Algebraic Number Theory (88-798)
5773 Semester A
Question Sheet 3
Due 22/11/2011, ח' בכסלו תשע"ג

(1) Let $r, s \geq 0$ be integers such that $r + 2s = n$. Let $t \in \mathbb{R}$, and let $X_t \subset \mathbb{R}^n$ be the set

$$X_t = \left\{ (x_1, \ldots, x_r, y_1, z_1, \ldots, y_s, z_s) : |x_1| + \cdots + |x_r| + 2\sqrt{y_1^2 + z_1^2} + \cdots + 2\sqrt{y_s^2 + z_s^2} < t \right\}.$$

Prove that X_t is bounded, convex (קמור), and symmetric. Recall that the volume of X_t is $\mathrm{vol}(X_t) = \int_{X_t} 1 \cdot dx_1 \cdots dz_s$. Prove that

$$\mathrm{vol}(X_t) = \frac{2^{r-s}\pi^s t^n}{n!}.$$

Hint: Change to polar coordinates and use induction on r.

(2) Let K be a number field. If $y \in \mathcal{O}_K$, prove that $N(y\mathcal{O}_K) = |N_{K/\mathbb{Q}}(y)|$.

(3) Let $I \subset \mathcal{O}_K$ be a non-zero ideal, and let $\mathcal{C} \in \mathrm{Cl}_K$ be any class in the class group. Prove that there exists an ideal $J \subset \mathcal{O}_K$ such that $J \in \mathcal{C}$ and J is co-prime (i.e. has no common prime factors) with I.

For the remaining exercises, you may assume Minkowski's determinant bound, which we will finish proving in class next week: If K is a number field such that $[K : \mathbb{Q}] = n$ and such that among the n distinct embeddings $K \hookrightarrow \mathbb{C}$ there are r_1 real embeddings and r_2 pairs of conjugate complex embeddings (in particular, $n = r_1 + 2r_2$), then every class in Cl_K contains an integral ideal $I \subset \mathcal{O}_K$ such that

$$N(I) \leq \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|d_K|}.$$

(4) Let K be a number field. Prove that if $|d_K| \leq 1$, then $K = \mathbb{Q}$.

(5) Prove that the rings of integers of the fields $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, and $\mathbb{Q}(\sqrt{-7})$ are all PID's.

(6) Find the class number of the field $\mathbb{Q}(\sqrt{6})$.

(7) Prove that if $K = \mathbb{Q}(\sqrt{-5})$, then $h_K = 2$.

(8) Prove that there are no integers x, y such that $x^2 + 5 = y^3$.

Hint: Suppose that such $x, y \in \mathbb{Z}$ do exist. Prove that they must be relatively prime, that x is even, and that y is odd. Consider the ideals $I, J \subset \mathbb{Z}[\sqrt{-5}]$ given by $I = (x + \sqrt{-5})$ and $J = (x - \sqrt{-5})$. Note that $IJ = (y)^3$. Prove that I and J must be relatively prime. It follows that $I = (I')^3$ for some ideal $I' \subset \mathcal{O}_K$. Now use the result of the previous exercise to deduce a contradiction.