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Galois Trimester — Institut Henri Poincaré

Lecture 1

~~Let $F/$~~ The two lectures will be independent.

Lect 1. Supersingular mod p reps. of $GL_2(F)$

2. Weights in Serre's conjecture for totally real fields in which p ramifies.

Let $F/\mathbb{Q}_p$ be a finite extension, $\mathcal{O}\subseteq F$, $\pi\in\mathcal{O}$ uniformizer, $k=\mathcal{O}/(\pi)\cong\mathbb{F}_q=\mathbb{F}_{p^f}$. The mod p local Langlands correspondence should be an injective map

$$\left\{\begin{array}{c}\text{continuous smooth}\\ \text{Galois reps}\\ \rho: \mathrm{Gal}(\bar{F}/F)\to GL_n(\bar{\mathbb{F}}_p)\end{array}\right\}\longrightarrow\left\{\begin{array}{c}\text{smooth admissible}\\ \cancel{\text{irreducible}}\ \bar{\mathbb{F}}_p\text{-reps. of}\\ GL_n(F)\end{array}\right\}.$$

$$\rho\longmapsto\pi(\rho)$$

What is the image of this map? In general, the set on the RHS is far too large. One restriction on the objects in the image comes from the weights in Serre's conjecture and has been mentioned in Diamond's lectures.

Serre's conjecture

Def Let $L/\mathbb{Q}$ be a global field, $\wp|p$ a place of L such that $L_\wp\cong F$. A Serre weight at $\wp$ is an irreducible $\bar{\mathbb{F}}_p$-rep rep. of $GL_n(\mathcal{O}_L/\wp)$, which necessarily factors through $GL_n(\mathcal{O}/\pi)$.

A global Serre weight is an irred. $\bar{\mathbb{F}}_p$-rep of $GL_n(\mathcal{O}_L/p)=GL_n(\mathcal{O}_L/\wp_i^{r_i})$ if $p\mathcal{O}_L=\wp_1^{r_1}\cdots\wp_s^{r_s}$. This factors through $\prod_{v|p}GL_n(k_v)$.

Given a global Galois rep $\tilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to GL_n(\overline{\mathbb{F}}_p)$ ~~in many cases~~ one can associate to it a set of global Serre weights $D(\tilde{\rho})$ — the modular weights. Various generalizations of Serre's conjecture specify these weights. For $n=2$, more about this in tomorrow's lecture.

In all conjectures, $D(\tilde{\rho}) = \{\sigma = \bigotimes_{v|p} \sigma_v : \sigma_v \in D_v(\tilde{\rho}|_{G_{L_v}})\}$ where $D(\tilde{\rho}|_{G_{L_v}})$ is a set of Serre weights at

v that depends only on $\tilde{\rho}|_{G_{L_v}}$, in fact on $\tilde{\rho}|_{I_{L_v}}$. So if $\tilde{\rho}|_{G_{L_\wp}} \cong \rho$, it makes sense to speak of $D(\rho)$.

Let $\mathbb{Q}_p \subseteq F_0 \subseteq F$ be the maximal unramified subextension of F. Let $e = [F:F_0]$. Then there are e reps $\overset{(n=2)}{\rho_1', \dots, \rho_e'} : \mathrm{Gal}(\overline{F}/F_0) \to GL_n(\overline{\mathbb{F}}_p)$ such that $D(\rho)$ is a multi-set

$$D(\rho) = \coprod_{i=1}^{e} D(\rho_i').$$

Generically the $D(\rho_i')$ have multiplicity 1. For e small and ρ sufficiently generic, $D(\rho)$ can also have multiplicity one.

Emerton: let
$$G = GL_n(F)$$
$$K = GL_n(\mathcal{O}) \subset G$$
$$Z = Z(G) \cong F^\times$$

View Serre weights at p as reps. of KZ via inflation $K \twoheadrightarrow GL_n(k)$ and letting π act trivially. Then we should have

$$\mathrm{soc}_{KZ}\, \pi(\rho) = \bigoplus_{\sigma \in D(\rho)} \sigma,$$

where $D(\rho)$ is viewed as a multiset.

