Algebraic Number Theory (88-798)
5773 Semester A
Question Sheet 8
Due 3/1/2013, כ"א בטבת תשע"ג

(1) Let $n > 2$. Prove that the cyclotomic field $\mathbb{Q}(\zeta_n)$ contains at least one quadratic subfield, i.e. that there exists a field $K \subset \mathbb{Q}(\zeta_n)$ such that $[K : \mathbb{Q}] = 2$.

(2) Let p be an odd prime. Determine the discriminant $d_{\mathbb{Q}(\zeta_p)}$ (in class we only determined it up to sign). Let q be such that $q = \pm p$ and $q \equiv 1 \bmod 4$. Show that $K = \mathbb{Q}(\sqrt{q})$ is the unique quadratic field contained in $\mathbb{Q}(\zeta_p)$.

Hint: Keep track of the sign in the calculation of the discriminant that we did in class. Observe that it is the square of something.

(3) Let G be a finite abelian group. Show that there exists a Galois extension $L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$.

(4) Let K be a field with a valuation. A sequence (a_n) of elements $a_n \in K$ is called a Cauchy sequence if for all $\varepsilon \in \mathbb{R}_{>0}$ there exists N such that for all $m, n > N$ the inequality $|a_m - a_n| < \varepsilon$ holds. A sequence (a_n) is called a null-sequence if for all $\varepsilon > 0$ there exists N such that $|a_n| < \epsilon$ for all $n > N$ (in other words, it satisfies the usual condition for convergence to the limit zero).

Consider the ring R_K of all Cauchy sequences, with component-wise addition and multiplication. Prove that the set $N_K \subset R_K$ of null-sequences is a maximal ideal.

(The field $\hat{K} = R_K/N_K$ is called the completion of K with respect to the valuation $|\cdot|$.)

(5) Let K be a field with a non-Archimedean valuation. Let (a_n) be a sequence of elements of K. Prove that the series $\sum_{n=0}^{\infty} a_n$ converges (i.e. its sequence of partial sums is Cauchy) if and only if (a_n) is a null-sequence.

(6) Let K be a field with a valuation $|\cdot|$. Prove that this valuation can be extended to the completion $\hat{K}$ by defining $|x| = \lim_{n\to\infty} |a_n|$, where $(a_n) \in R_K$ is a sequence representing $x \in \hat{K}$.

(7) Show that K is dense in the completion $\hat{K}$ and that $\hat{K}$ is indeed complete, i.e. for every Cauchy sequence (a_n) of elements of $\hat{K}$ there exists $\ell \in \hat{K}$ which is the limit of the sequence in the usual sense: for every $\varepsilon > 0$ there exists N such that $|a_n - \ell| < \varepsilon$ for all $n > N$.