NORMAL ZETA FUNCTIONS OF THE HEISENBERG GROUPS OVER NUMBER RINGS I – THE UNRAMIFIED CASE

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Abstract. Let K be a number field with ring of integers $\mathcal{O}_K$. We compute the local factors of the normal zeta functions of the Heisenberg groups $H(\mathcal{O}_K)$ at rational primes which are unramified in K. These factors are expressed as sums, indexed by Dyck words, of functions defined in terms of combinatorial objects such as weak orderings. We show that these local zeta functions satisfy functional equations upon the inversion of the prime.

1. Introduction

1.1. **Normal zeta functions of groups.** If G is a finitely generated group, then the numbers $a_n^{\triangleleft}(G)$ of normal subgroups of G of index n in G are finite for all $n \in \mathbb{N}$. In their seminal paper [7], Grunewald, Segal, and Smith defined the *normal zeta function* of G to be the Dirichlet generating function

$$\zeta_G^{\triangleleft}(s) = \sum_{n=1}^{\infty} a_n^{\triangleleft}(G) n^{-s}.$$

Here s is a complex variable. If G is a finitely generated nilpotent group, then its normal zeta function converges absolutely on some complex half-plane. In this case the Euler product decomposition

$$\zeta_G^{\triangleleft}(s) = \prod_{p \text{ prime}} \zeta_{G,p}^{\triangleleft}(s)$$

holds, where the product runs over all rational primes, and for each prime p,

$$\zeta_{G,p}^{\triangleleft}(s) = \sum_{k=0}^{\infty} a_{p^k}^{\triangleleft}(G) p^{-ks}$$

counts normal subgroups of G of p-power index in G; cf. [7, Proposition 4]. The Euler factors $\zeta_{G,p}^{\triangleleft}(s)$ are all rational functions in p^{-s}; cf. [7, Theorem 1].

Date: September 5, 2014.

2010 *Mathematics Subject Classification.* 20E07, 11M41, 05A10, 05A15, 20F18.

Key words and phrases. Finitely generated nilpotent groups, normal zeta functions, Dyck words, generating functions, functional equations.

Schein was supported by grant 2264/2010 from the Germany-Israel Foundation for Scientific Research and Development and a grant from the Pollack Family Foundation. We acknowledge support by the DFG Sonderforschungsbereich 701 "Spectral Structures and Topological Methods in Mathematics" at Bielefeld University.

For any ring R the Heisenberg group over R is defined as

$$H(R) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in R \right\}. \tag{1.1}$$

In this paper we study the normal zeta functions of the Heisenberg groups $H(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a number field K. The groups $H(\mathcal{O}_K)$ are finitely generated, nilpotent of class two, and torsion-free.

Let $n = [K : \mathbb{Q}]$ and $g \in \mathbb{N}$. Given g-tuples $\mathbf{e} = (e_1, \dots, e_g) \in \mathbb{N}^g$ and $\mathbf{f} = (f_1, \dots, f_g) \in \mathbb{N}^g$ satisfying $\sum_{i=1}^g e_i f_i = n$, we say that a (rational) prime p is of *decomposition type* $(\mathbf{e}, \mathbf{f})$ *in* K if

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g},$$

where the $\mathfrak{p}_i$ are distinct prime ideals in $\mathcal{O}_K$ with ramification indices e_i and inertia degrees $f_i = [\mathcal{O}_K/\mathfrak{p}_i : \mathbb{F}_p]$ for $i = 1, \dots, g$. Note that this notion of decomposition type features some redundancy reflecting the absence of a natural ordering of the prime ideals of $\mathcal{O}_K$ lying above p. One of the main results of [7] asserts that the Euler factors $\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}(s)$ are rational in the two parameters p^{-s} and p on sets of primes of fixed decomposition type in K:

Theorem 1.1. [7, Theorem 3] *Given* $(\mathbf{e}, \mathbf{f}) \in \mathbb{N}^g \times \mathbb{N}^g$ *with* $\sum_{i=1}^g e_i f_i = n$*, there exists a rational function* $W^{\triangleleft}_{\mathbf{e},\mathbf{f}}(X, Y) \in \mathbb{Q}(X, Y)$ *such that, for all number fields* K *of degree* $[K : \mathbb{Q}] = n$ *and for all primes* p *of decomposition type* $(\mathbf{e}, \mathbf{f})$ *in* K*, the following identity holds:*

$$\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}(s) = W^{\triangleleft}_{\mathbf{e},\mathbf{f}}(p, p^{-s}).$$

We write $\mathbf{1}$ for the vector $(1, \dots, 1) \in \mathbb{N}^g$, all of whose components are ones. We remark (see (1.3)) that if $H(\mathbb{Z})^g$ denotes the direct product of g copies of $H(\mathbb{Z})$, then for all primes p we have

$$W^{\triangleleft}_{\mathbf{1},\mathbf{1}}(p, p^{-s}) = \zeta^{\triangleleft}_{H(\mathbb{Z})^g,p}(s).$$

1.2. **Main results.** In Theorem 3.6 we explicitly compute the functions $W^{\triangleleft}_{\mathbf{1},\mathbf{f}}(X, Y)$, thereby finding the Euler factors $\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}$ at all rational primes p that are unramified in K, i.e. those for which $\mathbf{e} = 1$. The functions $W^{\triangleleft}_{\mathbf{1},\mathbf{f}}(X, Y)$ are expressed as sums, indexed by Dyck words, where each summand is a product of functions that can be interpreted combinatorially. We use the explicit formulae to prove the following functional equations:

Theorem 1.2. *Let* $\mathbf{f} \in \mathbb{N}^g$ *with* $\sum_{i=1}^g f_i = n$*. Then*

$$W^{\triangleleft}_{\mathbf{1},\mathbf{f}}(X^{-1}, Y^{-1}) = (-1)^{3n} X^{\binom{3n}{2}} Y^{5n} W^{\triangleleft}_{\mathbf{1},\mathbf{f}}(X, Y). \tag{1.2}$$

By [18, Theorem C], the Euler factors $\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}$ satisfy a functional equation upon inversion of the parameter p for all but finitely many p. However, the methods of that paper do not determine the finite set of exceptional primes. In general it is not known whether any functional equation obtains at the exceptional primes. For the Heisenberg groups, we establish such functional equations for non-split primes in the forthcoming paper [10]:

Theorem 1.3. [10, Theorem 1.1] *Let* $e, f \in \mathbb{N}$ *with* $ef = n$. *Then*

$$W^{\triangleleft}_{(e),(f)}(X^{-1}, Y^{-1}) = (-1)^{3n} X^{\binom{3n}{2}} Y^{5n+2(e-1)f} W^{\triangleleft}_{(e),(f)}(X, Y).$$

Based on Theorems 1.2 and 1.3 and computations of Euler factors that we have performed in other cases for $n = 4$, we conjecture the existence of a functional equation at *all* primes for Heisenberg groups over number rings.

Conjecture 1.4. *Let* $(\mathbf{e}, \mathbf{f}) \in \mathbb{N}^g \times \mathbb{N}^g$ *with* $\sum_{i=1}^{g} e_i f_i = n$. *Then*

$$W^{\triangleleft}_{\mathbf{e},\mathbf{f}}(X^{-1}, Y^{-1}) = (-1)^{3n} X^{\binom{3n}{2}} Y^{5n+\sum_{i=1}^{g} 2(e_i-1)f_i} W^{\triangleleft}_{\mathbf{e},\mathbf{f}}(X, Y).$$

In particular we conjecture that, for the groups $H(\mathcal{O}_K)$, the finite set of rational primes excluded in [18, Theorem C] consists precisely of the primes that ramify in K. The conjectured existence of a functional equation at all primes is remarkable, since in general this does not hold even for groups where a functional equation is satisfied at all but finitely many primes by [18, Theorem C].

Our methods in fact allow the rational functions $W^{\triangleleft}_{\mathbf{e},\mathbf{f}}(X, Y)$ to be determined explicitly for *any* decomposition type $(\mathbf{e}, \mathbf{f})$. However, if $g > 1$ and $\mathbf{e} \neq \mathbf{1}$, then we do not in general know how to interpret these explicit formulae in terms of functions that are known to satisfy a functional equation. Conjecture 1.4 has been verified for all cases occurring for $n \leq 4$.

Prior to this work, the functions $\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}$ had been known only in a very limited number of cases; see [5, Section 2] for a summary of the previously available results. In [7, Section 8] the local functions were computed for all primes when $n = 2$ and for the inert and totally ramified primes when $n = 3$. The remaining cases for $n = 3$ were computed in Taylor's thesis [16], using computer-assisted calculations of cone integrals; see [4]. Finally, Woodward determined $W^{\triangleleft}_{\mathbf{1},\mathbf{1}}(X, Y)$ for $n = 4$. The numerator of this rational function is the first polynomial in [5, Appendix A], where it takes up nearly a full page. Example 5.2 below exhibits how our method produces this function as a sum of fourteen well-understood summands.

1.3. **Related work and open problems.** In the recent past, zeta functions associated to Heisenberg groups and their various generalizations have often served as a test case for an ensuing general theory. For instance, the seminal paper [7] contains special cases of the computations done in the present paper as examples. Similarly, Ezzat [6] computed the representation zeta functions of the groups $H(\mathcal{O}_K)$ for quadratic number rings $\mathcal{O}_K$, enumerating irreducible finite-dimensional complex representations of such groups up to twists by one-dimensional representations. The paper [15] develops a general framework for the study of representation zeta functions of finitely generated nilpotent groups. Moreover, it generalized Ezzat's explicit formulae to arbitrary number rings and more general group schemes.

The current paper leaves open a number of challenges. One of them is the computation of the rational functions $W^{\triangleleft}_{\mathbf{e},\mathbf{f}}$ for general $\mathbf{e} \in \mathbb{N}^g$; in the special case $g = 1$, this has been achieved in [10]. Another one is the computation of the local factors of the *subgroup zeta function* $\zeta_{H(\mathcal{O}_K)}(s)$ enumerating *all* subgroups of finite index in $H(\mathcal{O}_K)$. This has not even been fully achieved for quadratic number rings $\mathcal{O}_K$.

More generally, it is of interest to compute the (normal) subgroup zeta functions of other finitely generated nilpotent groups, and their behavior under base extension. We refer to [5] for a comprehensive list of examples. In his M.Sc. thesis [1], Bauer has generalized many of our results to the normal zeta functions of the higher Heisenberg groups $H_m(\mathcal{O}_K)$ for all $m \in \mathbb{N}$, where H_m is a centrally amalgamated product of m Heisenberg groups. In other words, if R is a ring and we view elements of R^m as row vectors, and if I_m denotes the $m \times m$ identity matrix, then

$$H_m(R) = \left\{ \begin{pmatrix} 1 & \mathbf{a} & c \\ 0 & I_m & \mathbf{b}^T \\ 0 & 0 & 1 \end{pmatrix} \mid \mathbf{a}, \mathbf{b} \in R^m,\, c \in R \right\}.$$

The paper [8] arose from the (uncompleted) project to compute the subgroup zeta functions $\zeta_{H_m(\mathbb{Z})}(s)$.

1.4. **Structure of the proofs of the main results.** The problem of counting normal subgroups in a finitely generated torsion-free nilpotent group of nilpotency class 2 is known to be equivalent to that of counting ideals in a suitable Lie ring; cf. [7, Section 4]. Specifically, let Z be the center of $H(\mathcal{O}_K)$; it is easy to see that this is the subgroup of matrices satisfying $a = b = 0$ in the notation of (1.1), and that it coincides with the derived subgroup of $H(\mathcal{O}_K)$. Define the Lie ring

$$L = Z \oplus (H(\mathcal{O}_K)/Z),$$

with Lie bracket induced by commutators in the group $H(\mathcal{O}_K)$. It is easy to verify that $L \cong L(\mathcal{O}_K)$ where, more generally and in analogy with (1.1), the Heisenberg Lie ring over an arbitrary ring R is defined as

$$L(R) = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in R \right\},$$

with Lie bracket induced from $\mathfrak{gl}_3(R)$. The *ideal zeta function* of $L(\mathcal{O}_K)$ is the Dirichlet generating function

$$\zeta^{\triangleleft}_{L(\mathcal{O}_K)}(s) = \sum_{n=1}^{\infty} a^{\triangleleft}_n(L(\mathcal{O}_K)) n^{-s},$$

where $a^{\triangleleft}_n(L(\mathcal{O}_K))$ denotes the number of ideals of $L(\mathcal{O}_K)$ of index n in $L(\mathcal{O}_K)$. This zeta function, too, satisfies an Euler product decomposition, of the form

$$\zeta^{\triangleleft}_{L(\mathcal{O}_K)}(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{L(\mathcal{O}_K),p}(s) = \sum_{k=0}^{\infty} a^{\triangleleft}_{p^k}(L(\mathcal{O}_K)) p^{-ks}.$$

By the remark following [7, Lemma 4.9] we have, for all primes p, that

$$\zeta^{\triangleleft}_{H(\mathcal{O}_K),p} = \zeta^{\triangleleft}_{L(\mathcal{O}_K),p}.$$

Now set $R_p = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $L_p = L(R_p)$ for every prime p. We write $L'_p = [L_p, L_p]$ for the derived subring and center of L_p, and denote by $\overline{L}_p$ the abelianization $L_p/[L_p, L_p]$. The $\mathbb{Z}_p$-modules underlying L'_p and $\overline{L}_p$ have ranks n and $2n$, respectively. Then

$$L_p = L(R_p) \cong L'_p \oplus \overline{L}_p.$$

The Euler factor $\zeta^{\triangleleft}_{L(\mathcal{O}_K),p}$ may be identified with the ideal zeta function $\zeta^{\triangleleft}_{L_p}$ of the $\mathbb{Z}_p$-Lie lattice L_p, enumerating $\mathbb{Z}_p$-ideals of L_p of finite additive index in L_p. To summarize, the following equalities hold for all primes p:

$$\zeta^{\triangleleft}_{H(\mathcal{O}_K),p} = \zeta^{\triangleleft}_{L(\mathcal{O}_K),p} = \zeta^{\triangleleft}_{L(R_p)} = \zeta^{\triangleleft}_{L_p}. \tag{1.3}$$

Essentially by [7, Lemma 6.1], we have that

$$\zeta^{\triangleleft}_{L_p}(s) = \sum_{\overline{\Lambda} \leq_f \overline{L}_p} |\overline{L}_p : \overline{\Lambda}|^{-s} \sum_{[\overline{\Lambda}, L_p] \leq M \leq L'_p} |L'_p : M|^{2n-s}. \tag{1.4}$$

Here the outer sum runs over all $\mathbb{Z}_p$-sublattices $\overline{\Lambda} \leq \overline{L}_p$ of finite additive index. We briefly summarize our strategy for computing the right hand side of (1.4). Let p be a prime of decomposition type $(\mathbf{e}, \mathbf{f})$ in K. In Lemma 2.2 we determine the isomorphism type of the finite p-group $L'_p/[\overline{\Lambda}, L_p]$ for every finite-index sublattice $\overline{\Lambda} \leq_f \overline{L}_p$. More precisely, we associate to $\overline{\Lambda}$ an n-tuple $\ell = \ell(\overline{\Lambda}) = (\ell_1, \dots, \ell_n) \in \mathbb{N}_0^n$ such that

$$L'_p/[\overline{\Lambda}, L_p] \simeq \mathbb{Z}/p^{\ell_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\ell_n}\mathbb{Z}.$$

Noting that the inner sum of (1.4) depends only on ℓ and not on $\overline{\Lambda}$, we proceed to evaluate the outer sum in terms of the parameters ℓ; cf. Lemma 2.4. By this point, we are able to transform (1.4) into the equation

$$\zeta^{\triangleleft}_{L_p}(s) = \left(\prod_{i=1}^{g} (1 - p^{-2f_i s}) \right) \zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s)\, D^{\mathbf{e},\mathbf{f}}(p, p^{-s}),$$

where

$$D^{\mathbf{e},\mathbf{f}}(p, p^{-s}) = \sum_{\ell \in \mathrm{Adm}_{\mathbf{e},\mathbf{f}}} p^{-2s\sum_{i=1}^n \ell_i} \sum_{\mu \leq \lambda(\ell)} \alpha(\lambda(\ell), \mu; p)\; p^{(2n-s)\sum_{i=1}^n \mu_i}; \tag{1.5}$$

cf. Lemma 2.19. The zeta function $\zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s)$ is well-known; cf. (2.9). We now explain the meanings of the terms in (1.5).