Let π be a rep. of G. Let $\sigma|_{KZ} \subseteq \pi|_{KZ}$ be an irreducible subrep. Then, up to twist by an unramified character, σ is a Serre weight. By Frobenius reciprocity, we have a map $\mathrm{ind}_{KZ}^{G}\sigma \longrightarrow \pi$. Here we have compact induction.

<u>Partial classification of reps. of G</u>

<u>Thm</u> (Barthel–Livné for $n=2$, Herzig for $n \geq 3$)

$$\mathrm{End}_G(\mathrm{ind}_{KZ}^{G}\sigma) \cong \overline{\mathbb{F}}_p[T_1, \ldots, T_{n-1}]$$

where $T_1, \ldots, T_{n-1}$ are explicit generators.

Let $$I = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \bmod \pi \right\} \subseteq K$$

$I(1)$ the pro-p-Iwahori subgroup.

Let W be an admissible rep. of G. For any Serre weight σ,

$$\mathrm{Hom}_G(\mathrm{ind}_{KZ}^{G}\sigma, W) = \mathrm{Hom}_{KZ}(\sigma, W|_{KZ}) \subseteq W^{I(1)} \quad \text{finite-dimensional.}$$

$$\circlearrowleft \qquad\qquad \varphi \longmapsto \varphi(\text{highest wt. vector})$$

$$\mathrm{End}_G(\mathrm{ind}\,\sigma)$$

There is an eigenvector since $\mathrm{End}_G(\mathrm{ind}\,\sigma)$ is commutative. Therefore, if there is a non-zero map $\mathrm{ind}\,\sigma \to W$, there is also one factoring

$$\mathrm{ind}_{KZ}^{G}\sigma \Big/ (T_1 - \lambda_1, \ldots, T_{n-1} - \lambda_{n-1})\,\mathrm{ind}_{KZ}^{G}\sigma \qquad \lambda_i \in \overline{\mathbb{F}}_p.$$

Note: if $\dim \mathrm{Hom}_{KZ}(\sigma, W|_{KZ}) = 1$, then every map is an eigenvector.

Def W is called supersingular if we may take $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$.

Now go to the case $n=2$ to make things simpler.

Thm (Barthel–Livné). If $\lambda \neq 0$, then $\mathrm{ind}_{KZ}^{G} \sigma / (T-\lambda)$ has a unique irreducible quotient (and is itself irreducible most of the time).

The irreducible $\overline{\mathbb{F}}_p$-reps. of G are:

1) principal series $\mathrm{ind}_B^G (\chi_1 \otimes \chi_2)$ $\quad \chi_1 \neq \chi_2$

$$\chi_1 \otimes \chi_2 : B \longrightarrow \overline{\mathbb{F}}_p^{\times}$$
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \chi_1(a)\,\chi_2(d)$$

2) characters $\chi \circ \det$, $\quad \chi : F^{\times} \to \overline{\mathbb{F}}_p^{\times}$

3) Steinberg $(\chi \circ \det) \otimes \mathrm{St}$

$$0 \to \mathbb{1} \to \mathrm{ind}_B^G \mathbb{1} \to \mathrm{St} \to 0$$

4) supersingulars

A similar situation holds for all $n \geq 2$ by work of Herzig. All irred. reps are either supersingular or sit inside parabolic inductions whose composition series may be described explicitly.

Thm (Breuil, 2001). If $G = GL_2(\mathbb{Q}_p)$, then all $\mathrm{ind}\, \sigma / T$ are irreducible, and there is a bijection

$$\left\{ \begin{array}{c} \text{irreducible} \\ \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p) \to GL_2(\overline{\mathbb{F}}_p) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible} \\ \text{supersingular } \overline{\mathbb{F}}_p\text{-reps} \\ \text{of } GL_2(\mathbb{Q}_p) \end{array} \right\}$$

given by $\pi(\rho) = \mathrm{ind}_{KZ}^{G} \sigma / T$ where $\sigma \in D(\rho)$.

Recall from Diamond's lectures that $D(\rho)$ usually has two elements, but Breuil shows that exactly the right isomorphisms between the $\mathrm{ind}\, \sigma/T$ hold.

