and τ factors through V_i. If $\lambda \neq 0$, then we get a non-zero map $\operatorname{ind} \sigma_i' / (T-\lambda) \operatorname{ind} \sigma_i' \xrightarrow{\tau \circ \Phi_i} W$

For our generic ρ, the source of this map is an irreducible principal series, hence $\tau \circ \Phi_i$ is injective.

Def Let $X_i'(\sigma_i')$ be the "minimal" generator of σ_{i-1} in $\operatorname{ind}_{KZ}^{G} \sigma_i' / T$. Let

$$Z_i = \Phi_i(X_i'(\sigma_i')) \in V_i.$$

Then Z_i generates an irreducible KZ-module in V_i isomorphic to σ_{i-1}.

Def A map $V_0 \to V_{e-1} \xrightarrow{\tau} W$ is amenable if $\tau(Z_i) \neq 0$ for all i.

Thm If W has good socle and $\tau : V_{e-1} \to W$ is amenable and surjective, then W is irreducible.

Proof Let $U \subseteq W$ be an irreducible G-module. Since X_0 generates V_0, it suffices to show $\tau(X_0) \in U$. Consider socle of U, some $\tau(X_i) \in U$, hence $\tau(Z_i) \in U$ as $X_i' = \beta X_i$ generates image of Φ_i.

By multiplicity one in socle of W, $\tau(Z_i) = c_i \tau(X_{i-1})$ for some scalar $c_i \neq 0$, hence $\tau(X_{i-1}) \in U$, keep going.

Rmk The family of B-P representations is parametrized by parameters in $\overline{\mathbb{F}}_p$. We expect that amenability corresponds to simultaneous non-vanishing of a finite set of polynomials in these parameters, hence Breuil–Paskunas reps. are generically irreducible.

Rmk Amenability condition is sufficient but not necessary. Work of L. Morra on socle filtration of $\mathrm{ind}_{KZ}^{G}\sigma/_{T}$, get fractal behavior.

Question. Let $c_1, \dots, c_{e-1} \in \overline{\mathbb{F}}_p^{\times}$ and consider

$$B(\vec{c}) = V_{e-1} / \langle Z_1 - c_1 X_0, \dots, Z_{e-1} - c_{e-1} X_{e-2} \rangle.$$

Is this admissible?

Any quotient with good socle is irreducible. What are they?

Lecture 2

Weights in Serre's conjecture for totally real fields where ρ is ramified.

1. Conj (Serre, 1970's). Let $\rho: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\bar{\mathbb{F}}_p)$ be a continuous, irreducible, and odd ($\det \rho(cc) = -1$) Galois representation. Then $\rho \cong \rho_f$, for f a modular form and ρ_f the Galois rep arising from it by the Eichler-Shimura-Deligne construction.

Th Serre also specified the weights of f.
The full conjecture is a thm. of Khare-Wintenberger-Kisin. The implication ρ modular $\Rightarrow$ ρ modular of right weights was known earlier. Deligne, Fontaine, etc. Will see example of one such theorem later.

Let p be prime, F totally real field, $p\mathcal{O}_F = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$.

Def A Serre weight is an irreducible $\bar{\mathbb{F}}_p$-rep of $GL_2(\mathcal{O}_F/p)$. These factor through $\prod_{i=1}^{r} GL_2(\mathcal{O}_F/\mathfrak{p}_i)$.
A Serre weight at $\mathfrak{p}$ is an irreducible $\bar{\mathbb{F}}_p$-rep of $GL_2(\mathcal{O}_F/\mathfrak{p})$.

$\rho: \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(\bar{\mathbb{F}}_p)$

We will define what it means for ρ to be modular of a Serre weight σ. For each $\mathfrak{p} | p$ will define a set $W_{\mathfrak{p}}(\rho)$ of Serre weights at $\mathfrak{p}$, determined by $\rho|_{I_{\mathfrak{p}}}$, such that

Conj ~~The modular weights of ρ~~

Let $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\overline{\mathbb{F}}_p)$ be continuous irreducible, and totally odd, and tamely ramified at all $\mathfrak{p}|p$. Then it is modular of weights

$$W(\rho) = \{\sigma = \bigotimes_{\mathfrak{p}|p} \sigma_{\mathfrak{p}} : \sigma_{\mathfrak{p}} \in W_{\mathfrak{p}}(\rho)\}.$$

If ρ is not tame, expect the same structure but with $W_{\mathfrak{p}}(\rho) \subsetneq W_{\mathfrak{p}}(\rho|_{I_p}^{ss})$.

Conjectures

Conjectures	$F=\mathbb{Q}$	Serre
	F totally real, ρ unram	BDJ
	F totally real, ρ tame	MMS
	GL_n, $n\geq 3$, $F=\mathbb{Q}$, ρ tame	Ash et al Herzig
	All, but less explicit	Gee.

