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Generalized Igusa functions and ideal growth in nilpotent Lie rings

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Abstract. We introduce a new class of combinatorially defined rational functions and apply them to deduce explicit formulae for local ideal zeta functions associated to the members of a large class of nilpotent Lie rings which contains the free class-2-nilpotent Lie rings and is stable under direct products. Our results unify and generalize a substantial number of previous computations. We show that the new rational functions, and thus also the local zeta functions under consideration, enjoy a self-reciprocity property, expressed in terms of a functional equation upon inversion of variables. We establish a conjecture of Grunewald, Segal, and Smith on the uniformity of normal zeta functions of finitely generated free class-2-nilpotent groups.

Keywords: Igusa functions, combinatorial reciprocity theorem, weak orders, subgroup growth, normal zeta functions

This abstract is an exposition of some results from the preprint [5]. The objective of our work is twofold. The first aim is to introduce a new class of combinatorially defined multivariate rational functions and to indicate that they satisfy a self-reciprocity property, expressed in terms of a functional equation upon inversion of variables. The second is to apply these rational functions to describe explicitly the local ideal zeta functions associated to a class of combinatorially defined Lie rings.

1 Generalized Igusa functions

In Section 1.1 we introduce generalized Igusa functions and state the functional equations that they satisfy. In Section 1.2 we record an identity involving weak order zeta functions, motivated by our applications of Igusa functions in ideal growth. We collect here some notation and fundamental notions.

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We write $\mathbb{N} = \{1, 2, \dots\}$ and, for a subset $X \subseteq \mathbb{N}$, set $X_0 = X \cup \{0\}$. For $n \in \mathbb{N}_0$ we denote $[n] = \{1, \dots, n\}$. Given a finite subset $J \subseteq \mathbb{N}_0$, we write $J = \{j_1, \dots, j_r\}_<$ to imply that $j_1 < \dots < j_r$. The power set of a set S is denoted $\mathcal{P}(S)$.

For a variable Y and integers $a, b \in \mathbb{N}_0$ with $a \geq b$, the associated *Gaussian binomial* is

$$\binom{a}{b}_Y = \frac{\prod_{i=a-b+1}^{a}(1 - Y^i)}{\prod_{i=1}^{b}(1 - Y^i)} \in \mathbb{Z}[Y].$$

Given $n \in \mathbb{N}$ and a subset $J = \{j_1, \dots, j_r\}_< \subseteq [n-1]$, the associated *Gaussian multinomial* is defined as

$$\binom{n}{J}_Y = \binom{n}{j_r}_Y \binom{j_r}{j_{r-1}}_Y \cdots \binom{j_2}{j_1}_Y \in \mathbb{Z}[Y]. \tag{1.1}$$

The following Coxeter group theoretic interpretation of the Gaussian multinomials is used in the proof of our reciprocity theorem. Recall that the symmetric group $W = S_n$ of degree n is a Coxeter group, with Coxeter generating system $S = (s_1, \dots, s_{n-1})$, where $s_i = (i\, i+1)$ denotes the standard transposition. The *Coxeter length* $\ell(w)$ of an element $w \in S_n$ is the length of a shortest word for w with elements from S. We define the (*right*) *descent set* $\mathrm{Des}(w) = \{i \in [n-1] \mid \ell(ws_i) < \ell(w)\}$. It is well-known ([17, Proposition 1.7.1]) that the Gaussian multinomials (1.1) satisfy

$$\binom{n}{J}_Y = \sum_{w \in S_n,\, \mathrm{Des}(w) \subseteq J} Y^{\ell(w)}.$$

1.1 Generalized Igusa functions and their functional equations

Let $\underline{n} = (n_1, \dots, n_m)$ be a composition of $N = \sum_{i=1}^m n_i$ with m parts. Consider the poset $C_{\underline{n}}$ of subwords of the word $v_{\underline{n}} := a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$ in "letters" $a_1, a_2, \dots, a_m$, each occurring with respective multiplicity n_i. This poset is naturally isomorphic to the lattice

$$C_{n_1} \times \dots \times C_{n_m},$$

the product of the chains of lengths n_i with the product order, which we denote by "$\leq$". We write $\widehat{1} = v_{\underline{n}}$ and $\widehat{0}$ for the empty word.