The set $\mathrm{Adm}_{\mathbf{e},\mathbf{f}} \subseteq \mathbb{N}_0^n$ of *admissible* n-tuples only depends on the decomposition type $(\mathbf{e}, \mathbf{f})$ of p in K; cf. Definition 2.3. For an n-tuple $\ell \in \mathbb{N}_0^n$, we define $\lambda(\ell)$ to be the partition $\lambda_1 \geq \cdots \geq \lambda_n$ obtained by arranging the components of ℓ in non-ascending order. As ℓ runs over $\mathrm{Adm}_{\mathbf{e},\mathbf{f}}$, the partitions $\lambda(\ell)$ run over all the possible elementary divisor types of commutator lattices $[\overline{\Lambda}, L_p] \leq L'_p$. The inner sum on the right hand side of (1.5) runs over all partitions μ which are dominated by $\lambda(\ell)$. Finally, $\alpha(\lambda(\ell), \mu; p)$ denotes the number of abelian p-groups of type μ contained in a fixed abelian p-group of type $\lambda(\ell)$. A classical formula of Birkhoff expresses this number in terms of the dual partitions of $\lambda(\ell)$ and μ; see Proposition 2.15.

So far, everything we have said holds for all decomposition types $(\mathbf{e}, \mathbf{f})$. The difficulty in evaluating (1.5) comes from the strong dependence of $\alpha(\lambda(\ell), \mu; p)$ on the relative sizes of the parts of the partitions $\lambda(\ell)$ and μ. For unramified primes, we overcome this difficulty by splitting $D^{\mathbf{1},\mathbf{f}}$ into a finite sum of more tractable functions. Indeed, the different ways in which the partition $\lambda(\ell)$ can "overlap" the partition μ are parametrized by Dyck words of length $2n$; see Section 2.4 for details. Given such a Dyck word w, we define a sub-sum $D_w^{\mathbf{1},\mathbf{f}}$ of $D^{\mathbf{1},\mathbf{f}}$ running over pairs of partitions $(\lambda(\ell),\mu)$ whose overlap is captured by w, so that

$$D^{\mathbf{1},\mathbf{f}} = \sum_{w\in\mathcal{D}_{2n}} D_w^{\mathbf{1},\mathbf{f}},$$

where $\mathcal{D}_{2n}$ is the set of Dyck words of length $2n$; see Section 2.6. The cardinality of $\mathcal{D}_{2n}$ is the n-th Catalan number $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$.

Each $D_w^{\mathbf{1},\mathbf{f}}$ can be expressed in terms of the Igusa functions introduced in [17] and their partial generalizations defined in Section 2.3. The latter may be interpreted as fine Hilbert series of Stanley-Reisner rings of barycentric subdivisions of simplices. Stanley proved that these rational functions satisfy a functional equation upon inversion of their variables. We deduce that the functions $D_w^{\mathbf{1},\mathbf{f}}$ all satisfy a functional equation whose symmetry factor is independent of the Dyck word w. This allows us to prove Theorem 1.2.

Acknowledgements. We are grateful to Mark Berman for bringing us together to work on this project, to Kai-Uwe Bux for conversations about face complexes, and to the referee for helpful comments.

2. Preliminaries

Throughout this paper, K is a number field of degree $n = [K : \mathbb{Q}]$ with ring of integers $\mathcal{O}_K$. By p we denote a rational prime, and we fix the abbreviation $t = p^{-s}$. For an integer $m \geq 1$, we write $[m]$ for $\{1, 2, \dots, m\}$ and $[m]_0$ for $\{0, 1, \dots, m\}$. Given integers a, b with $a \leq b$, we write $[a, b]$ for $\{a, a+1, \dots, b\}$, and $]a, b]$ for $\{a+1, a+2, \dots, b\}$.

2.1. **Lattices.** Suppose that p has decomposition type $(\mathbf{e},\mathbf{f}) \in \mathbb{N}^g \times \mathbb{N}^g$ in K, in the sense defined in Section 1.1. Then p decomposes in K as $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ are distinct prime ideals in $\mathcal{O}_K$. For each $i \in [g]$, let $k_i = \mathcal{O}_K/\mathfrak{p}_i$ be the corresponding residue field. Then $f_i = [k_i : \mathbb{F}_p]$. We define $C_i = \sum_{j=1}^{i} e_j f_j$ for each $i \in [g]_0$, so that $0 = C_0 < C_1 < \cdots < C_g = n$.

Let $R_p = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This ring is a free $\mathbb{Z}_p$-module of rank n. It splits into a direct product $R_p = R_p^{(1)} \times \cdots \times R_p^{(g)}$, where for each $i \in [g]$ the component $R_p^{(i)}$ is just the local ring $\mathcal{O}_{K,\mathfrak{p}_i}$. For each $i \in [g]$, we choose a uniformizer $\pi_i \in R_p^{(i)}$, an $\mathbb{F}_p$-basis $\{\overline{\beta}_1^{(i)}, \dots, \overline{\beta}_{f_i}^{(i)}\}$ of k_i, and a lift $\beta_j^{(i)} \in R_p^{(i)}$ of $\overline{\beta}_j^{(i)} \in k_i$ for each $j \in [f_i]$. Then the set

$$\mathcal{B}_i := \left\{\beta_j^{(i)}\pi_i^s \mid j \in [f_i], s \in [e_i - 1]_0\right\}$$

is a basis of $R_p^{(i)}$ as a $\mathbb{Z}_p$-module; see, for instance, the proof of [9, Proposition II.6.8]. The union of the bases $\mathcal{B}_i$, for $i \in [g]$, constitutes a basis $\{\alpha_1, \dots, \alpha_n\}$ of R_p as a $\mathbb{Z}_p$-module. We index it as follows:

$$\beta_j^{(i)}\pi_i^s = \alpha_{C_{i-1}+sf_i+j}.$$

We define structure constants $c_u^{km} \in \mathbb{Z}_p$, for $k, m, u \in [n]$, with respect to this basis, via

$$\alpha_k\alpha_m = \sum_{u=1}^{n} c_u^{km}\alpha_u. \tag{2.1}$$

Note that $c_u^{km} = 0$ unless there exists an $i \in [g]$ such that $k, m \in]C_{i-1}, C_i]$.

Hence we obtain the following presentation of the $\mathbb{Z}_p$-Lie ring $L_p = H(R_p)$:

$$L_p = \left\langle x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n \mid [x_k, y_m] = \sum_{u=1}^{n} c_u^{km} z_u, \text{ for } k, m \in [n] \right\rangle.$$

Here it is understood that all unspecified Lie brackets vanish. It is clear that the center of this Lie ring, which is equal to the derived subring, is spanned by $\{z_1, \ldots, z_n\}$. Similarly, the abelianization $\overline{L}_p = L_p/[L_p, L_p]$ is spanned by the images of the elements $x_1, \ldots, x_n, y_1, \ldots, y_n$. We abuse notation and denote these elements of $\overline{L}_p$ by $x_1, \ldots, x_n, y_1, \ldots, y_n$ as well.

Let $\overline{\Lambda} \leq \overline{L}_p$ be a sublattice of finite index. Then $\overline{\Lambda}$ is a free $\mathbb{Z}_p$-module of rank $2n$. Let $(b_1, \ldots, b_{2n})$ be an ordered $\mathbb{Z}_p$-basis for $\overline{\Lambda}$. Observe that each b_j can be expressed uniquely in the form

$$b_j = \sum_{k=1}^{n} b_{2k-1,j} x_k + \sum_{k=1}^{n} b_{2k,j} y_k. \tag{2.2}$$

for some $b_{1,j}, \ldots, b_{2n,j} \in \mathbb{Z}_p$. We set $B(\overline{\Lambda}) = (b_{k,j}) \in \mathrm{Mat}_{2n}(\mathbb{Z}_p)$. Conversely, the columns of any matrix $B \in \mathrm{Mat}_{2n}(\mathbb{Z}_p)$ with $\det B \neq 0$ encode generators of a sublattice of $\overline{L}_p$ of finite index in $\overline{L}_p$, by means of (2.2). The matrix $B(\overline{\Lambda})$ depends on the choice of basis; indeed, two matrices B, B' represent the same lattice if and only if there exists some $A \in \mathrm{GL}_{2n}(\mathbb{Z}_p)$ such that $B' = BA$.

If $F/\mathbb{Q}_p$ is a finite extension, we denote by val_F the normalized valuation on F. We simply write val instead of $\mathrm{val}_{\mathbb{Q}_p}$. For each $i \in [g]$ we define the following two parameters:

$$\varepsilon_i(\overline{\Lambda}) = \min \{\mathrm{val}(b_{k,j}) \mid k \in]2C_{i-1}, 2C_i],\ j \in [2n]\}, \tag{2.3}$$

$$\begin{aligned}\delta_i(\overline{\Lambda}) = \min \{& d \in [e_i - 1]_0 \mid \\ & \exists k \in]2C_{i-1} + 2df_i, 2C_{i-1} + 2(d+1)f_i], j \in [2n] :\ \mathrm{val}(b_{k,j}) = \varepsilon_i(\overline{\Lambda})\}.\end{aligned} \tag{2.4}$$

Informally, $\varepsilon_i(\overline{\Lambda})$ is the smallest valuation of any element appearing on or between the $(2C_{i-1}+1)$-st and $(2C_i)$-th rows of the matrix $B(\overline{\Lambda})$. If we split this range of $2e_i f_i$ rows into e_i blocks of $2f_i$ consecutive rows each, then $\delta_i(\overline{\Lambda})$ is the largest number such that the first $\delta_i(\overline{\Lambda})$ blocks contain no matrix elements of minimal valuation $\varepsilon_i(\overline{\Lambda})$. It is easy to see that $\varepsilon_i(\overline{\Lambda})$ and $\delta_i(\overline{\Lambda})$ are independent of the choice of basis and so are well-defined.

Definition 2.1. For $j \in [n]$ we set

$$\ell_j = \begin{cases} \varepsilon_i(\overline{\Lambda}) + 1, & \text{if } j \in]C_{i-1}, C_{i-1} + \delta_i(\overline{\Lambda}) f_i], \\ \varepsilon_i(\overline{\Lambda}), & \text{if } j \in]C_{i-1} + \delta_i(\overline{\Lambda}) f_i, C_i]. \end{cases}$$

and set $\ell(\overline{\Lambda}) = (\ell_1, \ldots, \ell_n) \in \mathbb{N}_0^n$.

Informally, the n-tuple $\ell(\overline{\Lambda})$ is a concatenation of g blocks of lengths $e_1 f_1, \ldots, e_g f_g$. Within each block, the components are all equal, except that for each $i \in [g]$ the first $\delta_i(\overline{\Lambda}) f_i$ components of the i-th block are incremented by 1. Thus $\ell(\overline{\Lambda})$ just depends on the ramification type $(\mathbf{e}, \mathbf{f})$ and the parameters $\varepsilon_i(\overline{\Lambda}), \delta_i(\overline{\Lambda})$ for each $i \in [g]$.

Lemma 2.2. *Let $\overline{\Lambda} \leq \overline{L}_p$ be a sublattice of finite index, and let $\ell(\overline{\Lambda})$ be as in Definition 2.1. Then*

$$L'_p/[\overline{\Lambda}, L_p] \cong \prod_{j=1}^{n} \mathbb{Z}/p^{\ell_j}\mathbb{Z}.$$

Proof. It is clear that

$$[\overline{\Lambda}, L_p] = \mathrm{span}_{\mathbb{Z}_p} \{[b_j, x_k], [b_j, y_k] \mid j \in [2n],\, k \in [n]\}. \tag{2.5}$$

For each $i \in [g]$, we define the following sublattice of $[\overline{\Lambda}, L_p]$:

$$[\overline{\Lambda}, L_p]_i = \mathrm{span}_{\mathbb{Z}_p} \{[b_j, x_k], [b_j, y_k] \mid j \in [2n], k \in]C_{i-1}, C_i]\}.$$

By the observation following (2.1), $[\overline{\Lambda}, L_p] = \bigoplus_{i=1}^{g} [\overline{\Lambda}, L_p]_i$. Moreover, if we set $L'_p(i)$ to be the $\mathbb{Z}_p$-submodule of L'_p generated by $\{z_{C_{i-1}+1}, \ldots, z_{C_i}\}$, then it is clear that

$$L'_p/[\overline{\Lambda}, L_p] \simeq \prod_{i=1}^{g} L'_p(i)/[\overline{\Lambda}, L_p]_i.$$

We have thus reduced to the case where p is non-split in K, i.e. $g = 1$. So suppose that $p\mathcal{O}_K = \mathfrak{p}^e$ is non-split in K and write ε, δ for $\varepsilon_1(\overline{\Lambda}), \delta_1(\overline{\Lambda})$ as in (2.3), (2.4). Then R_p is a local ring with residue field $k \simeq \mathbb{F}_{p^f}$, where $ef = n$. Let $\pi \in R_p$ be a uniformizer. Let F be the fraction field of R_p, and note that $(\mathrm{val}_F)_{|\mathbb{Q}_p} = e \cdot \mathrm{val}$. As before, we choose a $\mathbb{Z}_p$-basis $(\alpha_1, \ldots, \alpha_n)$ of the form $\alpha_{sf+j} = \beta_j \pi^s$, where $j \in [f]$ and $s \in [e-1]_0$, and the image in k of $\{\beta_1, \ldots, \beta_f\}$ is an $\mathbb{F}_p$-basis of k.

Let $\overline{\Lambda}$ be given by a matrix $B(\overline{\Lambda}) \in \mathrm{Mat}_{2n}(\mathbb{Z}_p)$ as above. Then ε is just the minimal valuation attained by the entries of $B(\overline{\Lambda})$. To prove the lemma, it suffices to establish the following claim.

Claim. *Let $(v_1, \ldots, v_{2n}) \in \mathbb{Z}_p^{2n}$. Set*

$$\varepsilon' = \min\{\mathrm{val}(v_{2k-1}) \mid k \in [n]\}, \quad \varepsilon'' = \min\{\mathrm{val}(v_{2k}) \mid k \in [n]\}$$

and define

$$\delta' = \min\{d \in [e-1]_0 \mid \exists k \in]df, (d+1)f] : \; \mathrm{val}(v_{2k-1}) = \varepsilon'\},$$
$$\delta'' = \min\{d \in [e-1]_0 \mid \exists k \in]df, (d+1)f] : \; \mathrm{val}(v_{2k}) = \varepsilon''\}.$$

Consider the element $v = \sum_{k=1}^{n} (v_{2k-1}x_k + v_{2k}y_k) \in \overline{\Lambda}$. Then

(2.6)
$$\mathrm{span}_{\mathbb{Z}_p}\{[v, y_1], \ldots, [v, y_n]\} = \mathrm{span}_{\mathbb{Z}_p}\{p^{\varepsilon'+1}z_1, \ldots, p^{\varepsilon'+1}z_{\delta' f}, p^{\varepsilon'}z_{\delta' f+1}, \ldots, p^{\varepsilon'}z_n\},$$
$$\mathrm{span}_{\mathbb{Z}_p}\{[v, x_1], \ldots, [v, x_n]\} = \mathrm{span}_{\mathbb{Z}_p}\{p^{\varepsilon''+1}z_1, \ldots, p^{\varepsilon''+1}z_{\delta'' f}, p^{\varepsilon''}z_{\delta'' f+1}, \ldots, p^{\varepsilon''}z_n\}.$$

Indeed, assuming the claim, it easily follows from (2.5) that

$$[\overline{\Lambda}, L] = \mathrm{span}_{\mathbb{Z}_p}\{p^{\varepsilon+1}z_1, \ldots, p^{\varepsilon+1}z_{\delta f}, p^{\varepsilon}z_{\delta f+1}, \ldots, p^{\varepsilon}z_n\}.$$

Now we prove the claim. We only consider the statement involving ε' and δ', since the other half of the claim is dealt with analogously. It is clear that neither side of (2.6) changes if we replace v by $v' = \sum_{k=1}^{n} v_{2k-1}x_k$. Moreover, replacing v' with $p^{-\varepsilon'}v'$ we may assume without loss of generality that $\varepsilon' = 0$.

