$$\rho|_{I_p} \sim \begin{pmatrix} \omega_{2,\tau}^{k-1} & \\ & \omega_{2,\tau}^{p(k-1)} \end{pmatrix}.$$

Thm (MMS) Suppose $\rho|_{G_{\wp}}$ is irreducible and

$$\sigma_{\wp} = \bigotimes_{\tau : k_{\wp} \hookrightarrow \overline{\mathbb{F}}_p} \det{}^{w_\tau} \otimes \mathrm{Sym}^{r_\tau} k_\wp \otimes_\tau \overline{\mathbb{F}}_p$$

is the $\wp$-component of a modular weight. ~~$\sigma(\wp)$~~.
Suppose $r_\tau + e < p$ for all $\tau \in I$. Then $\sigma \in W_\wp(\rho)$.

Proof Generalization of Fontaine's method. See below.

Under slightly stronger hypotheses on σ and technical hypotheses on ρ, can use modularity lifting to prove σ modular $\Leftrightarrow$ $\sigma_\wp \in W_\wp(\rho)$ in some cases:

p unram. Gee

p tot. ramified Gee-Savitt.

Finally we define modularity. Let D/F be a quaternion algebra split at all places above p and exactly one infinite place. $G = \mathrm{Res}_{F/\mathbb{Q}} D^\times$ algebraic group.

Consider compact open $U \subseteq G(\mathbb{A}^\infty)$, $U = \prod_v U_v$.
Get Shimura curve X_U/F,

$$X_U(\mathbb{C}) = {}_{G(\mathbb{Q})}\backslash G(\mathbb{A}^\infty)/U.$$

We work with U of the form

$$\prod_{\wp | p} GL_2(\mathcal{O}_\wp) \times U^p \rightsquigarrow X_{0,U}$$

$$\prod_{\wp | p} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_\wp) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \wp \right\} \times U_p^p \rightsquigarrow X^{bad}_{U_1(p),U}$$

$$\prod_{\wp | p} K_1(\wp) \times U^p \rightsquigarrow X_{1,U} \qquad K_1(\wp) = \ker\left(GL_2(\mathcal{O}_\wp) \to GL_2(k_\wp)\right)$$

If U^p is sufficiently small, then $X_{1,U} \to X_{0,U}$ is a Galois cover with group $\prod_{v|p} GL_2(k_v)$.

__Def__ A Galois rep $\rho : Gal(\bar{\mathbb{Q}}/F) \longrightarrow GL_2(\bar{\mathbb{F}}_p)$ is modular of weight σ (irred. $\bar{\mathbb{F}}_p$-rep. of $\prod_{v|p} GL_2(k_v)$) if there exist D/F as above and $U \subseteq G(\mathbb{A}^\infty)$ s.t.

$$\rho \subseteq \left(\mathrm{Pic}^\circ(X_{1,U})[p] \otimes_{\mathbb{F}_p} \sigma\right)^{\prod_{v|p} GL_2(k_v)}$$

There is a Hecke algebra $\mathbb{T}$ (away from bad primes) acting on all these curves by correspondences.

If ρ is modular of wt. σ, then there exists a maximal ideal $\mathfrak{m} \subseteq \mathbb{T}$ s.t. and a "mod p Hilbert modular eigenform" $f \in H^1_{et}(X_{0,U} \times \bar{F}, \mathcal{F}_\sigma) \otimes \mathbb{T}_\mathfrak{m}$ s.t. $\rho \cong \bar{\rho}_f$ (ρ_f as in Carayol).

Let $B(k_p) \subseteq GL_2(k_p)$ upper triangular matrices.

Let $\theta : B(k_p) \longrightarrow \bar{\mathbb{F}}_p^\times$ be a character such that σ is a Jordan–Hölder constituent of $\mathrm{Ind}_{B(k_p)}^{GL_2(k_p)} \theta$.

Then ρ irreducible $\Rightarrow$ $\mathfrak{m}$ non-Eisenstein $\Rightarrow$ can find a lift $f \in H^1_{et}(X_{0,H}\otimes\bar{\mathbb{Q}}, \mathcal{F}_{\mathrm{Ind}\,\theta}) = H^1_{et}(X^{bal}_{U_1(p),U}\otimes\bar{\mathbb{Q}}, \mathcal{F}_\theta)$

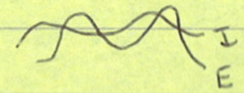
with same Hecke eigenvalues.

Moreover, $X^{bal}_{U_1(p),U}$ has an integral model over $\mathcal{O}_\wp \cap F$. Let

$D = W(k_\wp^{\bar{\mathbb{F}}_p})$ $\qquad D' = \mathcal{O}_{K'}$

$K = \mathrm{Frac}\, D = F_\wp^{nr}$ $\qquad K' = K(\sqrt[q-1]{\pi})$

Over D', $\underline{X}^{bal}_{U_1(p),U}$ has semistable reduction, with special fiber consisting of two Igusa curves intersecting transversally at finitely many points



(Katz-Mazur, Carayol, Jarvis, Gee).

$$\text{LCFT:} \quad \mathrm{Gal}(K'/K) = \mathcal{O}_\wp^*/_{1+\wp\mathcal{O}_\wp} = k_\wp^*.$$
$$\sigma \longmapsto j(\sigma)$$

$\mathrm{Gal}(K'/K)$ acts on the special fiber $\underline{X}^{bal}_{U_1(p),U}\otimes_D D'\otimes_{D'}\bar{\mathbb{F}}_p$.