We denote by $\mathrm{WO}_{\underline{n}}$ the chain (or order) complex of $C_{\underline{n}}$. An element $V \in \mathrm{WO}_{\underline{n}}$ is a (possibly empty) chain, or flag, of non-empty subwords of $v_{\underline{n}}$, of the form $V = \{v_1 < \dots < v_t\}$. On $\mathrm{WO}_{\underline{n}}$ we consider the partial order defined by refinement of flags, also denoted by "$\leq$". Consider the natural map

$$\begin{aligned} \underline{\pi} : C_{\underline{n}} &\to [n_1]_0 \times \dots \times [n_m]_0, \\ v = a_1^{\alpha_1} \dots a_m^{\alpha_m} &\mapsto (\alpha_1, \dots, \alpha_m) =: (\pi_1(v), \dots, \pi_m(v)). \end{aligned}$$

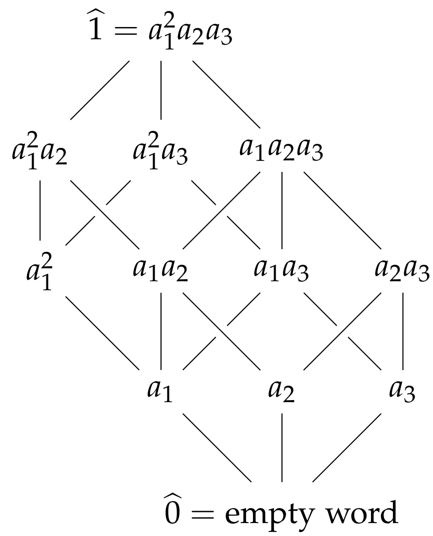

Figure 1: The poset $C_{\underline{n}}$ for $\underline{n} = (2, 1, 1)$.

Definition 1.1. We consider the induced morphism of posets

$$\begin{aligned} \underline{\varphi} : \mathrm{WO}_{\underline{n}} &\to \prod_{i=1}^{m} \mathcal{P}([n_i - 1]), \\ V = \{v_1 < \cdots < v_t\} &\mapsto \left(\{\pi_i(v_j) \mid j \in [t]\} \cap [n_i - 1]\right)_{i=1}^{m} =: (\varphi_i(V))_{i=1}^{m}. \end{aligned}$$

Definition 1.2. Let $V = \{v_1 < \cdots < v_t\} \in \mathrm{WO}_{\underline{n}}$. We define

$$W_V(\mathbf{X}) = \prod_{j=1}^{t} \frac{X_{v_j}}{1 - X_{v_j}} \in \mathbb{Q}(X_{v_1}, \ldots, X_{v_t})$$

and

$$\binom{\underline{n}}{V}_{\mathbf{Y}} = \prod_{i=1}^{m} \binom{n_i}{\varphi_i(V)}_{Y_i} \in \mathbb{Q}(Y_1, \ldots, Y_m),$$

where $\underline{\varphi}(V) = (\varphi_1(V), \ldots, \varphi_m(V))$.

Example 1.3. Let $\underline{n} = (3, 2, 2)$. For the flag $V = \{a_2 a_3 < a_1 a_2^2 a_3\} \in \mathrm{WO}_{(3,2,2)}$ we find that

$$W_V(\mathbf{X}) = \frac{X_{a_2 a_3} X_{a_1 a_2^2 a_3}}{(1 - X_{a_2 a_3})(1 - X_{a_1 a_2^2 a_3})}$$

and

$$\binom{\underline{n}}{V}_{\mathbf{Y}} = \binom{3}{1}_{Y_1} \binom{2}{1}_{Y_2} \binom{2}{1}_{Y_3} = (1 + Y_1 + Y_1^2)(1 + Y_2)(1 + Y_3).$$

The following is the key combinatorial tool of this paper.

Definition 1.4. The *generalized Igusa function associated with the composition $\underline{n}$* is

$$I^{\mathrm{wo}}_{\underline{n}}(\mathbf{Y};\mathbf{X}) := \sum_{V\in \mathrm{WO}_{\underline{n}}} \binom{\underline{n}}{V}_{\mathbf{Y}} W_V(\mathbf{X}) \in \mathbb{Q}(Y_1,\dots,Y_m,(X_r)_{r\leq v_{\underline{n}}}),$$

Example 1.5.