Now let $l \in [n]$ be the smallest number such that $\mathrm{val}(v_{2l-1}) = 0$. Then l satisfies $l \in \,]\delta' f, (\delta'+1)f]$ by the definition of δ'. Observe that for each $m \in [n]$ we have, by (2.1),

$$[v', y_m] = \left[\sum_{k=1}^{n} v_{2k-1} x_k, y_m\right] = \sum_{u=1}^{n} \left(\sum_{k=1}^{n} v_{2k-1} c_u^{km}\right) z_u. \tag{2.7}$$

It follows from our definition of the basis $(\alpha_1, \dots, \alpha_n)$ that $\mathrm{val}_F(\alpha_k) = d$ if $k \in \,]df, (d+1)f]$. In particular, if $k > \delta' f$, then $\mathrm{val}_F(\alpha_k) \geq \delta'$ and hence $\mathrm{val}_F(\alpha_k \alpha_m) \geq \delta'$ for all m. Since the α_k are linearly independent over $\mathbb{Z}_p$, it follows that $\mathrm{val}_F(c_u^{km} \alpha_u) \geq \delta'$ for all $u \in [n]$. Thus, if $k > \delta' f$ but $u \leq \delta' f$, then we must have $\mathrm{val}_F(c_u^{km}) > 0$ and hence $\mathrm{val}(c_u^{km}) > 0$. On the other hand, if $k \leq \delta' f$ then $\mathrm{val}(v_{2k-1}) > 0$ by the definition of δ'. Therefore, if $u \leq \delta' f$, then $\mathrm{val}\left(\sum_{k=1}^{n} v_{2k-1} c_u^{km}\right) \geq 1$. It follows by (2.7) that the left hand side of (2.6) is contained in the right hand side.

Let $M = (M_{um}) \in \mathrm{Mat}_n(\mathbb{Z}_p)$ be the matrix whose columns are $[v', y_1], \dots, [v', y_n]$, with respect to the basis $(z_1, \dots, z_n)$ of L'_p. Then $M_{um} = \sum_{k=1}^{n} v_{2k-1} c_u^{km}$, and it follows from the definition of the structure constants that M is the matrix of the $\mathbb{Z}_p$-linear operator

$$R_p \to R_p, \quad x \mapsto \left(\sum_{k=1}^{n} v_{2k-1} \alpha_k\right) x$$

with respect to the basis $(\alpha_1, \dots, \alpha_n)$ of R_p. Hence $\det M = \mathrm{N}_{F/\mathbb{Q}_p}\left(\sum_{k=1}^{n} v_{2k-1} \alpha_k\right)$, where $\mathrm{N}_{F/\mathbb{Q}_p}$ denotes the norm function. By the considerations in the previous paragraph we see that all the entries in the first $\delta' f$ rows of M are divisible by p.

Let $\Delta_{\delta' f} \in \mathrm{GL}_n(\mathbb{Q}_p)$ be the diagonal matrix such that the first $\delta' f$ diagonal entries are p^{-1} and the remaining diagonal entries are 1. Let $M' = \Delta_{\delta' f} M$. Then $M' \in M_n(\mathbb{Z}_p)$. As $\mathrm{val}_F\left(\sum_{k=1}^{n} v_{2k-1} \alpha_k\right) = \delta'$, it follows that $\mathrm{val}(\det M') = \mathrm{val}(\det M) - \delta' f = 0$. Thus the matrix M' is invertible, and the space spanned by its columns is just L'_p. It follows that $pz_1, \dots, pz_{\delta' f}$ are contained in the span of the columns of M. Hence the right hand side of (2.6) is contained in the left hand side. This completes the proof of the claim. $\square$

Definition 2.3. Let $(\mathbf{e}, \mathbf{f}) \in \mathbb{N}^g \times \mathbb{N}^g$. We say that an n-tuple $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{N}_0^n$ is *admissible for* $(\mathbf{e}, \mathbf{f})$ if there exists a sublattice $\overline{\Lambda} \leq \overline{L}_p$ of finite index such that $\ell(\overline{\Lambda}) = \ell$. This is equivalent to the condition that for, each $i \in [g]$, there exist $\delta_i \in [e_i - 1]_0$ such that

$$\ell_{C_{i-1}+1} = \dots = \ell_{C_{i-1}+\delta_i f_i} = \ell_{C_{i-1}+\delta_i f_i + 1} + 1 = \dots = \ell_{C_i} + 1. \tag{2.8}$$

We denote the set of admissible n-tuples by $\mathrm{Adm}_{\mathbf{e},\mathbf{f}} \subseteq \mathbb{N}_0^n$.

We sometimes make use of the fact that an admissible n-tuple ℓ determines, and is determined by, the pair of g-tuples $((\ell_{C_1}, \dots, \ell_{C_g}), (\delta_1, \dots, \delta_g))$ in (2.8). Note that $\mathrm{Adm}_{\mathbf{1},\mathbf{1}} = \mathbb{N}_0^n$. The opposite extreme occurs for $(\mathbf{e}, \mathbf{f}) = ((1), (n))$, where $\mathrm{Adm}_{(1),(n)} = \mathbf{1}\mathbb{N}_0$ consists of n-tuples all of whose components are equal.

Recall that, for $d \in \mathbb{N}$,

$$\zeta^{\triangleleft}_{\mathbb{Z}_p^d}(s) = \prod_{i=0}^{d-1} \zeta_p(s-i) = \frac{1}{\prod_{i=0}^{d-1}(1-p^{i-s})}, \tag{2.9}$$

where $\zeta_p(s) = (1-p^{-s})^{-1}$ is the local Riemann zeta function; cf., for instance, [7, Proposition 1.1]. Since $\overline{L}_p$ is a free abelian Lie ring of rank $2n$ over $\mathbb{Z}_p$, we have that $\zeta^{\triangleleft}_{\overline{L}_p}(s) = \zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s)$.

Lemma 2.4. *Let p be a prime of decomposition type $(\mathbf{e},\mathbf{f})$ in K. Given an n-tuple $\ell = (\ell_1,\dots,\ell_n) \in \mathrm{Adm}_{\mathbf{e},\mathbf{f}}$, we have*

$$\sum_{\overline{\Lambda}\leq_f \overline{L}_p,\, \ell(\overline{\Lambda})=\ell} |\overline{L}_p : \overline{\Lambda}|^{-s} = \frac{\left(\prod_{i=1}^{g}(1-t^{2f_i})\right) t^{2\sum_{i=1}^n \ell_i}}{\prod_{i=0}^{2n-1}(1-p^i t)} = \left(\prod_{i=1}^{g}(1-t^{2f_i})\right)\zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s) t^{2\sum_{i=1}^n \ell_i}.$$

Proof. Denote the leftmost object in the equality above by Σ_ℓ. We first prove that

$$\Sigma_\ell = t^{2\sum_{i=1}^n \ell_i}\Sigma_{\mathbf{0}}, \tag{2.10}$$

where $\mathbf{0}$ denotes the zero vector $(0,\dots,0) \in \mathbb{N}_0^n$. Indeed, there is a bijection ψ from matrices representing finite-index sublattices with $\ell(\overline{\Lambda}) = \mathbf{0}$ to those representing finite-index sublattices with $\ell(\overline{\Lambda}) = \ell$ given as follows. Given a matrix $B \in \mathrm{Mat}_{2n}(\mathbb{Z}_p)$, we define $\psi(B) = DPB$, where P is the permutation matrix representing the permutation

$$\prod_{i=1}^{g}(2C_{i-1}+1 \; 2C_{i-1}+2 \; \cdots \; 2C_i)^{2\delta_i f_i} \in S_{2n},$$

and D is the diagonal matrix $\mathrm{diag}(d_1,\dots,d_{2n})$ whose entries are

$$d_k = \begin{cases} p^{\ell_{C_i}+1}, & \text{if } k \in]2C_{i-1}, 2(C_{i-1}+\delta_i f_i)], \\ p^{\ell_{C_i}}, & \text{if } k \in]2(C_{i-1}+\delta_i f_i), 2C_i]. \end{cases}$$

Informally, within each block of $2e_if_i$ rows of B, we multiply everything by $p^{\ell_{C_i}}$, then we cyclically move each row down $2\delta_i f_i$ places and multiply the top $2\delta_i f_i$ rows of the resulting matrix by p. It is easy to see that this yields a bijection as claimed, and, since left multiplication commutes with right multiplication, it obviously induces a bijection between lattices with $\ell(\overline{\Lambda}) = \mathbf{0}$ and those with $\ell(\overline{\Lambda}) = \ell$; we also denote this bijection by ψ. Moreover, we observe that if the matrix B represents a finite-index sublattice $\overline{\Lambda} \leq \overline{L}_p$, then $|\overline{L}_p : \overline{\Lambda}| = p^{\mathrm{val}(\det B)}$. Since $\det\psi(B) = p^{2\sum_{i=1}^n \ell_i}\det B$, we conclude that indeed

$$\Sigma_\ell = \sum_{\overline{\Lambda}\leq_f \overline{L}_p,\, \ell(\overline{\Lambda})=\mathbf{0}} |\overline{L}_p : \psi(\overline{\Lambda})|^{-s} = t^{2\sum_{i=1}^n \ell_i} \sum_{\overline{\Lambda}\leq_f \overline{L}_p,\, \ell(\overline{\Lambda})=\mathbf{0}} |\overline{L}_p : \overline{\Lambda}|^{-s} = t^{2\sum_{i=1}^n \ell_i}\Sigma_{\mathbf{0}}.$$

We observe that

$$\sum_{\substack{\overline{\Lambda}\leq_f \overline{L}_p \\ \ell(\overline{\Lambda})\in\mathrm{Adm}_{\mathbf{e},\mathbf{f}}}} |\overline{L}_p : \overline{\Lambda}|^{-s} = \zeta^{\triangleleft}_{\overline{L}_p}(s)$$

by definition, since the sum runs over all finite-index sublattices of $\overline{L}_p$; since $\overline{L}_p$ is abelian, they are all ideals. Using the characterization of $\ell \in \mathrm{Adm}_{\mathbf{e},\mathbf{f}}$ via the ℓ_{C_i} and δ_i in (2.8), we see that

$$\begin{aligned}
\frac{\zeta^{\triangleleft}_{L_p}(s)}{\Sigma_{\mathbf{0}}} &= \sum_{\ell \in \mathrm{Adm}_{\mathbf{e},\mathbf{f}}} t^{2\sum_{i=1}^{n} \ell_i} \\
&= \sum_{(\ell_{C_1},\dots,\ell_{C_g}) \in \mathbb{N}_0^g} \sum_{\delta_1=0}^{e_1-1} \cdots \sum_{\delta_g=0}^{e_g-1} t^{2\sum_{i=1}^{g} f_i(e_i \ell_{C_i} + \delta_i)} \\
&= \prod_{i=1}^{g} \left(\sum_{\ell_{C_i}=0}^{\infty} (t^{2e_i f_i})^{\ell_{C_i}} \right) (1 + t^{2f_i} + (t^{2f_i})^2 + \cdots + (t^{2f_i})^{e_i-1}) \\
&= \prod_{i=1}^{g} \frac{(1 + t^{2f_i} + (t^{2f_i})^2 + \cdots + (t^{2f_i})^{e_i-1})}{1 - (t^{2f_i})^{e_i}} = \prod_{i=1}^{g} \frac{1}{1 - t^{2f_i}}.
\end{aligned}$$

Therefore $\Sigma_{\mathbf{0}} = \left(\prod_{i=1}^{g}(1 - t^{2f_i})\right) \zeta^{\triangleleft}_{L_p}(s)$. Together with (2.10), this establishes the lemma. □

2.2. **Igusa functions.** Recall that, for a variable Y and integers $a, b \in \mathbb{N}_0$ with $a \geq b$, the *Gaussian polynomial* (or *Gaussian binomial coefficient*) is defined to be

$$\binom{a}{b}_Y = \frac{\prod_{i=a-b+1}^{a}(1 - Y^i)}{\prod_{i=1}^{b}(1 - Y^i)} \in \mathbb{Z}[Y].$$

Given an integer $n \in \mathbb{N}$ and a subset $I \subseteq [n-1]$ whose elements are $i_1 < i_2 < \cdots < i_m$, the associated *Gaussian multinomial* is defined as

$$\binom{n}{I}_Y = \binom{n}{i_m}_Y \binom{i_m}{i_{m-1}}_Y \cdots \binom{i_2}{i_1}_Y \in \mathbb{Z}[Y].$$

Definition 2.5. Let $h \in \mathbb{N}$. Given variables Y and $\mathbf{X} = (X_1, \dots, X_h)$, we set

$$\begin{aligned}
I_h(Y; \mathbf{X}) &= \frac{1}{1 - X_h} \sum_{I \subseteq [h-1]} \binom{h}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_h), \\
I_h^{\circ}(Y; \mathbf{X}) &= \frac{X_h}{1 - X_h} \sum_{I \subseteq [h-1]} \binom{h}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_h).
\end{aligned}$$

As mentioned in the introduction, an important feature of these functions for us is that they satisfy a functional equation upon inversion of the variables; see Proposition 4.2.

Remark 2.6. The function I_h is – up to the factor $\frac{1}{1-X_h}$ – equal to the function F_h defined in [17, Theorem 4]. We consider it more natural to include the factor in the definition here.

Example 2.7.

$$I_1(Y; X_1) = \frac{1}{1 - X_1}, \quad I_2(Y; X_1, X_2) = \frac{1}{1 - X_2}\left(1 + (1 + Y)\frac{X_1}{1 - X_1}\right).$$

2.3. **Weak orderings, flag complexes, and generalized Igusa functions.** When dealing with unramified primes which are not totally split, we will need to work with a larger class of rational functions than the Igusa functions of Definition 2.5. These variant Igusa functions, which generalize the functions $I_h(1;\mathbf{X})$ by Lemma 2.11, will be defined in the terminology of weak orderings and flag complexes. We now explain these notions.

Let $h \in \mathbb{N}$. The symmetric group S_h of degree h is a Coxeter group, with Coxeter generating set $\mathcal{S} = \{s_1, \dots, s_{h-1}\}$, where s_i corresponds to the transposition $(i\ i+1)$ in the standard permutation representation of S_h. The (*Coxeter*) *length* $\operatorname{len}(\sigma)$ of an element $\sigma \in S_h$ is the length of a shortest word representing σ as a product of elements of $\mathcal{S}$. Given $\sigma \in S_h$, we define its (*right*) *descent set*

$$\operatorname{Des}(\sigma) = \{i \in [h-1] \mid \operatorname{len}(\sigma s_i) < \operatorname{len}(\sigma)\}.$$

It is well known that $\operatorname{Des}(\sigma) = \{i \in [h-1] \mid \sigma(i) > \sigma(i+1)\}$; see, for instance, [2, Proposition 1.5.3]. Given a set A, we denote by 2^A the set of all subsets of A.

Definition 2.8. A *weak ordering on* h is a pair $(\sigma, J) \in S_h \times 2^{[h-1]}$ such that $\operatorname{Des}(\sigma) \subseteq J$. We set

$$\mathrm{WO}_h = \{(\sigma, J) \in S_h \times 2^{[h-1]} \mid \operatorname{Des}(\sigma) \subseteq J\}.$$

Informally, a weak ordering is a possible outcome of a race among h contestants, if ties are permitted. Given $(\sigma, J) \in \mathrm{WO}_h$, where the elements of J are $j_1 < \cdots < j_\ell$, the contestants $\sigma(1), \dots, \sigma(j_1)$ share the first place, $\sigma(j_1+1), \dots, \sigma(j_2)$ share the second place, etc.

