-sided ideal.) Consequently any ring $R$ satisfying the conditions of theorem 2.5.22 and having no nonzero nil ideals is a subdirect product of special primitive rings. (Hint: Requires the proof of proposition 2.6.7.)

**2.10.0** The following properties of a module $M$ are equivalent: (i) $M$ is injective; (ii) Any map $f : N \to E$ satisfies $f(N) \subseteq M$, where $M$ is viewed as a submodule of its injective hull $E$; (iii) As in (ii), but for any essential extension $E$ of $M$. Note that complications arise when we dualize condition (iii) to check projectivity, since $M$ could be a non-projective module without any proper covers.

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**2.5.39** Suppose that $R$ is a domain such that for any $a, b$ in $R$ there is $n = n(a, b)$ such that $[a, [a, \dots, [a, b] \dots ]] = 0$ (taken $n$ times.) Then $R$ is commutative.

**2.5.40** Here is an interesting application of exercise 39, due to Avram Klein, extending an earlier result of Makar-Limanov: Suppose $R$ is a noncommutative domain. Then the polynomial ring $R[\lambda, \mu]$ contains a free multiplicative semigroup. (Hint: Write $Z = Z(R)$, and ad $a$ for $[a, \_]$. Take $a, b$ such that $(\mathrm{ad}^n a)b \neq 0$ for all $n$. (Such $a, b$ exist by exercise 39.) The claim is that $a' = a + \lambda$ and $b' = b + (a + \lambda)\mu$ generate a free semigroup. Indeed otherwise there are monomials $f \neq g$ in noncommuting indeterminates such that $f(a', b') = g(a', b')$; matching degrees first in $\mu$ and then in $\lambda$ enables one to assume $\deg_Y f = \deg_Y g = d$ and $\deg_X f = \deg_X g$. One may assume $f$ ends in $Y$ and $g$ ends in $X$. Write $f(X, Y - X\mu - g(X, Y - X\mu) = \sum h_i \mu^{d-i}$, where $h_i \in ZX, Y$. Then $h_1(a+\lambda, b) = 0$, and by induction on $\deg h_1$ one can show $(\mathrm{ad}\ ^n a)b = 0$. The trick is to rewrite $h_1 = \tilde{h}_1(X, [X, Y]) + \beta Y X^n$, using the equation $XY = YX + [X, Y]$, and note $\tilde{h}_1(a + \lambda, [x, b]) = 0$; clearly $\deg \tilde{h}_1 \leq \deg h - 1$, providing the inductive step.

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**2.8.2′** Given $R$-modules $M, N$, say $M$ is $N$*-projective* if every map $M \to N/L$ lifts to a map $M \to N$. (Thus "projective" means $N$-projective for every $N$.) Prove that if $M$ is $N$-projective and $\pi : M' \to M$ is any cover (i.e. $K = \ker \pi$ is small in $M'$ then for any map $f\colon M' \to N$ one has $fK = 0$. (Hint: f induces an epic $\bar{f}\colon M = M'/K \to N/f(K)$, so there is $g\colon M \to N$ such that $\eta g = f$, where $\eta\colon N \to fK$ is the canonical map. Let $h = g\pi$. For any $x$ in $M'$, $(f - h)x = fy$ for some $y$ in $K$, so $(f - h)(x - y) = 0$.

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**2.8.33** In Schanuel's lemma, show that there is an isomporphism $P_1 \oplus K_2 \to K_1 \oplus P_2$ which lifts to an isomorphism $P_1 \oplus P_2 \to P_1 \oplus P_2$.

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**Artinian modules are semi-LE.** The following exercises sketch a proof of the Camps-Dicks theorem that if $M$ is an Artinian $R$-module then $T = \mathrm{End}_R(M)$ is a semilocal ring.

**2.9.24** For $M$ Artinian, iff $f\colon M \to M$ is monic then $f$ is an isomorphism. (Hint: $f^iT = f^{i+1}T$ for some $i$.) **25** (Does not require $M$ Artinian.) For any $f, g$ in $T$ show $\ker(f - fgf) = \ker f \oplus \ker(1 - gf)$. (Hint: $x = (1 - gf)x + gfx$.)