1. For $\underline{n}=(N)$, the trivial composition of N, we recover $I^{\mathrm{wo}}_{(N)}(\mathbf{Y};\mathbf{X}) = I_N(Y;\mathbf{X})$, the classical Igusa zeta function

$$I_n(Y;\mathbf{X}) = \sum_{I\subseteq[n]} \binom{n}{I}_Y \prod_{i\in I}\frac{X_i}{1-X_i} = \frac{\sum_{\sigma\in S_n} Y^{\ell(\sigma)}\prod_{i\in \mathrm{Des}(\sigma)} X_i}{\prod_{i=1}^n(1-X_i)} \in \mathbb{Q}(Y,X_1,\dots,X_n).$$

   In this case, the functional equation upon inversion of the variables was found by Igusa; cf. [19, Theorem 4].

2. For $\underline{n}=(1,\dots,1)$, the all-one composition of N, we recover $I^{\mathrm{wo}}_{(1,\dots,1)}(\mathbf{Y};\mathbf{X}) = I^{\mathrm{wo}}_N(\mathbf{X})$, the weak order zeta function introduced in [15, Definition 2.9]. The variables $\mathbf{Y}$ do not appear in this case, as all the polynomials $\binom{\underline{n}}{V}_{\mathbf{Y}}$ are equal to the constant 1. These functions also coincide with certain instances of generating functions associated with chain partitions in [2, Section 4.9].

3. For $\underline{n}=(2,1)$ we obtain

$$\begin{aligned} I^{\mathrm{wo}}_{(2,1)}(\mathbf{Y};\mathbf{X}) = \frac{1}{1-X_{a_1^2a_2}}\Bigg(1+\frac{X_{a_2}}{1-X_{a_2}}+\frac{X_{a_1^2}}{1-X_{a_1^2}}+ \\ (1+Y_1)\left(\frac{X_{a_1}}{1-X_{a_1}}+\frac{X_{a_1a_2}}{1-X_{a_1a_2}}+\frac{X_{a_1}}{1-X_{a_1}}\frac{X_{a_1a_2}}{1-X_{a_1a_2}}+\right. \\ \left.\frac{X_{a_1}}{1-X_{a_1}}\frac{X_{a_1^2}}{1-X_{a_1^2}}+\frac{X_{a_2}}{1-X_{a_2}}\frac{X_{a_1a_2}}{1-X_{a_1a_2}}\right)\Bigg). \end{aligned}$$

The following "combinatorial reciprocity theorem" is the main result of this section.

Theorem 1.6. *The generalized Igusa function associated with the composition $\underline{n}$ of $N=\sum_{i=1}^m n_i$ satisfies the following functional equation:*

$$I^{\mathrm{wo}}_{\underline{n}}(\mathbf{Y}^{-1};\mathbf{X}^{-1}) = (-1)^N X_{v_{\underline{n}}}\left(\prod_{i=1}^m Y_i^{-\binom{n_i}{2}}\right) I^{\mathrm{wo}}_{\underline{n}}(\mathbf{Y};\mathbf{X}).$$

The proof of Theorem 1.6 builds on a number of crucial "inversion properties" satisfied by the rational functions $W_V(\mathbf{X})$. The first is rather simple.

Lemma 1.7. *For all $V \in \mathrm{WO}_{\underline{n}}$,*

$$W_V(\mathbf{X}^{-1}) = (-1)^{|V|} \sum_{Q \leq V} W_Q(\mathbf{X}).$$

To formulate the second, more sophisticated inversion property we denote by $\mathrm{WO}_{\underline{n}}^{\times}$ the subcomplex of $\mathrm{WO}_{\underline{n}}$ of flags of *proper* subwords of $v_{\underline{n}}$. Also, for $I = (I_1, \dots, I_m) \in \prod_{i=1}^m \mathcal{P}([n_i - 1])$ we write $I^c := K \setminus I$ for $([n_1 - 1] \setminus I_1, \dots, [n_m - 1] \setminus I_m)$.