Weak orderings may be also interpreted in terms of face complexes. Consider Γ_h, the first barycentric subdivision of the boundary D_h of the $(h-1)$-simplex on h vertices. Let P_h be its face complex. Thus $P_h = \mathcal{F}(\Gamma_h)$ and $\Gamma_h = \Gamma(P_h)$ in the notation of [11, Section 1]. We may interpret P_h as the poset of chains of nontrivial and proper subsets of $[h]$. The empty chain plays the role of the initial object $\widehat{0}$. A general element $y \in P_h$ has the form

$$y = (\mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \cdots \subsetneq \mathcal{J}_\ell),$$

where $\mathcal{J}_i \subsetneq [h]$ for each $i \in [\ell]$. The map

$$\varphi : \mathrm{WO}_h \to P_h, \quad (\sigma, J) \mapsto (\{\sigma(1), \dots, \sigma(j)\})_{j \in J} \tag{2.11}$$

is a poset isomorphism.

Next we define a class of functions, partially generalizing the Igusa functions introduced in Definition 2.5. Given $\mathcal{J} \subseteq [h]$, we say $\mathcal{J} \in y = (\mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \cdots \subsetneq \mathcal{J}_\ell)$ if $\mathcal{J} = \mathcal{J}_i$ for some $i \in [\ell]$.

Definition 2.9. Let $\mathbf{X} = (X_{\mathcal{J}})_{\mathcal{J} \in 2^{[h]} \setminus \{\varnothing\}}$ be a collection of variables parametrized by the non-empty subsets of $[h]$. Define

$$I_h^{\mathrm{wo}}(\mathbf{X}) = \frac{1}{1 - X_{[h]}} \sum_{y \in P_h} \prod_{\mathcal{J} \in y} \frac{X_{\mathcal{J}}}{1 - X_{\mathcal{J}}}.$$

Remark 2.10. Alternatively, one may view $I_h^{\mathrm{wo}}(\mathbf{X})$ as a fine Hilbert series of a face ring. Indeed, let k be any field, Δ_h the first barycentric subdivision of the $(h-1)$-simplex, and $k[\Delta_h]$ the associated face (Stanley-Reisner) ring; cf. [12, Chapter II, Section 1]. One verifies easily that $I_h^{\mathrm{wo}}(\mathbf{X})$ is the fine Hilbert series of $k[\Delta_h]$.

Lemma 2.11. *Given variables* $\mathbf{X} = (X_{\mathcal{J}})_{\mathcal{J}\in 2^{[h]}\setminus\{\varnothing\}}$ *and* $\mathbf{Z} = (Z_1, \dots, Z_h)$*, the substitutions*

$$X_{\mathcal{J}} \to Z_{|\mathcal{J}|}, \quad \mathcal{J} \subseteq [h]$$

map $I_h^{\mathrm{wo}}(\mathbf{X})$ *to* $I_h(1; Z_1, \dots, Z_h)$.

Proof. It is well known (see, for instance, [14, Proposition 1.4.1]) that, given $J \subseteq [h-1]$,

$$\#\{\sigma \in S_h \mid \mathrm{Des}(\sigma) \subseteq J\} = \binom{h}{J}. \tag{2.12}$$

Since the map of (2.11) is a poset isomorphism, this implies that

$$\begin{aligned} I_h^{\mathrm{wo}}((Z_{|\mathcal{J}|})_{\mathcal{J}\subseteq 2^{[h]}\setminus\{\varnothing\}}) &= \frac{1}{1-Z_h} \sum_{I\subseteq[h-1]} \#\{\sigma \in S_h \mid \mathrm{Des}(\sigma) \subseteq I\} \prod_{i\in I} \frac{Z_i}{1-Z_i} \\ &= \frac{1}{1-Z_h} \sum_{I\subseteq[h-1]} \binom{h}{I} \prod_{i\in I} \frac{Z_i}{1-Z_i} = I_h(1; Z_1, \dots, Z_h), \end{aligned}$$

as claimed. □

Example 2.12. Let $h = 3$. For a variable Y, denote $\mathrm{gp}(Y) = \frac{Y}{1-Y}$. We have

$$\begin{aligned} &I_3^{\mathrm{wo}}(X_1, X_2, X_3, X_{12}, X_{13}, X_{23}, X_{123}) = \\ &\quad \frac{1}{1-X_{123}} \left(1 + \mathrm{gp}(X_1) + \mathrm{gp}(X_2) + \mathrm{gp}(X_3) + \mathrm{gp}(X_{12}) + \mathrm{gp}(X_{13}) + \mathrm{gp}(X_{23})\right. \\ &\qquad + \mathrm{gp}(X_1)\mathrm{gp}(X_{12}) + \mathrm{gp}(X_1)\mathrm{gp}(X_{12}) + \mathrm{gp}(X_2)\mathrm{gp}(X_{12}) \\ &\qquad\qquad \left. + \mathrm{gp}(X_2)\mathrm{gp}(X_{13}) + \mathrm{gp}(X_3)\mathrm{gp}(X_{13}) + \mathrm{gp}(X_3)\mathrm{gp}(X_{23})\right), \end{aligned}$$

whereas

$$\begin{aligned} &I_3(1; Z_1, Z_2, Z_3) = \\ &\quad \frac{1}{1-Z_3}\left(1 + \binom{3}{1}\frac{Z_1}{1-Z_1} + \binom{3}{2}\frac{Z_2}{1-Z_2} + \binom{3}{\{1,2\}}\frac{Z_1 Z_2}{(1-Z_1)(1-Z_2)}\right). \end{aligned}$$

Remark 2.13. We note a consequence of (2.12) for future use. Let $w_0 \in S_h$ be the unique element of highest Coxeter length; it corresponds to the permutation $i \mapsto h+1-i$ and has order two. It is easy to check that for any $\sigma \in S_h$, we have $\mathrm{Des}(w_0\sigma w_0) = h - \mathrm{Des}(\sigma)$. Here for any subset $J \subseteq [h-1]$ we denote $h - J = \{h - j \mid j \in J\}$. Since conjugation by w_0 is an automorphism of S_h, it is immediate from (2.12) that

$$\binom{h}{h-J} = \binom{h}{J}.$$

More generally, for a variable Y, by means of the identity [14, Proposition 1.7.1]

$$\binom{h}{J}_Y = \sum_{\substack{\sigma\in S_h \\ \mathrm{Des}(\sigma)\subseteq J}} Y^{\mathrm{len}(\sigma)}$$

and the observation that $\mathrm{len}(w_0\sigma w_0) = \mathrm{len}(\sigma)$ for all $\sigma \in S_h$, we obtain that

$$\binom{h}{h-J}_Y = \binom{h}{J}_Y.$$

2.4. **Pairs of partitions and Dyck words.** Let $\mu = (\mu_1, \dots, \mu_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ be partitions of n non-negative parts such that λ dominates μ, that is $\mu_1 \geq \cdots \geq \mu_n \geq 0$ and $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\mu_i \leq \lambda_i$ for all $i \in [n]$. This last condition is abbreviated by $\mu \leq \lambda$. There are uniquely determined integers $r \in \mathbb{N}_0$ and $M_i, L_i \in \mathbb{N}$ ($i = 1, \dots, r$), such that

(2.13)
$$\lambda_1 \geq \cdots \geq \lambda_{L_1} \geq \mu_1 \geq \cdots \geq \mu_{M_1} > \lambda_{L_1+1} \geq \cdots \geq \lambda_{L_2} \geq \mu_{M_1+1} \geq \cdots \geq \mu_{M_2} > \cdots$$
$$> \lambda_{L_{r-1}+1} \geq \cdots \geq \lambda_n \geq \mu_{M_{r-1}+1} \geq \cdots \geq \mu_n.$$

Define $L_r = M_r = n$ and $L_0 = M_0 = 0$, and observe that the condition $\mu \leq \lambda$ is equivalent to the condition that $L_i \geq M_i$ for all $i \in [r]$.

A *Dyck word of length* $2n$ is a word w in the letters $\mathbf{0}$ and $\mathbf{1}$, such that $\mathbf{0}$ and $\mathbf{1}$ each occur n times in w and no initial segment of w contains more ones than zeroes. Equivalently, a Dyck word is a well-parsed sequence of n open parentheses and n closed parentheses. We denote the set of Dyck words of length $2n$ by $\mathcal{D}_{2n}$ and note that the cardinality of $\mathcal{D}_{2n}$ is the n-th Catalan number $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$. For example, $\mathcal{D}_6 = \{\mathbf{000111}, \mathbf{001011}, \mathbf{001101}, \mathbf{010011}, \mathbf{010101}\}$. See [13, Example 6.6.6] for more details about Dyck words.

Given a pair of partitions $\mu \leq \lambda$ of at most n parts as above, define the Dyck word

$$w(\mu, \lambda) = \mathbf{0}^{L_1}\mathbf{1}^{M_1}\mathbf{0}^{L_2-L_1}\mathbf{1}^{M_2-M_1}\cdots\mathbf{0}^{n-L_{r-1}}\mathbf{1}^{n-M_{r-1}} \in \mathcal{D}_{2n}.$$

In other words, the word $w(\mu, \lambda)$ consists of L_1 zeroes followed by M_1 ones, followed by $L_2 - L_1$ zeroes, etc. The condition $\mu \leq \lambda$ ensures that $w(\mu, \lambda)$ is indeed a Dyck word. Observe that the Dyck word $w(\mu, \lambda) \in \mathcal{D}_{2n}$ determines, and is determined by, the collection of integers $\{L_i, M_i\}_{i\in[r]}$ from (2.13). It is useful for us to have notation for the successive differences of the parts of λ and μ. We set, for $j \in [n]$,

$$r_j = \begin{cases} \mu_j - \mu_{j+1}, & \text{if } j \notin \{M_1, \dots, M_r\}, \\ \mu_{M_i} - \lambda_{L_i+1}, & \text{if } j = M_i. \end{cases} \tag{2.14}$$

where we define $\lambda_{n+1} = 0$. Similarly, we recall that $M_0 = 0$ and put, for $j \in [n]$,

$$s_j = \begin{cases} \lambda_j - \lambda_{j+1}, & \text{if } j \notin \{L_1, \dots, L_r\}, \\ \lambda_{L_i} - \mu_{M_{i-1}+1}, & \text{if } j = L_i. \end{cases} \tag{2.15}$$

Note that $r_j > 0$ for $j \in \{M_1, \dots, M_{r-1}\}$ and observe that $\mu_{M_i} > \mu_{M_i+1}$ and $\lambda_{L_i} > \lambda_{L_i+1}$ for each $i \in [r-1]$. Finally, for each $i \in [r]$ we define

$$\begin{aligned} J_i^\mu &= \{j \in [M_i - M_{i-1} - 1] \mid \mu_{M_i-j} > \mu_{M_i-j+1}\}, \\ J_i^\lambda &= \{j \in [L_i - L_{i-1} - 1] \mid \lambda_{L_i-j} > \lambda_{L_i-j+1}\}. \end{aligned} \tag{2.16}$$

Given a partition λ, we set, for $i \in \mathbb{N}$,

$$\lambda'_i := \#\{j \in \mathbb{N} \mid \lambda_j \geq i\}.$$

The partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is called the *dual partition* of λ. Observe that, if λ has at most n parts, then the parts of λ' are bounded by n. In this case we write $\mathcal{J}(\lambda) = \{j \in [n-1] \mid \lambda_j > \lambda_{j+1}\}$ for the set of positive parts of λ'.

Given $\ell \in \mathbb{N}_0^n$ we let $\lambda(\ell)$ be the partition obtained by arranging the entries of ℓ in non-ascending order. We let $\beta(\lambda)$ be the number of n-tuples $\ell \in \mathbb{N}_0^n$ such that $\lambda(\ell) = \lambda$.

Lemma 2.14. *Let $\mathcal{L} = \{L_1, \dots, L_{r-1}\} \subseteq [n-1]$ be as above. Then*

$$\beta(\lambda) = \binom{n}{\mathcal{J}(\lambda)} = \binom{n}{\mathcal{L}} \prod_{i=1}^{r} \binom{L_i - L_{i-1}}{J_i^\lambda}.$$

Proof. The first equation is clear. The second follows from the observation that

$$\mathcal{J}(\lambda) = \mathcal{L} \cup \bigcup_{i=1}^{r} \{L_i - j \mid j \in J_i^\lambda\}.$$

□

2.5. **Subgroups of abelian p-groups.** In order to evaluate sums like (1.5), we need to understand, given a pair of partitions $\mu \leq \lambda$, the numbers $\alpha(\lambda, \mu; p)$ of abelian p-groups of type μ contained in a fixed abelian p-group of type λ. We recall here an explicit formula for these numbers, attributed to Birkhoff in [3].

Proposition 2.15 (Birkhoff). *Let $\mu \leq \lambda$ be partitions, with dual partitions $\mu' \leq \lambda'$. Then*

$$\alpha(\lambda, \mu; p) = \prod_{k \geq 1} p^{\mu'_k(\lambda'_k - \mu'_k)} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{p^{-1}}.$$

Lemma 2.16. *Let $\mu \leq \lambda$ be partitions, and let $r \in \mathbb{N}$ and $\{L_i, M_i\}_{i \in [r]}$ be the parameters associated to them in (2.13). Then, for $i \in [r-1]$,*

$$\prod_{k=\lambda_{L_i+1}+1}^{\mu_{M_{i-1}+1}} p^{\mu'_k(\lambda'_k - \mu'_k)} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{p^{-1}} = \prod_{j=1}^{M_i - M_{i-1}} p^{(M_{i-1}+j)(L_i - M_{i-1} - j) r_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^\mu}_{p^{-1}} \cdot \binom{L_i - M_{i-1}}{L_i - M_i}_{p^{-1}}. \tag{2.17}$$

Proof. Observe that all the indices k appearing in the product on the left hand side satisfy $\lambda_{L_i+1} < k \leq \mu_{M_{i-1}+1} \leq \lambda_{L_i}$, and hence $\lambda'_k = L_i$. Moreover, it is easy to see that $\mu'_k = M_{i-1} + j$ when $\mu_{M_{i-1}+j+1} < k \leq \mu_{M_{i-1}+j}$ holds; observe that it may be the case for some j that no index k satisfies this condition. As a result, we see that for each $j \in [M_i - M_{i-1}]$, there are exactly $r_{M_{i-1}+j}$ elements k of the segment $]\lambda_{L_i+1}, \mu_{M_{i-1}+1}]$ for which $\mu'_k = M_{i-1} + j$.

Observe that the Gaussian binomial coefficients on the left hand side of (2.17) differ from 1 only when $\mu'_k \neq \mu'_{k+1}$, namely when k is a part of the partition μ, i.e. there exists an i such that $\mu_i = k$. It follows that if $J_i^\mu = \{j_{i,1}, \cdots, j_{i,\gamma_i}\}$, with $j_{i,1} < \cdots < j_{i,\gamma_i}$, then

$$\prod_{k=\lambda_{L_i+1}+1}^{\mu_{M_{i-1}}} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{p^{-1}} = \tag{2.18}$$

$$\binom{L_i - M_i + j_{i,\gamma_i}}{L_i - M_i}_{p^{-1}} \cdot \prod_{m=1}^{\gamma_i - 1} \binom{L_i - M_i + j_{i,m+1}}{L_i - M_i + j_{i,m}}_{p^{-1}} \cdot \binom{L_i - M_{i-1}}{L_i - M_i + j_{i,\gamma_i}}_{p^{-1}}.$$

We make use of the well-known identity

$$\binom{\alpha}{\beta}_Y = \frac{1 - Y^\alpha}{1 - Y^{\alpha - \beta}} \binom{\alpha - 1}{\beta}_Y$$

for Gaussian binomial coefficients. Applying it inductively, we see that for all $m \in [\gamma_i - 1]$,

$$\binom{L_i - M_i + j_{i,m+1}}{L_i - M_i + j_{i,m}}_{p^{-1}} = \binom{j_{i,m+1}}{j_{i,m}}_{p^{-1}} \frac{\binom{L_i - M_i + j_{i,m+1}}{L_i - M_i}_{p^{-1}}}{\binom{L_i - M_i + j_{i,m}}{L_i - M_i}_{p^{-1}}}.$$

Hence the right hand side of (2.18) is equal to

$$\binom{M_i - M_{i-1}}{J_i^\mu}_{p^{-1}} \cdot \binom{L_i - M_{i-1}}{L_i - M_i}_{p^{-1}}$$

and our claim follows. ☐

Lemma 2.17. *Let $\mu \leq \lambda$ be partitions, with dual partitions $\mu' \leq \lambda'$. Then, for $i \in [r-1]$,*

$$\prod_{k=\mu_{M_{i-1}+1}+1}^{\lambda_{L_{i-1}+1}} p^{\mu'_k(\lambda'_k - \mu'_k)} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{p^{-1}} = \prod_{j=1}^{L_i - L_{i-1}} p^{M_{i-1}(L_{i-1} - M_{i-1} + j) s_{L_{i-1}+j}}.$$

Proof. Note that the product on the left hand side may be empty; this happens in the case $\lambda_{L_{i-1}+1} = \cdots = \lambda_{L_i} = \mu_{M_{i-1}+1}$. All of the Gaussian binomial coefficients on the left hand side are equal to 1, since the interval $]\mu_{M_{i-1}+1}, \lambda_{L_{i-1}+1}]$ contains no parts of the partition μ. Moreover, we observe that $\mu'_k = M_{i-1}$ for all k in this interval. Finally, observe that for $j \in [L_i - L_{i-1}]$ we have $\lambda'_k = L_{i-1} + j$ when $\lambda_{L_{i-1}+j+1} < k \leq \lambda_{L_{i-1}+j}$ holds. The claim follows as in the proof of the previous lemma. ☐

2.6. **Rewriting the zeta function.** Let p be a prime of decomposition type $(\mathbf{e}, \mathbf{f})$ in K. We put our work so far to use to rewrite the zeta function $\zeta^{\triangleleft}_{L_p}(s)$.