Carayol congruence relations $\Rightarrow$ $\sigma \in \mathrm{Gal}(K'/K)$ acts on I (resp E) by $\begin{pmatrix}1 & 0\\ 0 & j(\sigma)^{-1}\end{pmatrix}$ (resp. $\begin{pmatrix}j(\sigma)^{-1} & 0\\ 0 & 1\end{pmatrix}$).

<u>Raynaud theory</u> Consider $J = \mathrm{Pic}^0(\underline{X}^{bal}_{U_1(p),U})$ (Néron model over D').

Have the p-divisible group $J[p^\infty]$. Pick out a piece of $J[p^\infty]\otimes \mathbb{T}/\mathfrak{m}$ on which the diagonal matrices act via θ. For suitable finite piece $G[\mathfrak{m}]$ of this, Eichler relations and Boston-Lenstra-Ribet $\Rightarrow$ $G[\mathfrak{m}]_K = \oplus(\bar\rho|_{I_p})$.

Let $\mathbb{F}$ be a finite field such that $\operatorname{im}\rho \subseteq \mathbb{F}$ and $\mathbb{F}_{q^2} \subseteq \mathbb{F}$.

Def A vector space scheme is a commutative group scheme W/D with action of a finite field $\mathbb{F}$. Let $I \subseteq \mathcal{O}_W$ be the augmentation ideal. Say W satisfies $(**)$ if I_χ is invertible for all $\chi: \mathbb{F}^\times \to D^\times$.

$$I_\chi = \{f \in I : a^* f = \chi(a) f \quad \forall a \in \mathbb{F}^\times\}.$$

$G[m]$ has a piece H/K of rank q^2 s.t.
$\mathrm{Gal}(\bar{K}/K) = I_\ell$ acts on $H(\bar{K})$ by φ $\quad \left(\rho|_{I_\ell} \sim \begin{pmatrix} \varphi & 0 \\ 0 & \varphi' \end{pmatrix}\right)$.

Raymond: have two Galois actions

1) $\mathrm{Gal}(\bar{K}/K)$ acts on $H(\bar{K}) = W(\bar{K})$ via a character

$$\varphi = \omega_{\tau_0}^{a_{2f-1}} \omega_{\tau_0^p}^{a_{2f-2}} \cdots \omega_{\tau_0^{p^{2f-1}}}^{a_0}.$$

2) $\mathrm{Gal}(K'/K)$ acts on $\cot(H_{D'}, X_{D'}, \bar{\mathbb{F}}_p)$ for a given extension $H_{D'}$ of H to D' (this isn't unique if $q > p$).

Define parameters $b_0, \dots, b_{2f-1}$ such that $\mathrm{Gal}(K'/K)$ acts on this cot via $j(\sigma)^{-b_i}$ (on appropriate generator of the cot when it is non-zero).

The parameters are related by

$$a_i' = b_{i+1} - p b_i + (q-1) a_i \quad \text{for } 0 \le a_i' \le e(q-1).$$

Assume $\theta: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d^{r_0 + p r_{f-1} + \dots + p^{f-1} r_1}$. Then, $\mathrm{Gal}(K'/K)$ acts

$\{1, \Theta(j(\sigma))^{-1}\}$

as set by $\{b_i, b_{i+g}\} = \{0, \overset{c_i}{\overset{\shortparallel}{r_i + p r_{i-1} + \dots + p^{g-1} r_{i+1}}}\}$.

<u>Cases</u> 1) $b_{i+1} = 0, \quad b_i = 0 \qquad a_i' = (q-1)a_i$

$a_i = 0, 1, \dots, e$

2) $b_{i+1} = 0 \quad b_i = c_i \qquad a_i' = -pc_i + (q-1)a_i$

$a_i = r_i + 1, \dots, r_i + e$

3) $b_{i+1} = c_{i+1} \quad b_i = 0 \qquad a_i' = c_{i+1} + (q-1)a_i$

$a_i = 0, \dots, (e-1)$

4) $b_{i+1} = c_{i+1} \quad b_i = c_i \qquad a_i' = (q-1)a_i - (q-1)r_{i+1}$

$a_i = r_{i+1}, \dots, r_{i+1} + e$.

We get some extraneous solutions. To get rid of them, consider all $\Theta: B(k_p) \to \mathbb{F}_p^\times$ such that $\sigma \in JH(\operatorname{Ind}_{B(k_p)}^{GL_2(k_p)} \Theta)$, intersect possible ρ.

Reason for the hypothesis $r_2 + e < p$ is that otherwise the intersection is too big. At the level of Θ, the above result is optimal.