Def A supersingular rep. W has good socle if

$$\mathrm{soc}_{KZ} W = \bigoplus_{\sigma \in D(\rho)} \sigma \quad \text{for a generic } \rho.$$

If $F/\mathbb{Q}_p$ is unramified, then Breuil and Paskunas construct a family of irreducible supersingular reps of $G = GL_2(F)$ with good socle. If $F/\mathbb{Q}_p$ ramifies, the same construction works but can't prove irreducibility. Motivation of the work discussed below is to prove it.

From now on, assume $F/\mathbb{Q}_p$ is totally ramified (everything we say can be done in general, but much messier).

Up to twist, for a generic irreducible $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to GL_2(\overline{\mathbb{F}}_p)$ have $D(\rho) = \{\sigma_0, \dots, \sigma_{e-1}\} \cup \{\sigma_0', \dots, \sigma_{e-1}'\}$

$$\sigma_0 = \mathrm{Sym}^r \, \overline{\mathbb{F}}_p^2 \qquad \sigma_0' = \det{}^r \, \mathrm{Sym}^{p-1-r} \, \overline{\mathbb{F}}_p^2$$

$$\sigma_i = \det{}^{-i} \otimes \mathrm{Sym}^{r+2i} \, \overline{\mathbb{F}}_p^2 \qquad \sigma_i' = \det{}^{r+i} \otimes \mathrm{Sym}^{p-1-r-2i} \, \overline{\mathbb{F}}_p^2.$$

In this case, the conjecture is proved by Gee–Savitt.

We will define a sequence of quotients

$V_0 \twoheadrightarrow V_1 \twoheadrightarrow \cdots \twoheadrightarrow V_{e-1}$ such that:

Thm 1 Let W be a supersingular admissible quotient of $V_0 = \mathrm{ind}\, \sigma_0 / T(\mathrm{ind}\, \sigma_0)$ with good socle. ~~Then~~ If

W does not contain a principal series as a G-subrepresentation (e.g. if W irreducible or if W arises from the Breuil–Paskunas construction) then W factors through V_{e-1}.

<u>Construction</u> $\mathrm{soc}_{KZ}(V_0) = \sigma_0 \oplus \sigma_0' \oplus \bigoplus_{i=1}^{\infty} \sigma_i'$.

Let $0 \neq X_0^{*} \in \sigma_0^{I(1)}$

$0 \neq X_0' \in \sigma_0'^{I(1)}$ $\quad X_0' = \beta X_0, \quad \beta = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$.

There is a "minimal" σ_1' subrep, namely one whose preimage sits closest to the origin of the Bruhat–Tits tree of GL_2 (viewing elements of $\mathrm{ind}_{KZ}^{G} \sigma_0$ as functions on vertices of the tree). This gives a function

$$\Phi_1 : \mathrm{ind}_{KZ}^{G} \sigma_1' \longrightarrow V_0$$

by Frobenius reciprocity. Define $V_1 = V_0 / \Phi_1(T(\mathrm{ind}\, \sigma_1'))$

In general, assume V_{i-1} is constructed. Weights appearing in socle are $\sigma_0, \sigma_0', \ldots, \sigma_{i-1}, \sigma_{i-1}'$ and σ_i'. There is a minimal σ_i', get $\Phi_i : \mathrm{ind}\, \sigma_i' \to V_{i-1}$, define $V_i = V_{i-1} / \Phi_i(T(\mathrm{ind}\, \sigma_i'))$.

<u>Proof of thm</u> Given $\tau : V_0 \twoheadrightarrow W$, assume it factors through V_{i-1}. Then $\mathrm{ind}\, \sigma_i' \xrightarrow{\Phi_i} V_{i-1} \xrightarrow{\tau} W$ is a ~~non-zero~~ map in $\mathrm{Hom}_G(\mathrm{ind}\, \sigma_i', W)$. If it is zero, we are done if not, By assumption on the socle of W this space is 1-dimensional, so $\tau \circ \Phi_i$ factors through some $\mathrm{ind}\, \sigma_i' / (T-\lambda)(\mathrm{ind}\, \sigma_i')$. If $\lambda = 0$ we are done