Definition 2.18. Given $(\mathbf{e}, \mathbf{f}) \in \mathbb{N}^g \times \mathbb{N}^g$ with $\sum_{i=1}^g e_i f_i = n$, we set

$$D^{\mathbf{e},\mathbf{f}}(p,t) = \sum_{\ell \in \mathrm{Adm}_{\mathbf{e},\mathbf{f}}} t^{2\sum_{i=1}^n \ell_i} \sum_{\mu \leq \lambda(\ell)} \alpha(\lambda(\ell), \mu; p) (p^{2n} t)^{\sum_{i=1}^n \mu_i}.$$

Lemma 2.19. *Let p be a prime of decomposition type $(\mathbf{e}, \mathbf{f})$ in K. Then*

$$\zeta^{\triangleleft}_{L_p}(s) = \left(\prod_{i=1}^g (1 - t^{2f_i}) \right) \zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s)\, D^{\mathbf{e},\mathbf{f}}(p,t).$$

Proof. Using (1.4) and Lemma 2.4, we obtain

$$\begin{aligned}\zeta^{\triangleleft}_{L_p}(s) &= \sum_{\overline{\Lambda}\leq_f \overline{L}_p} |\overline{L}_p : \overline{\Lambda}|^{-s} \sum_{[\overline{\Lambda},L_p]\leq M\leq L_p'} |L_p' : M|^{2n-s}\\ &= \sum_{\ell\in \mathrm{Adm}_{\mathbf{e},\mathbf{f}}} \sum_{\mu\leq\lambda(\ell)} \alpha(\lambda(\ell),\mu;p)\left(p^{2n}t\right)^{\sum_{i=1}^n \mu_i} \sum_{\overline{\Lambda}\leq_f \overline{L}_p,\ \ell(\overline{\Lambda})=\ell} |\overline{L}_p : \overline{\Lambda}|^{-s}\\ &= \left(\prod_{i=1}^{g}(1-t^{2f_i})\right)\zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s)\left(\sum_{\ell\in\mathrm{Adm}_{\mathbf{e},\mathbf{f}}} t^{2\sum_{i=1}^n \ell_i}\sum_{\mu\leq\lambda(\ell)}\alpha(\lambda(\ell),\mu;p)\left(p^{2n}t\right)^{\sum_{i=1}^n\mu_i}\right).\end{aligned}$$

The last bracketed factor above is exactly $D^{\mathbf{e},\mathbf{f}}(p,t)$, and our claim follows. □

Given $(\mathbf{e},\mathbf{f})\in\mathbb{N}^g\times\mathbb{N}^g$ as above and a Dyck word $w\in\mathcal{D}_{2n}$, we set

$$D_w^{\mathbf{e},\mathbf{f}}(p,t) = \sum_{\substack{\mu\leq\lambda\\ w(\mu,\lambda)=w}} \alpha(\lambda,\mu;p)\ (p^{2n}t)^{\sum_{i=1}^n\mu_i}\left(\sum_{\substack{\ell\in\mathrm{Adm}_{\mathbf{e},\mathbf{f}}\\ \lambda(\ell)=\lambda}} t^{2\sum_{i=1}^n\ell_i}\right), \tag{2.19}$$

so that $D^{\mathbf{e},\mathbf{f}} = \sum_{w\in\mathcal{D}_{2n}} D_w^{\mathbf{e},\mathbf{f}}$ and therefore

$$\zeta^{\triangleleft}_{L_p}(s) = \left(\prod_{i=1}^{g}(1-t^{2f_i})\right)\zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s)\sum_{w\in\mathcal{D}_{2n}} D_w^{\mathbf{e},\mathbf{f}}(p,t). \tag{2.20}$$

If $\mathbf{e}=\mathbf{1}$, then we write $D^{\mathbf{f}}$ instead of $D^{\mathbf{e},\mathbf{f}}$ and $D_w^{\mathbf{f}}$ instead of $D_w^{\mathbf{e},\mathbf{f}}$.

In the next section we compute explicit formulae for the generating functions $D_w^{\mathbf{f}}$. We work with the variables p and t, but it will be clear that the coefficients of the rational functions obtained depend only on $\mathbf{f}$ and w.

3. Computation of the functions $W^{\triangleleft}_{\mathbf{1},\mathbf{f}}(X,Y)$

3.1. **A special case: completely split primes ($\mathbf{f}=(1,\dots,1)$).** We start with the computation of the functions $W^{\triangleleft}_{\mathbf{1},\mathbf{1}}(X,Y)$, treating rational primes which split completely in K. Although this case is subsumed in the general unramified case presented in Section 3.2, we present it separately as it illustrates our method and serves as a template for the general case.

Recall that by (2.20) it suffices to compute the functions $D_w^{\mathbf{1}}$, indexed by Dyck words $w\in\mathcal{D}_{2n}$, that were defined in (2.19). Recall that $\mathrm{Adm}_{\mathbf{1},\mathbf{1}}=\mathbb{N}_0^n$.

Theorem 3.1. *Let $w=\prod_{i=1}^r\left(\mathbf{0}^{L_i-L_{i-1}}\mathbf{1}^{M_i-M_{i-1}}\right)\in\mathcal{D}_{2n}$ be a Dyck word and set $\mathcal{L}=\{L_1,\dots,L_{r-1}\}\subseteq[n-1]$. Then*

$$\begin{aligned}D_w^{\mathbf{1}}(p,t) = \binom{n}{\mathcal{L}}\prod_{i=1}^r\binom{L_i-M_{i-1}}{L_i-M_i}_{p^{-1}}\prod_{i=1}^r I_{L_i-L_{i-1}}(1;y_{L_{i-1}+1},\dots,y_{L_i})\,\cdot\\ \left(\prod_{i=1}^{r-1} I^{\circ}_{M_i-M_{i-1}}(p^{-1};x_{M_{i-1}+1},\dots,x_{M_i})\right) I_{n-M_{r-1}}(p^{-1};x_{M_{r-1}+1},\dots,x_n),\end{aligned}$$

with the numerical data

$$x_j = p^{j(2n+L_i-j)}t^{2L_i+j} \quad \text{for} \quad j \in]M_{i-1}, M_i], \tag{3.1}$$

$$y_j = p^{(2n-M_{i-1}+j)M_{i-1}}t^{2j+M_{i-1}} \quad \text{for} \quad j \in]L_{i-1}, L_i]. \tag{3.2}$$

Proof. Our starting point is the defining expression (2.19) for the functions $D^{\mathbf{1}}_w$. Note that summing over all partitions $\mu \leq \lambda$ such that $w(\mu,\lambda) = w$ is equivalent to summing over all the successive differences r_j and s_j, for $j \in [n]$, as defined in (2.14) and (2.15). Observe that

$$\begin{aligned} \mu_1 + \cdots + \mu_n &= \sum_{j=1}^{n} j r_j + \sum_{i=1}^{r-1} M_i(s_{L_i+1} + \cdots + s_{L_{i+1}}), \\ \lambda_1 + \cdots + \lambda_n &= \sum_{j=1}^{n} j s_j + \sum_{i=1}^{r} L_i(r_{M_{i-1}+1} + \cdots + r_{M_i}). \end{aligned} \tag{3.3}$$

Given a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}_0^n$ we set, for each $i \in [r]$,

$$\begin{aligned} \operatorname{supp}_i^M(\mathbf{v}) &= \{ j \in [M_i - M_{i-1} - 1] \mid v_{M_{i-1}+j} > 0 \}, \\ \operatorname{supp}_i^L(\mathbf{v}) &= \{ j \in [L_i - L_{i-1} - 1] \mid v_{L_{i-1}+j} > 0 \}. \end{aligned} \tag{3.4}$$

In practice, $\mathbf{v}$ will be one of the vectors of successive differences $\mathbf{r} = (r_1, \ldots, r_n)$ or $\mathbf{s} = (s_1, \ldots, s_n)$. Given a pair of partitions $\mu \leq \lambda$, recall the sets J_i^μ and J_i^λ that were defined in (2.16) for each $i \in [r]$. It is easy to see that, for every $i \in [r]$, we have

$$\operatorname{supp}_i^M(\mathbf{r}) = M_i - M_{i-1} - J_i^\mu \quad \text{and} \quad \operatorname{supp}_i^L(\mathbf{s}) = L_i - L_{i-1} - J_i^\lambda,$$

in the notation of Remark 2.13. It follows from the same remark that

$$\binom{M_i - M_{i-1}}{J_i^\mu}_{p^{-1}} = \binom{M_i - M_{i-1}}{\operatorname{supp}_i^M(\mathbf{r})}_{p^{-1}} \quad \text{and} \quad \binom{L_i - L_{i-1}}{J_i^\lambda} = \binom{L_i - L_{i-1}}{\operatorname{supp}_i^L(\mathbf{s})}. \tag{3.5}$$

We let δ_{ij} be the usual Kronecker delta function. Substituting the results of Lemmata 2.14, 2.16, and 2.17 into the right hand side of (2.19), rewriting the expressions in terms of the r_j and s_j, and using (3.5), we find that the summands are products of $2r$ factors. For each $i \in [r]$, there are two factors, each involving either the terms r_j, where $M_{i-1} + 1 \leq j \leq M_i$, or the terms s_j, where $L_{i-1} + 1 \leq j \leq L_i$. More precisely, the formula (2.19) for $D^{\mathbf{1}}_w(p,t)$ splits into a product as follows:

$$D^{\mathbf{1}}_w(p,t) = \binom{n}{\mathcal{L}} \prod_{i=1}^{r} \binom{L_i - M_{i-1}}{L_i - M_i}_{p^{-1}} \cdot \prod_{i=1}^{r} A_i B_i, \tag{3.6}$$

where, for $i \in [r]$,

$$A_i = \sum_{r_{M_{i-1}+1}=0}^{\infty} \cdots \sum_{r_{M_i-1}=0}^{\infty} \sum_{r_{M_i}=1-\delta_{ir}}^{\infty} \binom{M_i - M_{i-1}}{\operatorname{supp}_i^M(\mathbf{r})}_{p^{-1}} \prod_{j=M_{i-1}+1}^{M_i} \left(p^{(j(L_i-j)+2nj)} t^{(2L_i+j)} \right)^{r_j}$$

$$B_i = \sum_{s_{L_{i-1}+1}=0}^{\infty} \cdots \sum_{s_{L_i}=0}^{\infty} \binom{L_i - L_{i-1}}{\operatorname{supp}_i^L(\mathbf{s})} \prod_{j=L_{i-1}+1}^{L_i} \left(p^{(2n-M_{i-1}+j)M_{i-1}} t^{(2j+M_{i-1})} \right)^{s_j}.$$

We now show that all of the factors A_i and B_i are products of Igusa functions and Gaussian binomial coefficients. Given $i \in [r]$ and $I \subseteq [L_i - L_{i-1} - 1]$, we define $\mathbf{S}^i(I)$ to be the set of vectors $\mathbf{s}^i = (s_{L_{i-1}+1}, \dots, s_{L_i}) \in \mathbb{N}_0^{L_i - L_{i-1}}$ such that $s_j = 0$ unless $j \in \{L_{i-1} + k \mid k \in I\} \cup \{L_i\}$. With the numerical data defined in (3.2), we have

$$\begin{aligned} B_i &= \sum_{I \subseteq [L_i - L_{i-1} - 1]} \binom{L_i - L_{i-1}}{I} \sum_{\mathbf{s}^i \in \mathbf{S}^i(I)} \prod_{j \in (I + L_{i-1}) \cup \{L_i\}} \left(p^{(2n - M_{i-1} + j) M_{i-1}} t^{2j + M_{i-1}}\right)^{s_j} \\ &= \sum_{I \subseteq [L_i - L_{i-1} - 1]} \binom{L_i - L_{i-1}}{I} \left(\prod_{\iota \in I} \left(\sum_{s_{L_{i-1}+\iota} = 1}^{\infty} (y_{L_{i-1}+\iota})^{s_{L_{i-1}+\iota}}\right)\right) \sum_{s_{L_i} = 0}^{\infty} (y_{L_i})^{s_{L_i}} \\ &= \frac{1}{1 - y_{L_i}} \sum_{I \subseteq [L_i - L_{i-1} - 1]} \binom{L_i - L_{i-1}}{I} \prod_{\iota \in I} \frac{y_{L_{i-1}+\iota}}{1 - y_{L_{i-1}+\iota}} \\ &= I_{L_i - L_{i-1}}(1; y_{L_{i-1}+1}, \dots, y_{L_i}), \end{aligned}$$

where the y_j are as defined in the statement of the theorem.

Analogously one shows that, with the numerical data defined in (3.1),

$$A_i = \begin{cases} I^{\circ}_{M_i - M_{i-1}}(p^{-1}; x_{M_{i-1}+1}, \dots, x_{M_i}) & \text{for } i \in [r-1], \\ I_{n - M_{r-1}}(p^{-1}; x_{M_{r-1}+1}, \dots, x_n) & \text{for } i = r. \end{cases}$$

This completes the proof. □

Example 3.2. Suppose that $n = g = 3$ and $\mathbf{e} = \mathbf{f} = (1, 1, 1)$. In other words, K is a cubic number field in which the prime p is totally split. The corresponding zeta factor was obtained in Taylor's Ph.D. thesis by an involved computation with cone integrals [16, Theorem 15]; the formula is reproduced in [5, Theorem 2.5]. We show how to recover it from Theorem 3.1.

