Algebraic Number Theory (88-798)
5773 Semester A
Question Sheet 6
Due 20/12/2012, ז' בטבת תשע"ג

(1) The following exercises involve the same set-up as in the lecture: A is a Dedekind domain, $K = \mathrm{Frac}(A)$, L/K is a finite separable extension, and B is the integral closure of A in L. Let $\theta \in B$ be such that $L = K(\theta)$. Let $B' = A[\theta] \subset B$, and recall that the conductor $\mathcal{F}_\theta$ is the largest ideal of B contained in B'.

Let $\mathfrak{p} \subset A$ be a prime ideal, as in class set $k = A/\mathfrak{p}$, and let $g(X) \in A[X]$ be the minimal polynomial of θ. Prove that evaluation at θ induces an isomorphism $B' \simeq A[X]/(g(X))$ and hence that $B'/\mathfrak{p}B' \simeq k[X]/(\overline{g}(X))$, where $\overline{g}(X) \in k[X]$ is the reduction of $g(X)$ modulo $\mathfrak{p}$.

(2) What are the prime ideals of the ring $k[X]/(\overline{g}(X))$? Prove that if $\overline{g}(X)$ is a separable polynomial, then the intersection of the non-zero prime ideals is trivial.

(3) Use the isomorphism $B'/\mathfrak{p}B' \simeq B/\mathfrak{p}B$ that was proved in class to study the prime ideals of $B/\mathfrak{p}B$. Finally, prove the following theorem: Assume that $\mathfrak{p}B$ has no common prime factors with $\mathcal{F}_\theta$. If $\overline{g}(X) = (\overline{g}_1(X))^{e_1} \cdots (\overline{g}_r(X))^{e_r}$ is the decomposition of $\overline{g}(X) \in k[X]$ into irreducible factors, and if $g_i(X) \in A[X]$ is a lift of $\overline{g}_i(X)$, where without loss of generality the g_i and $\overline{g}_i$ may be taken to be monic, then $\mathfrak{p}B = P_1^{e_1} \cdots P_r^{e_r}$, where $P_i = \mathfrak{p}B + g_i(\theta)B$.

(4) Prove moreover that $f_i = \dim_k B/P_i$ is equal to the degree of $\overline{g}_i(X)$.

(5) Let L and L' be finite Galois extensions of $\mathbb{Q}$ and suppose that $\gcd(d_L, d_{L'}) = 1$. Let LL' be the compositum. Prove that $[LL' : \mathbb{Q}] = [L : \mathbb{Q}][L' : \mathbb{Q}]$.

Hint: Prove that $L \cap L' = \mathbb{Q}$.