Proposition 1.8. *For all $I \in \prod_{i=1}^m \mathcal{P}([n_i - 1])$,*

$$\sum_{\substack{V \in \mathrm{WO}_{\underline{n}}^{\times} \\ \underline{\varphi}(V) \supseteq I}} W_V(\mathbf{X}^{-1}) = (-1)^{N-1} \sum_{\substack{V \in \mathrm{WO}_{\underline{n}}^{\times} \\ \underline{\varphi}(V) \supseteq I^c}} W_V(\mathbf{X}).$$

1.2 Weak order zeta functions and generalized Igusa functions

We record an identity between instances of weak order zeta functions which is used to show that our computations of the ideal zeta functions of base extensions of the Heisenberg Lie algebra match results obtained earlier by two of the authors [15]. The identity, which may be of independent interest, compares instances of weak order zeta functions associated with the all-one-compositions $\underline{g}$ and $\underline{2g}$, with g and $2g$ parts, respectively, and holds when substituting for the variables monomials satisfying certain relations.

We call a subword of the word $\widehat{1} = v_{\underline{2g}} := a_1 \cdots a_{2g}$ *radical* if it is of the form $w = \prod_{i \in \mathcal{J}} a_i a_{i+g}$ for some $\mathcal{J} \subseteq [g]$. We observe that any subword $r \leq v_{\underline{2g}}$ may be written uniquely in the form $r = \sqrt{r} \cdot r' r''$, where $\sqrt{r} = \prod_{i \in \mathcal{I}} a_i a_{i+g}$ is a radical word, whereas $r' = \prod_{i \in \mathcal{I}'} a_i$ and $r'' = \prod_{i \in \mathcal{I}''} a_{i+g}$, and the subsets $\mathcal{I}, \mathcal{I}', \mathcal{I}'' \subseteq [g]$ are disjoint.

In the following result, we omit the non-occurring variable Y from the generalized Igusa functions $I_{\underline{g}}^{\mathrm{wo}}$ and $I_{\underline{2g}}^{\mathrm{wo}}$; cf. our remark in Example 1.5 (2).

Proposition 1.9. *Let $g \in \mathbb{N}$. Suppose that $\mathbf{y} = (y_r \mid r \leq v_{\underline{2g}})$ comprises terms satisfying $y_r = y_{\sqrt{r}} \cdot \prod_{i \in \mathcal{I}' \cup \mathcal{I}''} y_{a_i}$. Then*

$$I_{\underline{2g}}^{\mathrm{wo}}(\mathbf{y}) = \left(\prod_{i=1}^{g} \frac{1 + y_{a_i}}{1 - y_{a_i}} \right) I_{\underline{g}}^{\mathrm{wo}}(\mathbf{z}),$$

where $\mathbf{z} = (z_{\prod_{i \in \mathcal{I}} a_i} \mid \prod_{i \in \mathcal{I}} a_i \leq v_{\underline{g}})$ is given by $z_{\prod_{i \in \mathcal{I}} a_i} = y_{\prod_{i \in \mathcal{I}} a_i a_{i+g}}$ for all $\mathcal{I} \subseteq [g]$.

2 Applications to ideal growth in Lie rings

2.1 Finite uniformity for ideal zeta functions of nilpotent Lie rings

Given an additively finitely generated ring $\mathcal{L}$, i.e. a finitely generated $\mathbb{Z}$-module with some bi-additive, not necessarily associative multiplication, the ideal zeta function of $\mathcal{L}$ is the Dirichlet generating series

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \sum_{I \triangleleft \mathcal{L}} |\mathcal{L} : I|^{-s}, \tag{2.1}$$

where I runs over the (two-sided) ideals of $\mathcal{L}$ of finite additive index in $\mathcal{L}$ and s is a complex variable. Prominent examples of ideal zeta functions include the Dedekind zeta functions, enumerating ideals of rings of integers of algebraic number fields and, in particular, Riemann's zeta function $\zeta(s)$.

It is not hard to verify that, for a general ring $\mathcal{L}$, the ideal zeta function $\zeta^{\triangleleft}_{\mathcal{L}}(s)$ satisfies an Euler product whose factors are indexed by the rational primes:

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s),$$

where, for a prime p,

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s) = \sum_{I \triangleleft \mathcal{L}(\mathbb{Z}_p)} |\mathcal{L}(\mathbb{Z}_p) : I|^{-s}$$

enumerates the ideals of finite index in the completion $\mathcal{L}(\mathbb{Z}_p) := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ or, equivalently, the ideals of finite p-power index in $\mathcal{L}$. Here $\mathbb{Z}_p$ denotes the ring of p-adic integers; note that ideals of $\mathcal{L}(\mathbb{Z}_p)$ are, in particular, $\mathbb{Z}_p$-submodules of $\mathcal{L}(\mathbb{Z}_p)$. It is, in contrast, a deep result that the Euler factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s)$ are rational functions in the parameter p^{-s}; cf. [8, Theorem 3.5].