Recall that $\mathcal{D}_6 = \{\mathbf{000111}, \mathbf{001011}, \mathbf{001101}, \mathbf{010011}, \mathbf{010101}\}$. We denote these Dyck words by A, B, C, D, and E, respectively. Writing out the Igusa functions and noting that $I_h(1; t^2, t^4, \dots, t^{2h}) = \frac{1}{(1-t^2)^h}$ for all $h \in \mathbb{N}$ (see Lemma 5.1), we obtain the following formulae for $D^{\mathbf{1}}_w(p, t)$. Here we use the notation $\mathrm{gp}(x) = \frac{x}{1-x}$ and $\mathrm{gp}_0(x) = \frac{1}{1-x}$.

w	$D^{\mathbf{1}}_w(p,t)$
A	$\mathrm{gp}_0(p^{18}t^9)\left(1 + \binom{3}{1}_{p^{-1}}\left(\mathrm{gp}(p^{14}t^8) + \mathrm{gp}(p^8t^7)\right) + \binom{3}{1,2}_{p^{-1}}\mathrm{gp}(p^{14}t^8)\mathrm{gp}(p^8t^7)\right)\frac{1}{(1-t^2)^3}$
B	$3\mathrm{gp}_0(p^{18}t^9)\left(1 + \binom{2}{1}_{p^{-1}}\mathrm{gp}(p^{14}t^8)\right)\mathrm{gp}_0(p^8t^7)\binom{2}{1}_{p^{-1}}\mathrm{gp}(p^7t^5)\frac{1}{(1-t^2)^2}$
C	$3\mathrm{gp}_0(p^{18}t^9)\mathrm{gp}_0(p^{14}t^8)\mathrm{gp}(p^{12}t^6)\left(1 + \binom{2}{1}_{p^{-1}}\mathrm{gp}(p^7t^5)\right)\frac{1}{(1-t^2)^2}$
D	$3\mathrm{gp}_0(p^{18}t^9)\left(1 + \binom{2}{1}_{p^{-1}}\mathrm{gp}(p^{14}t^8)\right)\mathrm{gp}_0(p^8t^7)\left(1 + 2\mathrm{gp}(p^7t^5)\right)\mathrm{gp}(p^6t^3)\frac{1}{1-t^2}$
E	$6\mathrm{gp}_0(p^{18}t^9)\mathrm{gp}_0(p^{14}t^8)\mathrm{gp}(p^{12}t^6)\mathrm{gp}_0(p^7t^5)\mathrm{gp}(p^6t^3)\frac{1}{1-t^2}$

Adding these five functions and multiplying the sum by $(1-t^2)^3\zeta^{\triangleleft}_{\mathbb{Z}_p^6}(s)$, as prescribed by (2.20), we indeed obtain Taylor's formula.

As a further application of Theorem 3.1, we recover, in Example 5.2, the function dealing with primes that are totally split in a quartic number field; Woodward [5, Theorem 2.6] computed it by different means. For $n \geq 5$ the formulae we obtain are new.

3.2. **The general unramified case.** From now on, we fix $g \in \mathbb{N}$ and a vector $\mathbf{f} = (f_1, \dots, f_g) \in \mathbb{N}_0^g$ such that $\sum_{i=1}^g f_i = n$. We aim to compute the functions $W_{\mathbf{1},\mathbf{f}}^{\triangleleft}(X, Y)$. The computation in this case is similar to the one carried out in the totally split case ($\mathbf{f} = \mathbf{1}$) in Section 3.1, which it generalizes. Recall from (2.20) and (2.19) that

$$\zeta_{L_p}^{\triangleleft}(s) = \left(\prod_{i=1}^{g} (1 - t^{2f_i}) \right) \zeta_{\mathbb{Z}_p^{2n}}^{\triangleleft}(s) \sum_{w \in \mathcal{D}_{2n}} D_w^{\mathbf{f}}(p, t), \tag{3.7}$$

where, for each Dyck word $w \in \mathcal{D}_{2n}$,

$$D_w^{\mathbf{f}}(p, t) = \sum_{\substack{\mu \leq \lambda \\ w(\mu, \lambda) = w}} \alpha(\lambda, \mu; p) \ (p^{2n} t)^{\sum_{i=1}^n \mu_i} \left(\sum_{\substack{\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}} \\ \lambda(\ell) = \lambda}} t^{2 \sum_{i=1}^n \ell_i} \right). \tag{3.8}$$

In the special case $\mathbf{f} = \mathbf{1}$ we have $\mathrm{Adm}_{\mathbf{1},\mathbf{1}} = \mathbb{N}_0^n$. Then the sum inside the parentheses on the right hand side of (3.8) is $\beta(\lambda) t^{2 \sum_{i=1} \lambda_i}$, and this quantity is easily computed, e.g. by means of Lemma 2.14. Thus in the computations in Section 3.1 we could view the right hand side of (3.8) as a sum over pairs of partitions (μ, λ) satisfying certain conditions.

The additional complication introduced when considering general $\mathbf{f}$ is that we must take into account the structure of $\mathrm{Adm}_{\mathbf{1},\mathbf{f}}$. The solution to the combinatorial problem of computing how many *admissible* n-tuples ℓ give rise to a given partition λ is not nearly as clean as Lemma 2.14. We avoid this issue by summing directly over pairs (ℓ, μ), where $\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}}$ and μ is a partition such that $\mu \leq \lambda(\ell)$.

3.3. **A refinement of the sums $D_w^{\mathbf{f}}$.** We require precise control over the relation between admissible n-tuples $\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}}$ and the corresponding partitions $\lambda(\ell)$. For every $i \in [g]$, we have $C_i = \sum_{j=1}^i f_j$, as defined at the beginning of Section 2.1. Observe that there is a natural bijection

$$\psi : \mathrm{Adm}_{\mathbf{1},\mathbf{f}} \to \mathbb{N}_0^g, \quad \ell \mapsto (\ell_{C_1}, \ell_{C_2}, \dots, \ell_{C_g}).$$

The g-tuple $\psi(\ell)$ naturally gives rise to a weak ordering $v_\ell = (\sigma_\ell, J_\ell) \in \mathrm{WO}_g \subseteq S_g \times 2^{[g-1]}$, obtained by arranging the components of $\psi(\ell)$ in non-ascending order. For instance, $\ell_{C_{\sigma_\ell(1)}}$ is maximal among the components of $\psi(\ell)$ and $\ell_{C_{\sigma_\ell(g)}}$ is minimal. It is easy to express the partition $\lambda(\ell)$ in terms of v_ℓ. Indeed, if we set $C_i^\ell = \sum_{j=1}^i f_{\sigma_\ell(j)}$ for $i \in [g]$, then

$$\lambda(\ell)_j = \ell_{C_{\sigma_\ell(i)}} \text{ if } j \in]C_{i-1}^\ell, C_i^\ell]. \tag{3.9}$$

Now fix a Dyck word $w \in \mathcal{D}_{2n}$; we compute $D_w^{\mathbf{f}}$ by partitioning the right hand side of (3.8) into summands parametrized by WO_g. Indeed, given $v \in \mathrm{WO}_g$, we define

$$D_{w,v}^{\mathbf{f}}(p, t) = \sum_{\substack{\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}} \\ v_\ell = v}} \sum_{\substack{\mu \leq \lambda(\ell) \\ w(\mu, \lambda(\ell)) = w}} \alpha(\lambda(\ell), \mu; p) \ (p^{2n} t)^{\sum_{i=1}^n \mu_i} t^{2 \sum_{i=1}^n \ell_i}, \tag{3.10}$$

so that

$$D_w^{\mathbf{f}}(p, t) = \sum_{v \in \mathrm{WO}_g} D_{w,v}^{\mathbf{f}}(p, t).$$

The functions $D^{\mathbf{f}}_{w,v}$ are computed in Lemma 3.5. Afterwards we will see that they can be grouped together into sums that are expressible in terms of the generalized Igusa functions defined in Definition 2.9; cf. (3.12) and Theorem 3.6.

Remark 3.3. We say a few words about the motivation behind the definition of the functions $D^{\mathbf{f}}_{w,v}$. The condition $\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}}$ amounts to the fact that the partition $\lambda(\ell)$ is made up of g "blocks," each consisting of $f_1, f_2, \ldots, f_g$ equal parts. The weak ordering $v_\ell = (\sigma_v, J_v) \in \mathrm{WO}_g$ keeps track of the situation where the largest parts of $\lambda(\ell)$ are the $f_{\sigma_v(1)}$ equal parts coming from the prime $\mathfrak{p}_{\sigma_v(1)}$, that the next-largest parts (possibly of equal sizes to the parts coming from $\mathfrak{p}_{\sigma_v(1)}$) come from $\mathfrak{p}_{\sigma_v(2)}$, etc. Moreover, J_v specifies when the parts coming from two different prime ideals are equal. Thus, v_ℓ tells us exactly which differences between adjacent blocks of parts of $\lambda(\ell)$ are zero and which are positive; this information is essential to our method.

Our first task is to see when the set of pairs (μ, ℓ) over which the sum (3.10) runs is non-empty. Let $w = \prod_{i=1}^r \left(\mathbf{0}^{L_i - L_{i-1}} \mathbf{1}^{M_i - M_{i-1}}\right)$. The condition $w(\mu, \lambda(\ell)) = w$ ensures in particular that $\lambda(\ell)_{L_i} > \lambda(\ell)_{L_i+1}$ for all $1 \le i \le r-1$. By (3.9) this in turn implies that for each $i \in [r-1]$ we have $L_i = C^\ell_{t_i}$ for some $t_i \in [g]$, and moreover that $t_i \in J_\ell$. Observe that this is a condition on v_ℓ; if it is satisfied, then we say that v is *compatible with* w. It is easy to see that v is compatible with w if and only if $D^{\mathbf{f}}_{w,v}(p,t)$ is a non-vacuous sum. It is useful to rephrase the condition above as follows.

Definition 3.4. By a *set partition of* $[g]$ we mean an ordered collection $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_s)$ of pairwise disjoint non-empty subsets $\mathcal{A}_1, \ldots \mathcal{A}_s \subseteq [g]$ such that $\bigcup_{i=1}^s \mathcal{A}_i = [g]$. Let $w = \prod_{i=1}^r \left(\mathbf{0}^{L_i - L_{i-1}} \mathbf{1}^{M_i - M_{i-1}}\right) \in \mathcal{D}_{2n}$. We say that $\mathcal{A}$ is *compatible* with w if $s = r$, and for each $i \in [r]$ we have $\sum_{j \in \mathcal{A}_i} f_j = L_i - L_{i-1}$. We denote by $\mathcal{P}_w$ the set of set partitions of $[g]$ that are compatible with w.

It is clear that a weak ordering $v = (\sigma_v, J_v) \in \mathrm{WO}_g$ is compatible with a Dyck word $w \in \mathcal{D}_{2n}$ if and only if there exists a sequence $0 = t_0 < t_1 < t_2 < \cdots < t_{r-1} < t_r = g$ such that $\{t_1, \ldots, t_{r-1}\} \subseteq J_v$ and such that the set partition $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_r)$ of $[g]$ is compatible with w, where for each $k \in [r]$,

$$\mathcal{A}_k = \{\sigma_v(t_{k-1}+1), \ldots, \sigma_v(t_k)\}.$$

If such a sequence $\{t_k\}$ exists, then it is unique, and we may denote $\mathcal{A} = \mathcal{A}(w, v)$.

Now, given a set partition $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_r)$ compatible with a Dyck word w, we want to parametrize all the weak orderings v such that $\mathcal{A}(w, v) = \mathcal{A}$. For all $i \in [r]$, define $t_i = \sum_{k=1}^i |\mathcal{A}_k|$. Let the elements of $\mathcal{A}_i$ be $a^{(i)}_1 < \cdots < a^{(i)}_{t_i - t_{i-1}}$.

Consider the map

$$\begin{aligned} \varphi_{\mathcal{A}} : \prod_{i=1}^r \mathrm{WO}_{t_i - t_{i-1}} &\to \mathrm{WO}_g \\ \mathbf{v} = ((\sigma_i, J_i))_i &\mapsto (\sigma_{\varphi_{\mathcal{A}}(\mathbf{v})}, J_{\varphi_{\mathcal{A}}(\mathbf{v})}), \end{aligned} \tag{3.11}$$

where $\sigma_{\varphi_{\mathcal{A}}(\mathbf{v})} \in S_g$ is given by $\sigma_{\varphi_{\mathcal{A}}(\mathbf{v})}(t_{i-1}+j) = a^{(i)}_{\sigma_i(j)}$ for all $i \in [r]$ and $j \in [t_i - t_{i-1}]$, and $J_{\varphi_{\mathcal{A}}(\mathbf{v})}$ is the disjoint union

$$J_{\varphi_{\mathcal{A}}(\mathbf{v})} = \{t_1, \dots, t_{r-1}\} \cup \bigcup_{i=1}^{r} \{t_{i-1} + j \mid j \in J_i\}.$$

It is easy to see that $\varphi_{\mathcal{A}}$ is injective and that its image consists precisely of the weak orderings $v \in \mathrm{WO}_g$ such that $\mathcal{A}(w, v) = \mathcal{A}$.

Lemma 3.5. *Let $w = \prod_{i=1}^{r} \left(\mathbf{0}^{L_i - L_{i-1}} \mathbf{1}^{M_i - M_{i-1}}\right) \in \mathcal{D}_{2n}$. Suppose $v \in \mathrm{WO}_g$ is a weak ordering compatible with w. Let $\mathcal{A} = \mathcal{A}(w, v)$, let t_i and $a^{(i)}_j$ be defined as above for all $i \in [r]$ and all $j \in [t_i - t_{i-1}]$, and let $\mathbf{v} = (v_1, \dots, v_r) \in \prod_{i=1}^{r} \mathrm{WO}_{t_i - t_{i-1}}$ be such that $\varphi_{\mathcal{A}}(\mathbf{v}) = v$. Consider the chains $\varphi(v_i) \in P_{t_i - t_{i-1}}$ as in (2.11). Then*

$$D^{\mathbf{f}}_{w,v}(p,t) = \prod_{i=1}^{r} \binom{L_i - M_{i-1}}{L_i - M_i}_{p^{-1}} \prod_{i=1}^{r} \left(\frac{1}{1 - y^{(i)}_{[t_i - t_{i-1}]}} \prod_{\mathcal{J} \in \varphi(v_i)} \frac{y^{(i)}_{\mathcal{J}}}{1 - y^{(i)}_{\mathcal{J}}} \right) \cdot$$
$$\prod_{i=1}^{r-1} I^{\circ}_{M_i - M_{i-1}}(p^{-1}; x_{M_{i-1}+1}, \dots, x_{M_i}) \cdot I_{n - M_{r-1}}(p^{-1}; x_{M_{r-1}+1}, \dots, x_n),$$

where for each $i \in [r]$ and for each subset $\mathcal{J} \subseteq [t_i - t_{i-1}]$ we set $\varepsilon^{(i)}(\mathcal{J}) = L_{i-1} + \sum_{j \in \mathcal{J}} f_{a^{(i)}_j}$ and define the numerical data

$$x_j = p^{j(2n + L_i - j)} t^{2L_i + j} \qquad \text{for } j \in \,]M_{i-1}, M_i],$$
$$y^{(i)}_{\mathcal{J}} = p^{(2n - M_{i-1} + \varepsilon^{(i)}(\mathcal{J}))M_{i-1}} t^{2\varepsilon^{(i)}(\mathcal{J}) + M_{i-1}} \qquad \text{for } \mathcal{J} \subseteq [t_i - t_{i-1}].$$

Proof. The relevant computations are very similar to those in the proof of Theorem 3.1. If $\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}}$ and $\mu \leq \lambda(\ell)$ is a partition such that $w(\mu, \lambda(\ell)) = w$, then define the successive differences $\{r_j, s_j \mid j \in [n]\}$ just as in (2.14) and (2.15). It follows from (3.9) and from unraveling the definitions that the conditions $\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}}$ and $v_\ell = v$ impose the following conditions on the s_j:

(1) For all $i \in [r]$, we have $s_{L_i} = s_{\varepsilon^{(i)}([t_i - t_{i-1}])} \geq 0$.
(2) For all $i \in [r]$ and all $\mathcal{J} \in \varphi(v_i)$, we have $s_{\varepsilon^{(i)}(\mathcal{J})} > 0$.
(3) All other s_j vanish.

