Algebraic Number Theory 88-798
Question Sheet 5

Please feel free to e-mail me at `mschein@math.biu.ac.il` with any questions.

(1) Let K be a number field and $\mathfrak{p} \subset \mathcal{O}_K$ a prime ideal. Recall that we defined an absolute value $|\cdot|_{\mathfrak{p}} : K \to \mathbb{R}_{\geq 0}$ associated to $\mathfrak{p}$. We can complete K with respect to this absolute value to obtain a field $K_{\mathfrak{p}}$, in the same way as we constructed $\mathbb{Q}_p$. Let L/K be an extension of number fields, and let $\mathcal{P}$ be a prime of L that divides $\mathfrak{p}$. Show that there is a natural embedding of fields $K_{\mathfrak{p}} \subset L_{\mathcal{P}}$.

(2) Let p be an odd prime and let $u \in \mathbb{Z}_p^*$ be an element that is not the square of any element of $\mathbb{Z}_p$. Let $K/\mathbb{Q}_p$ be a quadratic extension. Show that K is equal to one of $\mathbb{Q}_p(\sqrt{u})$, $\mathbb{Q}_p(\sqrt{p})$, or $\mathbb{Q}_p(\sqrt{up})$.

Note: This is another example of the behavior of $\mathbb{Q}_p$ being very different from that of $\mathbb{Q}$. Recall that the fields $\mathbb{Q}(\sqrt{d})$ are all non-isomorphic for distinct square-free integers d, so $\mathbb{Q}$ has infinitely many non-isomorphic quadratic extensions.

(3) Let K be field with a non-Archimedean valuation $|\cdot|$. Let $x, y \in K$ such that $|x| \neq |y|$. Show that $|x+y| = \max\{|x|, |y|\}$.

(4) For every $\lambda \in \mathbb{F}_p$, let $[\lambda] \in \mathbb{Z}_p$ be the canonical representative of the class $\lambda \in \mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p$ that was defined in class.

(a) Recall the isomorphism $\mathbb{Z}_p \simeq \varprojlim \mathbb{Z}/p^n\mathbb{Z}$. What element on the right-hand side corresponds to $[\lambda]$?

(b) Prove that $[\lambda_0] + p[\lambda_1] + 1 \equiv [\lambda_0 + 1] + p[\lambda_1 + \frac{\lambda_0^p + 1 - (\lambda_0+1)^p}{p}] \mod p^2$, for all $\lambda_0, \lambda_1 \in \mathbb{F}_p$.