Computing these rational functions explicitly for a given ring $\mathcal{L}$ is, in general, a very hard problem. Solving it is usually rewarded by additional insights into combinatorial, arithmetic, or asymptotic aspects of ideal growth. It was shown by du Sautoy and Grunewald [13] that the problem, in general, involves the determination of the numbers of $\mathbb{F}_p$-rational points of finitely many algebraic varieties defined over $\mathbb{Q}$. Only under additional assumptions on $\mathcal{L}$ may one hope that these numbers are given by finitely many polynomial functions in p. We say that the ideal zeta function of $\mathcal{L}$ is *finitely uniform* if there are finitely many rational functions $W^{\triangleleft}_1(X,Y), \dots, W^{\triangleleft}_N(X,Y) \in \mathbb{Q}(X,Y)$ such that for any prime p there exists $i \in \{1, \dots, N\}$ such that

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s) = W^{\triangleleft}_i(p, p^{-s}).$$

If a single rational function suffices (i.e. $N = 1$), we say that the ideal zeta function of $\mathcal{L}$ is *uniform*. While finite uniformity dominates among low-rank examples, including most of those included in the book [14] and those computed by Rossmann's computer algebra package Zeta [10, 11], it is not ubiquitous: for a non-uniform example in rank 9, see [12] and [18]. In general, the ideal zeta function of a direct product of rings is not given by a simple function of the ideal zeta functions of the factors. It is not even clear whether (finite) uniformity of the latter implies (finite) uniformity of the former.

2.2 Main results

We apply the generalized Igusa functions introduced in Definition 1.4 to give constructive proofs of (finite) uniformity of ideal zeta functions associated to the members of a large class of nilpotent Lie rings of nilpotency class at most 2.

Definition 2.1. Let $\mathfrak{L}$ denote the class of nilpotent Lie rings of nilpotency class at most 2 which is closed under direct products and contains the following Lie rings:

1. the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ on d generators, for $d \geq 2$.

2. the free class-2-nilpotent products $\mathfrak{g}_{d,d'} = \mathbb{Z}^d * \mathbb{Z}^{d'}$, for $d, d' \geq 0$. These have $\mathbb{Z}$-basis $\{x_i, y_j, z_{ij} \mid i \in [d], j \in [d']\}$, with the relations $[x_i, y_j] = z_{ij}$; all other pairs of basis elements commute.

3. the higher Heisenberg Lie rings $\mathfrak{h}_d$ for $d \geq 1$; these are d copies of the Heisenberg Lie ring amalgamated over their centres and have $\mathbb{Z}$-basis $\{x_1, \ldots, x_d, y_1, \ldots, y_d, z\}$, with relation $[x_i, y_i] = z$ for all $i \in [d]$. All other pairs of basis elements commute.

Note that $\mathfrak{L}$ contains the free abelian Lie rings $\mathbb{Z}^d = \mathfrak{g}_{d,0} = \mathfrak{g}_{0,d}$.

Our main "global" result produces explicit formulae for almost all Euler factors of the ideal zeta functions associated to Lie rings obtained from the members of $\mathfrak{L}$ by base extension with general rings of integers of number fields. In particular, we show that these zeta functions are finitely uniform and, more precisely, that the variation of the Euler factors is uniform among unramified primes with the same decomposition behaviour in the relevant number field.