Note that (3.3) expresses $\sum_{i=1}^{n} \mu_i$ and $\sum_{i=1}^{n} \lambda_i$ in terms of the successive differences s_j and r_j, whereas (3.5) and Lemmata 2.16 and 2.17 imply that

$$\alpha(\lambda(\ell), \mu; p) = \left(\prod_{i=1}^{r} \binom{L_i - M_{i-1}}{L_i - M_i}_{p^{-1}} \binom{M_i - M_{i-1}}{\mathrm{supp}^M_i(\mathbf{r})}_{p^{-1}} \right) \cdot$$
$$p^{\sum_{i=1}^{r} \left(\sum_{j=1}^{M_i - M_{i-1}} (M_{i-1} + j)(L_i - M_{i-1} - j) r_{M_{i-1}+j} + \sum_{j=1}^{L_i - L_{i-1}} M_{i-1}(L_{i-1} - M_{i-1} + j) s_{L_{i-1}+j} \right)},$$

where the sets $\mathrm{supp}^M_i(\mathbf{r}) \subseteq [M_i - M_{i-1} - 1]$ are defined in (3.4). Substituting all this into (3.10) and observing that some of the s_j vanish as above, we obtain the decomposition

$$D^{\mathbf{f}}_{w,v}(p,t) = \prod_{i=1}^{r} \binom{L_i - M_{i-1}}{L_i - M_i}_{p^{-1}} \cdot \prod_{i=1}^{r} A_i B_i,$$

where the functions A_i are defined as in (3.6) and

$$B_i = \sum_{\mathbf{s}^i \in \mathbf{S}_v^i} \sum_{s_{L_i}=0}^{\infty} \left(\prod_{\mathcal{J} \in \varphi(v_i) \cup [t_i - t_{i-1}]} (p^{(2n-M_{i-1}+\varepsilon^{(i)}(\mathcal{J}))M_{i-1}} t^{2\varepsilon^{(i)}(\mathcal{J})+M_{i-1}})^{s_{\varepsilon^{(i)}(\mathcal{J})}} \right)$$
$$= \frac{1}{1 - y^{(i)}_{[t_i - t_{i-1}]}} \prod_{\mathcal{J} \in \varphi(v_i)} \frac{y^{(i)}_{\mathcal{J}}}{1 - y^{(i)}_{\mathcal{J}}},$$

where the $y^{(i)}_{\mathcal{J}}$ are defined as in the statement of the lemma. Here, for each $i \in [r]$, we define $E_v^i = \{\varepsilon^{(i)}(\mathcal{J}) \mid \mathcal{J} \in \varphi(v_i)\}$ and let $\mathbf{S}_v^i$ be the collection of vectors $\mathbf{s}^i = (s_k)_{k \in E_v^i} \in \mathbb{Z}^{E_v^i}$ such that $s_k \geq 1$ for all $k \in E_v^i$. The functions A_i were already computed in the proof of Theorem 3.1. □

The functions $D^{\mathbf{f}}_{w,v}$ computed in Lemma 3.5 split $D^{\mathbf{f}}_w$ into too many summands to be useful; in particular, $D^{\mathbf{f}}_{w,v}$ need not satisfy any functional equation. Therefore we introduce a coarser decomposition of $D^{\mathbf{f}}_w$ as follows. Given a set partition $\mathcal{A} \in \mathcal{P}_w$ of $[g]$ that is compatible with the Dyck word w, we define

$$D^{\mathbf{f}}_{w,\mathcal{A}} = \sum_{\substack{v \in \mathrm{WO}_g \\ \mathcal{A}(w,v)=\mathcal{A}}} D^{\mathbf{f}}_{w,v}. \tag{3.12}$$

We will prove in Section 4 that $D^{\mathbf{f}}_{w,\mathcal{A}}$ satisfies a functional equation whose symmetry factor is independent of w and $\mathcal{A}$; cf. Proposition 4.3. Recall that (3.7) implies that

$$\zeta^{\triangleleft}_{L_p}(s) = \prod_{i=1}^{g} (1 - t^{2f_i}) \cdot \zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s) \sum_{w \in \mathcal{D}_{2n}} \sum_{\mathcal{A} \in \mathcal{P}_w} D^{\mathbf{f}}_{w,\mathcal{A}}(p,t).$$

Theorem 3.6. *Let Let $w = \prod_{i=1}^r \left(\mathbf{0}^{L_i - L_{i-1}} \mathbf{1}^{M_i - M_{i-1}}\right) \in \mathcal{D}_{2n}$ and $\mathcal{A} \in \mathcal{P}_w$. As before, let $t_i = \sum_{k=1}^i |\mathcal{A}_k|$ for $i \in [r]$. Then,*

$$D^{\mathbf{f}}_{w,\mathcal{A}}(p,t) = \prod_{i=1}^{r} \binom{L_i - M_{i-1}}{L_i - M_i}_{p^{-1}} \prod_{i=1}^{r} I^{\mathrm{wo}}_{t_i - t_{i-1}}(\mathbf{y}^{(i)}) \cdot$$
$$\prod_{i=1}^{r-1} I^{\circ}_{M_i - M_{i-1}}(p^{-1}; x_{M_{i-1}+1}, \ldots, x_{M_i}) \cdot I_{n - M_{r-1}}(p^{-1}; x_{M_{r-1}+1}, \ldots, x_n),$$

where $\mathbf{y}^{(i)} = (y^{(i)}_{\mathcal{J}})_{\mathcal{J} \in 2^{[t_i - t_{i-1}]} \setminus \{\varnothing\}}$, and the numerical data are

$$x_j = p^{j(2n+L_i-j)} t^{2L_i+j} \quad \text{for} \quad j \in]M_{i-1}, M_i],$$
$$y^{(i)}_{\mathcal{J}} = p^{(2n-M_{i-1}+\varepsilon^{(i)}(\mathcal{J}))M_{i-1}} t^{2\varepsilon^{(i)}(\mathcal{J})+M_{i-1}} \quad \text{for} \quad \mathcal{J} \in 2^{[t_i - t_{i-1}]} \setminus \{\varnothing\}.$$

Here $\varepsilon^{(i)}(\mathcal{J})$ is defined as in the statement of Lemma 3.5.

Proof. The weak orderings $v \in \mathrm{WO}_g$ such that $\mathcal{A}(w,v) = \mathcal{A}$ are parametrized by the r-tuples of weak orderings $(v_1, \ldots, v_r) \in \mathrm{WO}_{t_1} \times \mathrm{WO}_{t_2 - t_1} \times \cdots \times \mathrm{WO}_{g - t_{r-1}}$ via the map $\varphi_{\mathcal{A}}$ of (3.11). The claim is now immediate from Lemma 3.5 and Definition 2.9 of the generalized Igusa functions $I^{\mathrm{wo}}_{t_i - t_{i-1}}(\mathbf{y}^{(i)})$. □

Corollary 3.7. *Suppose that p is inert in K. Then*

$$\zeta^{\triangleleft}_{L_p}(s) = \zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s) I_n(p^{-1}; x_1, \dots, x_n),$$

where $x_j = p^{j(3n-j)} t^{2n+j}$ for all $j \in [n]$.

Proof. It follows from Lemma 2.2 that $\mathrm{Adm}_{(1),(n)}$ consists of all $\ell \in \mathbb{N}_0^n$ such that all the components of ℓ are equal. Thus D_w vanishes unless w is the "trivial" Dyck word $\mathbf{0}^n\mathbf{1}^n$. Moreover, $g = 1$ and there is only one set partition $\mathcal{A}$ of $[g]$. Thus, Theorem 3.6 reduces to the statement that

$$\zeta^{\triangleleft}_{L_p}(s) = (1 - t^{2n}) \zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s) I_1^{\mathrm{wo}}(y_{[1]}) I_n(p^{-1}; x_1, \dots, x_n),$$

where $x_j = p^{j(3n-j)} t^{2n+j}$ for $j \in [n]$ and $y_{[1]} = t^{2n}$. The result follows since $I_1^{\mathrm{wo}}(y_{[1]}) = \frac{1}{1-t^{2n}}$. □

Remark 3.8. Corollary 3.7 is also easily obtained with the methods of [17]. For details see [10].

Example 3.9. Observe that if p is totally split in K, then $f_1 = \cdots = f_n = 1$ and it is easy to see that $D^{\mathbf{f}}_{w,\mathcal{A}}$ is independent of the set partition $\mathcal{A}$. Since there are $\binom{n}{\mathcal{L}}$ partitions compatible with the Dyck word w and since in this case $\varepsilon^{(i)}(\mathcal{J}) = L_{i-1} + |\mathcal{J}|$ for all $\mathcal{J} \subseteq [t_i - t_{i-1}]$, we recover Theorem 3.1 in view of the relation between the generalized and "standard" Igusa functions given in Lemma 2.11.

4. The functional equation

We say that a rational function $W(X,Y) \in \mathbb{Q}(X,Y)$ satisfies a functional equation with symmetry factor $(-1)^a X^b Y^c$ if the following holds:

$$W(X^{-1}, Y^{-1}) = (-1)^a X^b Y^c W(X,Y).$$

We refer to the triple $(a,b,c) \in \mathbb{N}_0^3$ as the symmetry data of the functional equation.

In this section we prove that, if p is unramified in K, then the Euler factor $\zeta^{\triangleleft}_{H(\mathcal{O}_K),p}(s)$ satisfies a functional equation with symmetry data independent of p. Recall Definition 2.9 of the generalized Igusa zeta functions $I_h^{\mathrm{wo}}(\mathbf{X})$, for $h \in \mathbb{N}$ and variables $\mathbf{X} = (X_{\mathcal{J}})_{\mathcal{J} \in 2^{[h]} \setminus \{\varnothing\}}$.

Proposition 4.1. *For all $h \in \mathbb{N}$,*

$$I_h^{\mathrm{wo}}(\mathbf{X}^{-1}) = (-1)^h X_{[h]} I_h^{\mathrm{wo}}(\mathbf{X}).$$

Proof. Recall from Section 2.3 the interpretation of WO_h as the face complex P_h of the boundary D_h of the $(h-1)$-simplex. Let $\Delta(P_h)$ be the order complex of P_h. As a simplicial complex, $\Delta(P_h)$ is isomorphic to the second barycentric subdivision of D_h. The geometric realization of $\Delta(P_h)$ is, of course, isomorphic to the $(s-2)$-sphere S^{s-2}, as is the geometric realization of P_h. This implies that P_h is Gorenstein*; cf. [11, Section 4]. Noting that P_h has rank $h-1$, [11, Proposition 4.4] yields

$$\sum_{y \in P_h} \prod_{\mathcal{J} \in y} \frac{X_{\mathcal{J}}^{-1}}{1 - X_{\mathcal{J}}^{-1}} = (-1)^{h-1} \left(\sum_{y \in P_h} \prod_{\mathcal{J} \in y} \frac{X_{\mathcal{J}}}{1 - X_{\mathcal{J}}} \right).$$

The claim follows.

An alternative proof uses the interpretation of $I_h^{\mathrm{wo}}(\mathbf{X})$ as the fine Hilbert series of a face ring; cf. Remark 2.10. The proposition's statement follows from [12, Corollary 7.2], noting that the reduced Euler characteristic of the $(h-1)$-simplex vanishes. □

Proposition 4.2. *For all $h \in \mathbb{N}$,*

$$I_h(Y^{-1}; \mathbf{X}^{-1}) = (-1)^h X_h Y^{-\binom{h}{2}} I_h(Y; \mathbf{X}),$$
$$I_h^{\circ}(Y^{-1}; \mathbf{X}^{-1}) = (-1)^h X_h^{-1} Y^{-\binom{h}{2}} I_h^{\circ}(Y; \mathbf{X}).$$

Proof. This follows from [17, Theorem 4]; note Remark 2.6. □

Let $w \in \mathcal{D}_{2n}$ be a Dyck word and let $\mathcal{A} \in \mathcal{P}_w$ be a set partition of $[g]$ compatible with w; cf. Definition 3.4. Recall the definition (3.12) of the function $D^{\mathbf{f}}_{w,\mathcal{A}}$.

Proposition 4.3. *The function $D^{\mathbf{f}}_{w,\mathcal{A}}$ satisfies the functional equation*

$$D^{\mathbf{f}}_{w,\mathcal{A}}(p^{-1}, t^{-1}) = (-1)^{g+n} p^{\frac{5n^2-n}{2}} t^{5n} D^{\mathbf{f}}_{w,\mathcal{A}}(p, t).$$

Proof. This is a straightforward computation using the formula for $D^{\mathbf{f}}_{w,\mathcal{A}}$ from Theorem 3.6. Indeed, the Gaussian binomial coefficients clearly satisfy

$$\binom{a}{b}_{Y^{-1}} = Y^{b(b-a)} \binom{a}{b}_{Y}.$$

Combining this with the functional equations provided by Proposition 4.1 and 4.2, we see that each of the factors on the right hand side of the formula of Theorem 3.6 satisfies a functional equation. Hence $D^{\mathbf{f}}_{w,\mathcal{A}}$ also satisfies a functional equation whose symmetry factor is

$$\prod_{i=1}^{r} p^{(L_i - M_i)(M_i - M_{i-1})} \cdot \prod_{i=1}^{r} (-1)^{|\mathcal{A}_i|} y^{(i)}_{[t_i - t_{i-1}]} \cdot \prod_{i=1}^{r-1} (-1)^{M_i - M_{i-1}} p^{-\binom{M_i - M_{i-1}}{2}} x_{M_i}^{-1}$$
$$\cdot (-1)^{n - M_{r-1}} p^{-\binom{n - M_{r-1}}{2}} x_n.$$

Noting that $\sum_{i=1}^{r} |\mathcal{A}_i| = g$ and substituting the values of x_{M_i} and $y^{(i)}_{[t_i - t_{i-1}]}$ from Theorem 3.6, a simple calculation yields the claim. □

The following theorem is equivalent to Theorem 1.2.

Theorem 4.4. *Suppose that p is unramified in K. Then we have the functional equation*

$$\zeta^{\triangleleft}_{L_p}(s)|_{p \to p^{-1}} = (-1)^{3n} p^{\binom{3n}{2} - 5ns} \zeta^{\triangleleft}_{L_p}(s).$$

Proof. Consider the formula (2.20) for $\zeta^{\triangleleft}_{L_p}(s)$. The factor $\zeta^{\triangleleft}_{\mathbb{Z}_p^{2n}}(s) = \prod_{i=0}^{2n-1} \frac{1}{1-p^i t}$ satisfies a functional equation with symmetry factor $(-1)^{2n} p^{\binom{2n}{2}} t^{2n}$, while $\prod_{i=1}^{g}(1 - t^{2f_i})$ satisfies a functional equation with symmetry factor $(-1)^g t^{-2\sum_{i=1}^{g} f_i}$, which is equal to $(-1)^g t^{-2n}$ as p is unramified. Combining these facts with Proposition 4.3, we see that $\zeta^{\triangleleft}_{L_p}(s)$ satisfies a functional equation with symmetry factor $(-1)^{3n} p^{\binom{3n}{2}} t^{5n}$, and this is our claim. □

Remark 4.5. Conjecture 1.4 follows from the claim that the functions $D^{\mathbf{e},\mathbf{f}}(p,t)$ defined in (2.19) all satisfy a functional equation and that the symmetry data is, up to sign, independent of the decomposition type $(\mathbf{e},\mathbf{f})$. Indeed, if

$$D^{\mathbf{e},\mathbf{f}}(p^{-1},t^{-1}) = (-1)^{g+n} p^{\frac{5n^2-n}{2}} t^{5n} D^{\mathbf{e},\mathbf{f}}(p,t)$$

for all $(\mathbf{e},\mathbf{f})$, then Conjecture 1.4 follows from (2.20) and a computation analogous to that in the proof of Theorem 4.4.

5. Examples

In this section we present several instances of the results of this paper. Throughout the section we use the notation $\mathrm{gp}(x) = \frac{x}{1-x}$ and $\mathrm{gp}_0(x) = \frac{1}{1-x}$. Our computations in the first example rely on the following fact.