**2.9.26** $T$ is semilocal. (Extensive hint: Let $J = \mathrm{Jac}(T)$. One must show $\bar{T} = T/J$ is semisimple Artinian as a module over itself.

Step 1. Define a relation $<$ on $\mathcal{L}(M)$ by saying $K < N$ if $K$ is a proper submodule of $N$. Looking at complements, show $\mathcal{L}(M)$ satisfies ACC with respect to this relation.

Step 2. Suppose $f \in T \setminus J$. For any $g \in T$ such that $1 - gf$ is not invertible, one has 1-gf not monic, and thus $\ker(f - fgf) > \ker f$.

Step 3. Let $\mathcal{S} = \{f \in T \setminus J : \bar{f}$ is idedmpotent (in $\bar{T}$) and $\bar{T}/(1-\bar{f})$ is semisimple Artinian. $1 \in \mathcal{S}$. Take $f$ in $\mathcal{S}$ with $\ker f$ maximal with respect to $<$, and $g \in T$ with $gf \notin J$, such that $\ker gf$ is maximal possible with $\ker(gf - fhgf) > \ker gf$, so $gf - fhgf \in J$, i.e. $\overline{gf} = \overline{fhgf}$. Conclude that $\overline{fhgf}$ is idempotent in $\overline{fTf}$, so $\overline{f - fhgf}$ is idempotent. Furthermore $\overline{Tgf}$ is simple, since for any $a \in T$ for which $\overline{agf} \neq 0$ one has $h$ satisfying $\overline{gf} = \overline{fhagf} \in \overline{Tgf}$. Conclude $\overline{T}\overline{(1 - (f - fhgf))} = \overline{T}(1 - \overline{f}) \oplus \overline{Thgf}$ is semisimple Artinian, but $f - fhgf \in \mathcal{S}$, so $f - fhgf \in J$. Hence $\overline{T} = \overline{T(1 - (f - fhgf)}$ is semisimple Artinian.)

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**Matlis' conjecture.**

Suppose $M \oplus N$ is a direct sum of indecomposable injectives. Is $M$ a direct sum of indecomposable injectives? This is known as Matlis' conjecture (and sometimes is asked more generally for LE-modules, in view of exercise 2.10.9). The next two exercises give some sample results along these lines, under the above hypothesis.

**2.10.25.** $M$ is a sum (not necessarily direct) of indecomposable injectives. (Hint: Any $x$ in $M$ is contained in a finite direct sum of indecomposable injectives. But the kernel of the natural projection $E \to M$ intersects $Rx$ trivially, so is 0, i.e. $x$ is contained in a copy of $E$ inside $M$.)

**2.10.26.** Prove Matlis' conjecture when $R$ is left Noetherian. (Hint: $M$ has a large submodule which is a direct sum of indecomposable injectives and hence is injective, and thus equals $M$.)

p. 451 **0′** If $L$ is a maximal left ideal of $R$ and $a \in R \setminus L$ then $La^{-1}$ is a maximal left ideal of $R$.

## Volume II

p. 176 **0.** $R$ has subexponential growth iff $\lim_{n\to\infty}(\log G_S(n))/n = 0$.

(after corollary 6.2.25′) Surprisingly a PI-ring $R$ can have a nilpotent ideal $N$ for which GK-dim $R/N >$GK-dim $R$, as described in exercises 6.3.20ff .

