Algebraic Number Theory (88-798)
5773 Semester A
Question Sheet 2
Due 15/11/2011, א' בכסלו תשע"ג

(1) Let A be a Dedekind domain. Let $I, J \subset A$ be ideals. We say that $J|I$ if there exists an ideal $J' \subset A$ such that $JJ' = I$. Prove that $J|I$ if and only if $I \subseteq J$.
(2) Let A be a Dedekind domain. Prove that A is a PID if and only if it is a UFD.
(3) Let K be a number field with $n = [K : \mathbb{Q}]$, and let p be a prime number. Prove that there are at most n prime ideals $P \subset \mathcal{O}_K$ such that $p\mathcal{O}_K \subseteq P$.
(4) Let D be a square-free integer, and let $K = \mathbb{Q}(\sqrt{D})$. Prove that

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{D}] & : D \equiv 2, 3 \bmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right] & : D \equiv 1 \bmod 4. \end{cases}$$

Find the discriminant of K.
(5) Prove that every UFD is integrally closed.
(6) Let K be a number field with $n = [K : \mathbb{Q}]$, and let $\{x_1, \ldots, x_n\}$ be an integral basis for $\mathcal{O}_K$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be the collection of embeddings $K \hookrightarrow \overline{\mathbb{Q}}$. Let $M \in M_n(\overline{\mathbb{Q}})$ be the matrix given by $M_{ij} = \sigma_i(x_j)$, and recall from the first lecture that $d_K = d(x_1, \ldots, x_n) = (\det(M))^2$. Recall the definition of $\det(M)$ from your linear algebra course; it is a sum of $n!$ terms, each of which is a product of matrix entries times ± 1. Let P be the sum of terms with sign $+1$ and N be the sum of terms with sign -1, so that $\det(M) = P - N$. Prove that $P + N$ and PN are fixed by each $\sigma_i \in \Sigma$ and conclude that $P + N, PN \in \mathbb{Q}$.
(7) With notation as in the previous exercise, show that $P + N, PN \in \mathbb{Z}$. Conclude that either $d_K \equiv 0 \mod 4$ or $d_K \equiv 1 \mod 4$. This statement is called Stickelberger's discriminant relation.