Theorem 2.2. *Let $\mathcal{L}$ be an element of $\mathfrak{L}$, let $g \in \mathbb{N}$, and $\mathbf{f} = (f_1, \ldots, f_g) \in \mathbb{N}^g$. There exists an explicitly described rational function $W^{\triangleleft}_{\mathcal{L},\mathbf{f}} \in \mathbb{Q}(X, Y)$ such that the following holds:*

Let $\mathcal{O}$ be the ring of integers of a number field and set $\mathcal{L}(\mathcal{O}) = \mathcal{L} \otimes \mathcal{O}$. If a rational prime p factorizes in $\mathcal{O}$ as $p\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_g$, for pairwise distinct prime ideals $\mathfrak{p}_i$ in $\mathcal{O}$ of inertia degrees $(f_1, \ldots, f_g)$, then

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s) = W^{\triangleleft}_{\mathcal{L},\mathbf{f}}(p, p^{-s}).$$

In particular, $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O})}(s)$ is finitely uniform and $\zeta^{\triangleleft}_{\mathcal{L}}(s) = \zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z})}(s)$ is uniform.

A special case of Theorem 2.2 establishes part of a conjecture of Grunewald, Segal, and Smith on the normal subgroup growth of free nilpotent groups under extension of scalars. In [8], they introduced the concept of the *normal zeta function*

$$\zeta^{\triangleleft}_{G}(s) = \sum_{H \triangleleft G} |G : H|^{-s}$$

of a torsion-free finitely generated nilpotent group G, enumerating the normal subgroups of G of finite index in G. As G is nilpotent, it also satisfies an Euler product decomposition

$$\zeta^{\triangleleft}_{G}(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{G,p}(s),$$

whose factors enumerate the normal subgroups of G of p-power index. If G has nilpotency class two, then its normal zeta function coincides with the ideal zeta function of the associated Lie ring $\mathcal{L}_G := G/Z(G) \oplus Z(G)$; see [8, Remark on p. 206]. Thus, $\zeta^{\triangleleft}_{G}(s) = \zeta^{\triangleleft}_{\mathcal{L}_G}(s)$. Moreover, every class-2-nilpotent Lie ring $\mathcal{L}$ arises in this way and gives rise to a torsion-free finitely generated nilpotent group $G(\mathcal{L})$; see [21, Section 1.2] for details. Theorem 2.2 thus has a direct corollary pertaining to the normal zeta functions of the finitely generated class-2-nilpotent groups corresponding to the Lie rings in $\mathfrak{L}$. Since the groups associated to the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ are the finitely generated free class-2-nilpotent groups $F_{2,d} = G(\mathfrak{f}_{2,d})$, Theorem 2.2 implies the Conjecture on p. 188 of [8] for the case $* = \triangleleft$ and $c = 2$. The conjecture for normal zeta functions had previously been established only for $d = 2$ ([8, Theorem 3]; see also Section 2.3). We are not aware of any other case for which the conjecture has been proven or refuted.

Theorem 2.2 is a direct consequence of the following uniform "local" result. In the following, $\mathfrak{o}$ will denote a compact discrete valuation ring of arbitrary characteristic and residue field of characteristic p and cardinality q. Thus, $\mathfrak{o}$ may, for instance, be a finite extension of the ring $\mathbb{Z}_p$ of p-adic integers (of characteristic zero) or a ring of formal power series of the form $\mathbb{F}_q[\![T]\!]$ (of positive characteristic). The $\mathfrak{o}$-*ideal zeta function*

$$\zeta^{\triangleleft \mathfrak{o}}_{L}(s) = \sum_{I \triangleleft L} |L : I|^{-s}$$

of an $\mathfrak{o}$-algebra L of finite $\mathfrak{o}$-rank is defined as in (2.1), with I ranging over the $\mathfrak{o}$-ideals of L, viz. $(\operatorname{ad} L)$-invariant $\mathfrak{o}$-submodules of L. Note that every element $\mathcal{L}$ of $\mathfrak{L}$ may, after tensoring over $\mathbb{Z}$ with $\mathfrak{o}$, be considered a free and finitely generated $\mathfrak{o}$-Lie algebra. Given an $\mathfrak{o}$-module R, we write $L(R) = L \otimes_{\mathfrak{o}} R$.