**Digression 6.3.28′** For use in invariant theory one would like to adjoin the characteristic coefficients of *all* the elements of $R$ (not just "enough"); let us call this $\tilde{C}$, and let $\tilde{T}(R) = R\tilde{C} \subseteq Q$. We shall see now when $C$ is a field that also $\tilde{C}$ is affine over $C$, and thus Noetherian; since $\tilde{T}(R)$ is f.g. over $\tilde{C}$ (as in (iv)), it follows that both $\tilde{T}(R)$ and its center are affine over $C$. In fact, modulo a result from commutative algebra to be quoted in the proof, we have

**Proposition.** *(Notation as above) $\tilde{C}$ is f.g. as $C'$-module, and thus is affine over $C$.*

*Proof.* We view $R = C\{r_1, \dots, r_t\}$ in $Q \otimes_C K \approx M_n(K)$, where $K$ is the algebraic closure of $Z(Q)$. Then each $r_k$ can be identified with the matrix $(\xi_{ij}^{(k)}$, and the characteristic polynomial of $r_k$ is $\det(\lambda \cdot 1 - (\xi_{ij}^{(k)})$, whose coeffients are certainly in $C[\xi_{ij}^{(k)} : 1 \leq i, j \leq n,\ 1 \leq k \leq t]$. Now take any $\tilde{c}$ in $\tilde{C}$. Writing $c = \sum c_u \alpha_1^{u_1} \dots \alpha_t^{u_t}$, where each $\alpha_i$ is a characteristic coefficient of a suitable element $a_i$ of $R$, we see each $a_i$ is integral over $C'$ (by Shirshov's theorem), so its conjugates are also integral over $C'$, and thus each $\alpha_i$ is integral over $C'$, i.e. each $\alpha_i$ is contained in the integral closure $\bar{C}'$ of $C'$ in $C[\xi_{ij}^{(k)}]$. Hence $c \in \bar{C}'$, thereby proving $\tilde{C} \subseteq \bar{C}'$. But $\bar{C}'$ is f.g. as a $C'$-module, by a standard result of commutative algebra, cf. Zariski-Samuel, vol. 1, p. 267 or Matsamura [ ], p. 240. Hence $\bar{C}'$ is a Noetherian $C'$-module, so its submodule $\tilde{C}$ is Noetherian and in particular f.g.

□.

**The Gelfand-Kirillov dimension modulo nilpotent ideals.** The next few exercises describe how the GK dimension passes modulo a nilpotent ideal.

**6.3.20** Suppose $M$ is an $R-T$ bimodule, where $R, T$ are $F$-algebras. Let $W = \begin{smallmatrix} R & M \\ 0 & T \end{smallmatrix}$, cf. example 1.9. Then GK-dim $W \leq$GK-dim $R$+GK-dim $T$. (Hint: Any finite dimensional subspace is contained in a suitable space $V = \begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}$, where $A, B, C$ are respective finite dimensional subspaces of $R, M, T$. Then $V^n \subseteq \begin{smallmatrix} A^n & A^n BC^n \\ 0 & C^n \end{smallmatrix}$; take logarithms mod $n$ and then let $n \to \infty$.

**6.3.21** If $I, J \triangleleft R$ with $IJ = 0$ then GK-dim $R \leq$GK-dim $R/I$+GK-dim $R/J$. (Hint: Apply Proposition 6.3.14 to exercise 20.) **6.3.22** If $N \triangleleft R$ and $N^t = 0$ then GK-dim $R/N \leq t\cdot$GK-dim $R$.

**6.3.23** Let $R = FX_1, X_2/ < X_2 >^m$ . Then $R/\text{Nil}(R) \approx FX_1 = F[X_1]$ has GK-dimension 1, but GK-dim $(R) = m$. (Hint: "′ follows from exercise 22; "″ is because there are $\binom{n}{m-1}$ monomials of degreee $m-1$ in $X_2$.

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**40.** If k is relatively prime to index($R$) then $R^{\otimes k} \approx R$. (Hint: Reduce to symbols via Merkurjev-Suslin.)

**41.** If $D$ is a division algebra of degree $p^t$, $p$ prime, then there is a field $L \supseteq F = Z(D)$ such that $[L : F]$ is prime to $p$ and $D \otimes L$ has a maximal subfield $E_0$ with a chain $E_0 \supseteq E_1 \cdots \supseteq E_t = L$ in which each $[E_i : E_{i+1}] = p$. (Hint: In proposition 7.2.11 take $E_0 = KL$ in $E$, noting Gal$(E/L)$ is solvable.)