Lemma 5.1. *For all $h \in \mathbb{N}$,*

$$I_h(1; X, X^2, \dots, X^h) = \frac{1}{(1-X)^h}.$$

Proof. Bringing the left hand side to a common denominator, we observe that

$$I_h(1; X, X^2, \dots, X^h) = \frac{\sum_{I \subseteq [h-1]} \binom{h}{I} \left(\prod_{i\in I} X^i\right) \left(\prod_{i \notin I}(1-X^i)\right)}{\prod_{i=1}^h (1-X^i)}. \tag{5.1}$$

By (2.12) we have that

$$\binom{h}{I} = \sum_{\substack{\sigma \in S_h \\ \mathrm{Des}(\sigma)\subseteq I}} 1.$$

Thus the numerator of the right hand side of (5.1) may be rearranged as follows:

$$\begin{aligned}
&\sum_{\sigma\in S_h} \sum_{I \supseteq \mathrm{Des}(\sigma)} \left(\prod_{i\in I} X^i\right)\left(\prod_{i\notin I}(1-X^i)\right) \\
&= \sum_{\sigma\in S_h} \left(\prod_{i\in \mathrm{Des}(\sigma)} X^i\right) \sum_{J\subseteq [h-1]\setminus \mathrm{Des}(\sigma)} \prod_{j\in J} X^j \prod_{j\notin J}(1-X^j) \\
&= \sum_{\sigma\in S_h} \left(\prod_{i\in \mathrm{Des}(\sigma)} X^i\right) = \sum_{\sigma\in S_h} X^{\mathrm{maj}(\sigma)}.
\end{aligned}$$

Here $\mathrm{maj}(\sigma) = \sum_{i\in\mathrm{Des}(\sigma)} i$ is the major index, and the second equality follows because

$$\sum_{J\subseteq [h-1]\setminus \mathrm{Des}(\sigma)} \prod_{j\in J} X^j \prod_{j\notin J}(1-X^j) = (X + (1-X))^{\#([h-1]\setminus\mathrm{Des}(\sigma))} = 1.$$

However, we have

$$\sum_{\sigma\in S_h} X^{\mathrm{maj}(\sigma)} = \sum_{\sigma\in S_h} X^{\mathrm{len}(\sigma)} = \prod_{i=1}^h \frac{1-X^i}{1-X}.$$

Here the first equality is the equidistribution of Coxeter length and major index [14, (1.41)] and the second equality is [14, Corollary 1.3.13]. By (5.1) our claim follows immediately. □

Example 5.2. Consider the case of $n = [K : \mathbb{Q}] = 4$ and p totally split in K. The set $\mathcal{D}_8$ is comprised of fourteen Dyck words, listed here in lexicographical order.

	Dyck word	Overlap types of partitions $\mu \leq \lambda$
A	**00001111**	$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$
B	**00010111**	$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \mu_1 > \lambda_4 \geq \mu_2 \geq \mu_3 \geq \mu_4$
C	**00011011**	$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \mu_1 \geq \mu_2 > \lambda_4 \geq \mu_3 \geq \mu_4$
D	**00011101**	$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \mu_1 \geq \mu_2 \geq \mu_3 > \lambda_4 \geq \mu_4$
E	**00100111**	$\lambda_1 \geq \lambda_2 \geq \mu_1 > \lambda_3 \geq \lambda_4 \geq \mu_2 \geq \mu_3 \geq \mu_4$
F	**00101011**	$\lambda_1 \geq \lambda_2 \geq \mu_1 > \lambda_3 \geq \mu_2 > \lambda_4 \geq \mu_3 \geq \mu_4$
G	**00101101**	$\lambda_1 \geq \lambda_2 \geq \mu_1 > \lambda_3 \geq \mu_2 \geq \mu_3 > \lambda_4 \geq \mu_4$
H	**00110011**	$\lambda_1 \geq \lambda_2 \geq \mu_1 \geq \mu_2 > \lambda_3 \geq \lambda_4 \geq \mu_3 \geq \mu_4$
I	**00110101**	$\lambda_1 \geq \lambda_2 \geq \mu_1 \geq \mu_2 > \lambda_3 \geq \mu_3 > \lambda_4 \geq \mu_4$
J	**01000111**	$\lambda_1 \geq \mu_1 > \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \mu_2 \geq \mu_3 \geq \mu_4$
K	**01001011**	$\lambda_1 \geq \mu_1 > \lambda_2 \geq \lambda_3 \geq \mu_2 > \lambda_4 \geq \mu_3 \geq \mu_4$
L	**01001101**	$\lambda_1 \geq \mu_1 > \lambda_2 \geq \lambda_3 \geq \mu_2 \geq \mu_3 > \lambda_4 \geq \mu_4$
M	**01010011**	$\lambda_1 \geq \mu_1 > \lambda_2 \geq \mu_2 > \lambda_3 \geq \lambda_4 \geq \mu_3 \geq \mu_4$
N	**01010101**	$\lambda_1 \geq \mu_1 > \lambda_2 \geq \mu_2 > \lambda_3 \geq \mu_3 > \lambda_4 \geq \mu_4$

Below we list the functions $D_w^{\mathbf{1}}(p,t)$, for $w \in \mathcal{D}_8$, as obtained from Theorem 3.1. To simplify the expressions, we use the fact that $I_h(1; t^2, \ldots, t^{2h}) = \frac{1}{(1-t^2)^h}$ by Lemma 5.1. One verifies easily that the sum of these fourteen functions, multiplied by $(1-t^2)^4 \zeta^{\triangleleft}_{\mathbb{Z}_p^8}(s)$ as in (2.20), agrees with the function computed in Woodward's thesis and stated in [5, Theorem 2.6].

$$D_A^{\mathbf{1}} = \frac{1}{(1-t^2)^4} I_4(p^{-1}; p^{11}t^9, p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12})$$

$$D_B^{\mathbf{1}} = \frac{4}{(1-t^2)^3} \binom{3}{2}_{p^{-1}} \mathrm{gp}(p^{10}t^7)\mathrm{gp}_0(p^{11}t^9) I_3(p^{-1}; p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12})$$

$$D_C^{\mathbf{1}} = \frac{4}{(1-t^2)^3} \binom{3}{1}_{p^{-1}} I_2^{\circ}(p^{-1}; p^{10}t^7, p^{18}t^8)\mathrm{gp}_0(p^{20}t^{10}) I_2(p^{-1}; p^{27}t^{11}, p^{32}t^{12})$$

$$D_D^{\mathbf{1}} = \frac{4}{(1-t^2)^3} I_3^{\circ}(p^{-1}; p^{10}t^7, p^{18}t^8, p^{24}t^9)\mathrm{gp}_0(p^{27}t^{11})\mathrm{gp}_0(p^{32}t^{12})$$

$$D_E^{\mathbf{1}} = \frac{6}{(1-t^2)^2} \binom{2}{1}_{p^{-1}} \mathrm{gp}(p^9t^5) I_2(1; p^{10}t^7, p^{11}t^9) I_3(p^{-1}; p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12})$$

$$D_F^{\mathbf{1}} = \frac{12}{(1-t^2)^2} \binom{2}{1}_{p^{-1}}^2 \mathrm{gp}(p^9t^5)\mathrm{gp}_0(p^{10}t^7)\mathrm{gp}(p^{18}t^8)\mathrm{gp}_0(p^{20}t^{10}) I_2(p^{-1}; p^{27}t^{11}, p^{32}t^{12})$$

$$D_G^{\mathbf{1}} = \frac{12}{(1-t^2)^2} \binom{2}{1}_{p^{-1}} \mathrm{gp}(p^9t^5)\mathrm{gp}_0(p^{10}t^7) I_2^{\circ}(p^{-1}; p^{18}t^8, p^{24}t^9)\mathrm{gp}_0(p^{27}t^{11})\mathrm{gp}_0(p^{32}t^{12})$$

$$D_H^{\mathbf{1}} = \frac{6}{(1-t^2)^2} I_2^{\circ}(p^{-1}; p^9t^5, p^{16}t^6) I_2(1; p^{18}t^8, p^{20}t^{10}) I_2(p^{-1}; p^{27}t^{11}, p^{32}t^{12})$$

$$D_I^{\mathbf{1}} = \frac{12}{(1-t^2)^2} I_2^{\circ}(p^{-1}; p^9t^5, p^{16}t^6)\mathrm{gp}_0(p^{18}t^8)\mathrm{gp}(p^{24}t^9)\mathrm{gp}_0(p^{27}t^{11})\mathrm{gp}_0(p^{32}t^{12})$$

$$D_J^{\mathbf{1}} = \frac{4}{1-t^2}\mathrm{gp}(p^8t^3)I_3(1; p^9t^5, p^{10}t^7, p^{11}t^9)I_3(p^{-1}; p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12})$$

$$D_K^{\mathbf{1}} = \frac{12}{1-t^2}\binom{2}{1}_{p^{-1}}\mathrm{gp}(p^8t^3)I_2(1; p^9t^5, p^{10}t^7)\mathrm{gp}(p^{18}t^8)\mathrm{gp}_0(p^{20}t^{10})I_2(p^{-1}; p^{27}t^{11}, p^{32}t^{12})$$

$$D_L^{\mathbf{1}} = \frac{12}{1-t^2}\mathrm{gp}(p^8t^3)I_2(1; p^9t^5, p^{10}t^7)I_2^{\circ}(p^{-1}; p^{18}t^8, p^{24}t^9)\mathrm{gp}_0(p^{27}t^{11})\mathrm{gp}_0(p^{32}t^{12})$$

$$D_M^{\mathbf{1}} = \frac{12}{1-t^2}\mathrm{gp}(p^8t^3)\mathrm{gp}_0(p^9t^5)\mathrm{gp}(p^{16}t^6)I_2(1; p^{18}t^8, p^{20}t^{10})I_2(p^{-1}; p^{27}t^{11}, p^{32}t^{12})$$

$$D_N^{\mathbf{1}} = \frac{24}{1-t^2}\mathrm{gp}(p^8t^3)\mathrm{gp}_0(p^9t^5)\mathrm{gp}(p^{16}t^6)\mathrm{gp}_0(p^{18}t^8)\mathrm{gp}(p^{24}t^9)\mathrm{gp}_0(p^{27}t^{11})\mathrm{gp}_0(p^{32}t^{12}).$$

Example 5.3. Consider the case $n = [K : \mathbb{Q}] = 4$ and $p\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathbf{f} = (f_1, f_2) = (2, 2)$. In this case,

$$\mathrm{Adm}_{\mathbf{1},\mathbf{f}} = \left\{\ell = (\ell_1, \ell_2, \ell_3, \ell_4) \in \mathbb{N}_0^4 \mid \ell_1 = \ell_2, \ell_3 = \ell_4\right\}.$$

The four parts of a partition $\lambda(\ell)$ arising from any $\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}}$ necessarily split into two pairs, with the parts in each pair being equal. Only three of the fourteen elements of $\mathcal{D}_8$ allow for this situation; these are the Dyck words labeled A, E, and H in the chart given in Example 5.2.

Only one set partition of [2] is compatible with the Dyck word A, namely the set partition $\mathcal{A} = (\{1, 2\})$. An easy computation shows $I_2^{\mathrm{wo}}(\mathbf{y}^{(1)}) = \frac{1}{(1-t^4)^2}$, and hence Theorem 3.6 yields

$$D_A^{(2,2)} = D_{A,\mathcal{A}}^{(2,2)} = \frac{1}{(1-t^4)^2} I_4(p^{-1}; p^{11}t^9, p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12}).$$

There are two set partitions of [2] compatible with each of the Dyck words E and H, namely $\mathcal{A}' = (\{1\}, \{2\})$ and $\mathcal{A}'' = (\{2\}, \{1\})$. Since the inertia degrees of the two prime ideals lying over p are equal, $D_{w,\mathcal{A}}^{(2,2)}(p, t)$ is independent of the set partition $\mathcal{A}$. Now Theorem 3.6 gives

$$D_E^{(2,2)} = 2D_{E,\mathcal{A}'}^{(2,2)} = \frac{2}{1-t^4}\binom{2}{1}_{p^{-1}}\mathrm{gp}(p^9t^5)\mathrm{gp}_0(p^{11}t^9)I_3(p^{-1}; p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12})$$

$$D_H^{(2,2)} = 2D_{H,\mathcal{A}'}^{(2,2)} = \frac{2}{1-t^4} I_2^{\circ}(p^{-1}; p^9t^5, p^{16}t^6)\mathrm{gp}_0(p^{20}t^{10})I_2(p^{-1}; p^{27}t^{11}, p^{32}t^{12}).$$

Adding these three functions and multiplying by $(1-t^4)^2\zeta_{\mathbb{Z}_p^8}^{\triangleleft}$ as in (2.20), we obtain

$$\zeta_{L_p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^8}^{\triangleleft}(s)\zeta_p(11s-27)\zeta_p(10s-20)\zeta_p(9s-11)\zeta_p(5s-9)\zeta_p(6s-16)^2 \cdot P(p, t),$$

where

$$\begin{aligned}P(p,t) =& p^{61}t^{35} + 2p^{53}t^{30} - p^{53}t^{26} + p^{52}t^{30} - p^{52}t^{26} + p^{51}t^{26} - p^{45}t^{25} + p^{44}t^{25} - \\ & p^{44}t^{21} + 2p^{43}t^{25} - p^{43}t^{21} + p^{42}t^{25} - p^{42}t^{21} - p^{37}t^{24} - p^{36}t^{24} + p^{36}t^{20} + \\ & p^{35}t^{24} - p^{35}t^{20} - p^{35}t^{16} - p^{34}t^{16} + p^{33}t^{20} - p^{33}t^{16} - p^{28}t^{19} + p^{28}t^{15} - \\ & p^{27}t^{19} - p^{26}t^{19} - p^{26}t^{15} + p^{26}t^{11} + p^{25}t^{15} - p^{25}t^{11} - p^{24}t^{11} - p^{19}t^{14} + \\ & p^{19}t^{10} - p^{18}t^{14} + 2p^{18}t^{10} - p^{17}t^{14} + p^{17}t^{10} - p^{16}t^{10} + p^{10}t^{9} - p^{9}t^{9} + \\ & p^{9}t^{5} - p^{8}t^{9} + 2p^{8}t^{5} + 1.\end{aligned}$$

Example 5.4. Let $[K:\mathbb{Q}] = 4$ and suppose $p\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathbf{f} = (f_1, f_2) = (3,1)$. In this case, at least three of the four parts of a partition $\lambda(\ell)$ arising from $\ell \in \mathrm{Adm}_{\mathbf{1},\mathbf{f}}$ must be equal to each other, and only the Dyck words A, B, C, D, and J allow for this. In each of these five cases, only one set partition $\mathcal{A}$ of $[2]$ is compatible with the given Dyck word, namely $\mathcal{A} = \{1,2\}$ for the word A, $\mathcal{A} = (\{1\},\{2\})$ for the words B, C, and D, and $\mathcal{A} = (\{2\},\{1\})$ for the word J. We apply Theorem 3.6 to compute the zeta function.

For the word A, we observe that $(y^{(1)}_{\{1\}}, y^{(1)}_{\{2\}}, y^{(1)}_{\{1,2\}}) = (t^6, t^2, t^8)$, and hence that

$$I_2^{\mathrm{wo}}(\mathbf{y}^{(1)}) = \frac{1}{1-t^8}\left(1 + \frac{t^6}{1-t^6} + \frac{t^2}{1-t^2}\right) = \frac{1}{(1-t^6)(1-t^2)}.$$

Therefore,

$$D_A^{(3,1)} = \frac{1}{(1-t^6)(1-t^2)} I_4(p^{-1}; p^{11}t^9, p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12}).$$

Similarly, for the other relevant Dyck words we obtain:

$$\begin{aligned}D_B^{(3,1)} =& \binom{3}{2}_{p^{-1}} \mathrm{gp}_0(t^6)\mathrm{gp}(p^{10}t^7)\mathrm{gp}_0(p^{11}t^9) I_3(p^{-1}; p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12}) \\ D_C^{(3,1)} =& \binom{3}{1}_{p^{-1}} \mathrm{gp}_0(t^6) I_2^{\circ}(p^{-1}; p^{10}t^7, p^{18}t^8)\mathrm{gp}_0(p^{20}t^{10}) I_2(p^{-1}; p^{27}t^{11}, p^{32}t^{12}) \\ D_D^{(3,1)} =& \mathrm{gp}_0(t^6) I_3^{\circ}(p^{-1}; p^{10}t^7, p^{18}t^8, p^{24}t^9)\mathrm{gp}_0(p^{27}t^{11})\mathrm{gp}_0(p^{32}t^{12}) \\ D_J^{(3,1)} =& \mathrm{gp}_0(t^2)\mathrm{gp}(p^8t^3)\mathrm{gp}_0(p^{11}t^9) I_3(p^{-1}; p^{20}t^{10}, p^{27}t^{11}, p^{32}t^{12}).\end{aligned}$$

By (2.20), the sum of these five functions is $\frac{\zeta^{\triangleleft}_{L_p}(s)}{(1-t^6)(1-t^2)\zeta^{\triangleleft}_{\mathbb{Z}_p^8}(s)}$. The numerator of the zeta function has 120 terms, so we do not reproduce it here.

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