Theorem 2.3. *Let $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \ldots, \mathcal{L}_g)$ be a family of elements of $\mathfrak{L}$ and $\boldsymbol{f} = (f_1, \ldots, f_g) \in \mathbb{N}^g$. There exists an explicit rational function $W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}} \in \mathbb{Q}(X, Y)$ such that the following holds:*

Let $\mathfrak{o}$ be a compact discrete valuation ring and $(\mathfrak{O}_1, \ldots, \mathfrak{O}_g)$ be a family of finite unramified extensions of $\mathfrak{o}$ with inertia degrees $(f_1, \ldots, f_g)$. Consider the $\mathfrak{o}$-Lie algebra

$$L = \mathcal{L}_1(\mathfrak{O}_1) \times \cdots \times \mathcal{L}_g(\mathfrak{O}_g).$$

For every finite extension $\mathfrak{O}$ of $\mathfrak{o}$, of inertia degree f over $\mathfrak{o}$, say, the $\mathfrak{O}$-ideal zeta function of $L(\mathfrak{O})$ satisfies

$$\zeta^{\triangleleft\mathfrak{O}}_{L(\mathfrak{O})}(s) = W^{\triangleleft}_{\mathcal{L},f}(q^f, q^{-fs}).$$

The rational function $W^{\triangleleft}_{\mathcal{L},f}$ satisfies the functional equation

$$W^{\triangleleft}_{\mathcal{L},f}(X^{-1}, Y^{-1}) = (-1)^{N_0} X^{\binom{N_0}{2}} Y^{N_0+N_1} W^{\triangleleft}_{\mathcal{L},f}(X, Y), \tag{2.2}$$

where

$$N_0 = \mathrm{rk}_{\mathfrak{o}} L = \sum_{i=1}^{g} f_i \,\mathrm{rk}_{\mathbb{Z}}(\mathcal{L}_i) \quad \text{and} \quad N_1 = \mathrm{rk}_{\mathfrak{o}}(L/Z(L)) = \sum_{i=1}^{g} f_i \,\mathrm{rk}_{\mathbb{Z}}(\mathcal{L}_i/Z(\mathcal{L}_i)).$$

Remark 2.4. In [15, Conjecture 1.4] it was suggested that a functional equation should hold for *all* local factors $\zeta^{\triangleleft}_{\mathfrak{f}_{2,2}(\mathcal{O}),p}(s)$, where $\mathfrak{f}_{2,2}$ is the Heisenberg Lie ring and $\mathcal{O}$ is a number ring; if p ramifies in $\mathcal{O}$, then the symmetry factor must be modified from that of (2.2). Some cases of the conjecture were proved in [16, Corollary 3.13]. There is computational evidence, due to T. Bauer, that other Lie rings in the class $\mathfrak{L}$ also exhibit the remarkable property of the local factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s)$ at ramified primes p being described by rational functions satisfying functional equations. Those computations, together with the results of this paper, suggest the following natural question: how do the local factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s)$ behave at ramified primes, and how does the structure of $\mathcal{L}$ govern their behaviour? A further generalization of our generalized Igusa functions will be necessary to provide explicit formulae in ramified cases.

2.3 Previous and related work

Theorems 2.2 and 2.3 generalize and unify several previously known results.

1. The most classical may be the formula for the $\mathfrak{o}$-ideal zeta function

$$\zeta_{\mathfrak{o}^n}(s) := \zeta^{\triangleleft\mathfrak{o}}_{\mathfrak{o}^n}(s) = \prod_{i=1}^{n} \frac{1}{1-q^{-s+i-1}}$$

   of the (abelian Lie) ring $\mathfrak{o}^n = \mathfrak{g}_{0,n}(\mathfrak{o}) = \mathfrak{g}_{n,0}(\mathfrak{o})$; cf. [8, Proposition 1.1].

2. The ideal zeta functions of the *Grenham Lie rings* $\mathfrak{g}_{1,d}$ were given in [19, Theorem 5].

3. Formulae for the ideal zeta functions of the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ on d generators are the main result of [20].

4. The paper [15] contains formulae for all local factors of the ideal zeta functions of the Lie rings $\mathfrak{f}_{2,2}(\mathcal{O}) = \mathfrak{g}_{1,1}(\mathcal{O}) = \mathfrak{h}_1(\mathcal{O})$, i.e. the *Heisenberg Lie ring* over an arbitrary number ring $\mathcal{O}$, which are indexed by primes unramified in $\mathcal{O}$. The uniform nature of these functions had already been established in [8, Theorem 3]. Formulae for factors indexed by non-split primes are given in [16].

5. The ideal zeta functions of the Lie rings $\mathfrak{h}_d \times \mathfrak{o}^r$ were computed in [8, Proposition 8.4], whereas for the direct products $\mathfrak{h}_d \times \cdots \times \mathfrak{h}_d$ they were computed in [1].