Fix $\mathfrak{p}|p$, let $k_{\mathfrak{p}} = \mathbb{F}_q = \mathbb{F}_{p^f}$. The Serre weights at $\mathfrak{p}$ are:

$$\bigotimes_{\tau: k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p} \det^{w_\tau} \otimes (\mathrm{Sym}^{r_\tau} k_{\mathfrak{p}}^2 \otimes_\tau \overline{\mathbb{F}}_p)$$

$$0 \leq r_\tau \leq p-1$$
$$0 \leq w_\tau \leq p-1.$$

Let $\tau: \mathbb{F}_{q^n}^{\times} \hookrightarrow \overline{\mathbb{F}}_p$. Let $P_{\mathfrak{p}} \subseteq I_{\mathfrak{p}}$ be the wild inertia. Let $\omega_{n,\tau}$ be the fundamental character of niveau n:

$$\omega_{n,\tau}: I_{\mathfrak{p}} \to I_{\mathfrak{p}}/P_{\mathfrak{p}} \simeq \varprojlim \mathbb{F}_{q^n}^{\times} \twoheadrightarrow \mathbb{F}_{q^n}^{\times} \xrightarrow{\tau} \overline{\mathbb{F}}_p^{\times}.$$

If ρ is tamely ramified, then $\rho|_{I_\mathfrak{p}}$ factors through $I_\mathfrak{p}/P_\mathfrak{p}$. ~~$G_\mathfrak{p}$ acts by conjugation~~ This is abelian, so $\rho|_{I_\mathfrak{p}} = \psi \oplus \psi'$. $G_\mathfrak{p}$ acts by conjugation, so $\{\psi^q, \psi'^q\} = \{\psi, \psi'\}$. Two cases:

1) $\rho|_{G_\mathfrak{p}}$ reducible, ψ and ψ' of niveau f

2) $\rho|_{G_\mathfrak{p}}$ irreducible, $\psi' = \psi^q$, $\psi^q \neq \psi'$, ψ, ψ' of niveau $2f$.

Let e be the ramification index of $F_\mathfrak{p}/\mathbb{Q}_p$. Then:

Def 2) In irreducible case: $\sigma = \bigotimes_{\tau: k_\mathfrak{p} \hookrightarrow \overline{\mathbb{F}}_p} \det^{w_\tau} \otimes \mathrm{Sym}^{r_\tau} k_\mathfrak{p}^2 \otimes \overline{\mathbb{F}}_p$

$\in W_\mathfrak{p}(\rho)$ if and only if for each $\tau \in I = \{\tau : k_\mathfrak{p} \hookrightarrow \overline{\mathbb{F}}_p\}$ there exists $\tilde{\tau}: \mathbb{F}_{q^2} \hookrightarrow \overline{\mathbb{F}}_p$ lifting τ and an integer $0 \le \delta_\tau \le e-1$ such that

$$\rho|_{I_\mathfrak{p}} \sim \prod_\tau \omega_{f,\tau}^{w_\tau} \begin{pmatrix} \prod_{\tau \in I} \omega_{2f,\tilde{\tau}}^{r_\tau + 1 + \delta_\tau + q(e-1-\delta_\tau)} & 0 \\ 0 & (\quad)^q \end{pmatrix}$$

1) If $\rho|_{I_\mathfrak{p}}$ is reducible, then $\sigma \in W_\mathfrak{p}(\rho) \iff$ there is a subset $S \subseteq I$ and $0 \le \delta_\tau \le e-1$ for all $\tau \in I$ such that

$$\rho|_{I_\mathfrak{p}} \sim \prod_{\tau \in I} \omega_{f,\tau}^{w_\tau} \begin{pmatrix} \prod_{\tau \in S} \omega_{f,\tau}^{r_\tau+1+\delta_\tau} \prod_{\tau \notin S} \omega_{f,\tau}^{e-1-\delta_\tau} & 0 \\ 0 & \prod_{\tau \notin S} \omega_{f,\tau}^{r_\tau+1+\delta_\tau} \prod_{\tau \in S} \omega_{f,\tau}^{e-1-\delta_\tau} \end{pmatrix}$$

Rmks 1) $W_\mathfrak{p}(\rho)$ should be viewed as a multiset, where the multiplicity of σ is the number of different collections of $\{\delta_\tau\}$ that give rise to it.

Note the possible $\rho|_{I_\wp}$ depend only on the residue field $k_\wp$, so they are the same for $\mathbb{Q} \subset F_0 \subset F$ max. subextension where $\wp$ is unramified. The weights arising from any given $\{\delta_\tau\}_\tau$ are a complete set of modular weights for a rep of G_{F_0}.

Yet another way to look at this: for simplicity assume there is only one prime $\wp$ of F lying above p.

$$J = \{\sigma : F_\wp \hookrightarrow \overline{\mathbb{Q}}_p\}$$
$$I = \{\tau : k_\wp \hookrightarrow \overline{\mathbb{F}}_p\} \qquad |J| = e|I|.$$

Then the conj says ρ is modular of weight $\sigma = \bigotimes_\tau \det^{w_\tau} \otimes (\mathrm{Sym}^{r_\tau} k_\wp^2 \otimes_\tau \overline{\mathbb{F}}_p) \iff$ ~~it~~ $\rho|_{G_\wp}$ has a crystalline lift $\tilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_\wp/F_\wp) \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$ with labelled Hodge-Tate weights $\{m_\sigma, n_\sigma\}_\sigma$, where for each $\tau \in I$,

$$\{m_\sigma, n_\sigma\}_\sigma = \begin{cases} \{w_\tau, w_\tau + r_\tau + 1\} & \text{for one } \sigma \text{ above } \tau \\ \{0, 1\} & \text{for the others.} \end{cases}$$

As mentioned yesterday, this conjecture (with multiplicity) specifies the K_Z-socle of $\pi(\rho|_{G_\wp})$, the rep. of $GL_2(F_\wp)$ associated to $\rho|_{G_\wp}$ by the mod p local Langlands correspondence.

Results towards the conjecture:

<u>Thm</u> (Fontaine) Suppose ~~p|~~ $F = \mathbb{Q}$ and $\rho|_{I_p}$ is irreducible. If $\rho \cong \bar{\rho}_{f,\lambda}$ for a modular form f of weight $2 \le k \le p+1$, then