**42.** If $p$ divides $m = \text{index}(R)$ then index$(R^{\otimes p})$ divides $m/p$. (Hint: Assume $R$ is a division ring and compute index$(R_1^{\otimes p})$ where $R_1 = C_{D \otimes L}(E_1)$, notation as in exercise 41.)

**Group Algebras satisfying a PI.** Passman [89] has found a much shorter proof of the Isaacs-Passman Theorem, that the group algebra $F[G]$ satisfies a PI (where char$(F) = 0$) iff $G$ has an Abelian subgroup of finite index; we present the proof here. By theorem 8.1.52, one may assume $G = \Delta(G)$.

**Lemma 8.1.53.** *Suppose $G = \triangle(G)$, and $W$ is a finite central subgroup of $G$. If $0 \neq \alpha \in F[W]$ and $\alpha f(X_1, \ldots, X_n)$ is a multilinear identity of $F[G]$ (over $F[W]$) then $G$ has subgroups $A \supseteq B$ such that $B$ is finite, $A$ has finite index in $G$, $B \subseteq Z(A)$, and $A/B$ is commutative.*

*Proof.* One is done unless $G' \not\subseteq W$. Take $x, y$ in $G$ such that $z = xyx^{-1}y^{-1} \notin W$. Let $C = C_G(x) \cap C_G(y)$, which has finite index in $G$.

We proceed by induction on the number of monomials $m = m(f)$ of $f$. Clearly $m \geq 2$, so one may assume that $X_1$ occurs before $X_2$ in some but not all monomials of $f$. Let $g$ be the sum of those monomials in which $X_1$ occurs before $X_2$, and $h = f - g$. Then

$$0 = yx\alpha f(c_1, \ldots, c_n) = yx\alpha(g(c_1, \ldots, c_n) + h(c_1, \ldots, c_n))$$

and

$$0 = \alpha f(xc_1, yc_2, \ldots, c_n) = xy\alpha g(c_1, \ldots, c_n) + yx\alpha h(c_1, \ldots, c_n)$$

Subtracting these two equations yields

$$0 = \alpha(xy - yx)g(c_1, \ldots, c_n) = \alpha yx(z - 1)g(c_1, \ldots, c_n).$$

If $z \notin C$ then the coefficient of 1 yields $\alpha yxg(c_1, \ldots, c_n) = 0$, so $g(c_1, \ldots, c_n) = 0$ for all $c_i$ in $C$; replacing $G$ by $C$ and $f$ by $g$, one is done by induction.

If $z \in C$ then $z \in Z(C)$ has finite period, by corollary 8.1.33, so $W' = W < z >$ is a finite subgroup of $C$, and we replace $G$ by $C$, $W$ by $W'$, and $f$ by $g$, and again are done by induction.

*Proof of the Isaacs-Passman Theorem.* Take $A, B, G$ as in exercise 27. Replacing $G$ by $A$, we may assume $G'$ is finite and central; take $HG$ of finite index, with $H'$ minimal possible. Then $K' = H'$ for any subgroup $K$ of $H$ having finite index. For any prime ideal $P$ of $F[H]$ let $S$ be the central localization of $F[H]/P$. $S$ is generated over its center by the image of a finite number of elements of $H$, whose common centralizer $C$ thus has finite index in $H$. Thus the image of $C$ in $S$ is central. Hence $H' = C' \subseteq 1 + P$. Since $F[H]$ is semiprime, we conclude $H' = 1$, i.e. $H$ is Abelian.

The original Isaacs-Passman proofs have explicit bounds on the index of the Abelian subgroup in terms of the PI-degree; part of this can be gleaned from exercise 27.

**Exercise 8.1.27** Using an ultraproduct argument, show that the index of the Abelian subgroup of $G$ can be bounded by a function of the PI-degree of $F[G]$.