6. The ideal zeta function of the Lie ring $\mathfrak{g}_{2,2}$ was computed in [9, Theorem 11.1].

2.4 Informal overview of the proofs of Theorems 2.2 and 2.3

Here we summarize the principal ideas behind our approach, which greatly generalize those of [15]. Let L be an $\mathfrak{o}$-Lie algebra with derived subalgebra $L' = [L, L]$. If L is class-2-nilpotent, then an $\mathfrak{o}$-sublattice $\Lambda \leq L$ is an $\mathfrak{o}$-ideal if $[\overline{\Lambda}, L] \leq \Lambda \cap L'$, where $\overline{\Lambda} = (\Lambda + L')/L'$. For simplicity of exposition we will assume that $L' = Z(L)$, i.e. that L has no abelian direct summands. By an argument going back to [8, Lemma 6.1], the computation of $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ is reduced to a summation over pairs $(\overline{\Lambda}, \mathrm{M})$, where $\overline{\Lambda} \leq L/L'$ and $\mathrm{M} \leq L'$ are $\mathfrak{o}$-sublattices such that $[\overline{\Lambda}, L] \leq \mathrm{M}$. Recall that the $\mathfrak{O}$-elementary divisor type of a finite-index $\mathfrak{O}$-sublattice $\Lambda \leq \mathfrak{O}^n$, where $\mathfrak{O}$ is a compact discrete valuation ring with maximal ideal $\mathfrak{M}$, is the partition $(\lambda_1, \dots, \lambda_n)$ such that

$$\mathfrak{O}^n/\Lambda \simeq \mathfrak{O}/\mathfrak{M}^{\lambda_1} \times \cdots \times \mathfrak{O}/\mathfrak{M}^{\lambda_n}.$$

Given the $\mathfrak{o}$-elementary divisor type $\lambda(\overline{\Lambda})$ of $[\overline{\Lambda}, L]$, the lattices M satisfying this condition are enumerated by a formula going back to work of Birkhoff [3] (see [4, Lemma 1.4.1] and also [7, 6, 22]).

An essential ingredient of our method, therefore, is an effective description of the $\mathfrak{o}$-elementary divisor type $\lambda(\overline{\Lambda})$ in terms of the structure of $\overline{\Lambda}$. For the $\mathfrak{o}$-Lie algebras considered in this paper, this is accomplished as follows. The quotient L/L' decomposes, as an $\mathfrak{o}$-module, into a direct sum of m components, which are viewed as free modules over finite extensions $\mathfrak{O}_1, \dots, \mathfrak{O}_m$ of $\mathfrak{o}$. For each component, we consider the $\mathfrak{O}_i$-elementary divisor type $\nu^{(i)}$ of the $\mathfrak{O}_i$-lattice generated by the projection of $\overline{\Lambda}$ onto that component. The Lie rings in the class $\mathfrak{L}$ of Definition 2.1 satisfy the combinatorial condition, crucial to our method, that the parts of the partition $\lambda(\overline{\Lambda})$ are given by the minima of term-by-term comparisons among the elementary divisor types $\nu^{(1)}, \dots, \nu^{(m)}$. Assuming this, we deduce a purely combinatorial expression for $\zeta_L^{\triangleleft \mathfrak{o}}(s)$.

Analogously to the argument of [15], we break up the ideal zeta function into a sum of finitely many pieces on which the Gaussian multinomial coefficients and the dual partitions occurring in Birkhoff's formula are constant. The sum over each piece yields a product of Gaussian multinomials and geometric progressions; these, in turn, are assembled into generalized Igusa functions introduced in Section 1. The combinatorial machinery of the weak orders of Section 1.1 is required to keep track of the relative sizes of the parts of the different partitions $\nu^{(i)}$; this is necessary in order to specify domains of summation over which the dual partition $\lambda(\overline{\Lambda})'$ is constant. As in [15], Dyck words of fixed length turn out to be suitable indexing objects for the finitely many pieces. An intrinsic advantage of this combinatorial point of view over the general (and typically immensely more powerful) algebro-geometric approach is that, structurally, $\mathbb{Z}_p$ only enters as a compact discrete valuation ring. The effect of passage to various other such local rings, including those of positive characteristic, is therefore easy to control.

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