Algebraic Number Theory (88-798)

5773 Semester A

Question Sheet 9

Due 17/1/2013, ו' בשבט תשע"ג

(1) Let $\pi_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ be the natural projection (reduction modulo p^{n-1}. Let $R = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ be the ring of sequences $(c_1, c_2, \dots)$ such that $c_n \in \mathbb{Z}/p^n\mathbb{Z}$ and $\pi_n(c_n) = c_{n-1}$, where addition and multiplication are defined term by term.

Prove that R is a local ring with maximal ideal pR, and prove that $R/p^nR \simeq \mathbb{Z}/p^n\mathbb{Z}$ for all $n \geq 1$.

(2) Recall that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : v_p(x) \geq 0\}$. Prove that R and $\mathbb{Z}_p$ are isomorphic as rings.

(3) Define the topology on R by giving it the weakest topology such that all the natural maps

$$\begin{array}{rcl} R & \to & \mathbb{Z}/p^n\mathbb{Z} \\ (c_1, c_2, \dots) & \mapsto & c_n \end{array}$$

are continuous. Find an isomorphism of rings between R and $\mathbb{Z}_p$ that is also a homeomorphism (where $\mathbb{Z}_p$ carries the metric topology that it inherited from $\mathbb{Q}_p$).

(4) Let K be a field that is complete with respect to the non-Archimedean valuation $|\cdot|$, and let $f(x) = a_nx^n + \cdots + a_1x + a_0 \in K[x]$ be an irreducible polynomial with $a_n \neq 0$. Prove that $\max\{|a_0|, |a_1|, \dots, |a_{n-1}|, |a_n|\} = \max\{|a_0|, |a_n|\}$. In particular, show that any monic irreducible $f(x) \in K[x]$ with $a_0 \in \mathcal{O}$ is automatically contained in $\mathcal{O}[x]$.

Hint: Hensel's lemma.

(5) Let K be a field with respect to the valuation $|\cdot|$. Let V be an n-dimensional K-vector space. Recall that a norm on V is a function $|\cdot| : V \to \mathbb{R}_{\geq 0}$ such that for all $v, w \in V$ and $a \in K$ we have

(a) $|v| = 0$ if and only if $v = 0$.

(b) $|v + w| \leq |v| + |w|$.

(c) $|av| = |a||v|$.

Let $\{v_1, \dots, v_n\}$ be a basis of V, and define

$$||a_1v_1 + \cdots + a_nv_n|| = \max\{|a_1|, \dots, |a_n|\}.$$

Prove that $||\cdot||$ is a norm on V.

(6) Now suppose that K is complete with respect to its valuation. Let $|\cdot|$ be an arbitrary norm on V. Choose a basis $\{v_1, \dots, v_n\}$ of V and define a norm $||\cdot||$ as above. The aim of the next exercises is to show that $|\cdot|$ and $||\cdot||$ are equivalent, in the sense that they define the same metric topology on V. For this is suffices to show that there exist constants $\rho, \rho' > 0$ such that $\rho||x|| \leq |x| \leq \rho'||x||$ for all $x \in V$.

Show that $\rho' = |v_1| + \cdots + |v_n|$ works.

(7) To prove the existence of ρ, use induction on n. First prove the claim for $n = 1$. Now, for each $i = 1, 2, \ldots, n$, let $V_i \subset V$ be the span of $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$. By induction, prove that each V_i is complete with respect to the norm $|\cdot|$. Deduce that $\bigcup_{i=1}^{n}(V_i + v_i)$ is closed in the $|\cdot|$-topology. Prove that there exists $\rho > 0$ such that $\rho \leq |v_i + w_i|$ for all i and all $w_i \in V_i$.

Now, let $x = a_1v_1 + \cdots + a_nv_n \in V - \{0\}$ and suppose $||x|| = |a_k|$. Prove that $\rho \leq |a_k^{-1}x|$ and conclude the desired claim.

(8) Let K be a number field, let p be a prime number, and suppose that $p\mathcal{O}_K = P_1^{e_1} \cdots P_r^{e_r}$. Let $\mathcal{O}_{P_i}$ be the valuation ring of K_{P_i}, the completion of K with respect to the P_i-valuation. Let M_i be the maximal ideal of $\mathcal{O}_{P_i}$. Show that $p\mathcal{O}_{P_i} = M_i^{e_i}$.

(9) If K is a valued field, let k_K be the residue field $\mathcal{O}/M$, where $\mathcal{O}$ is the valuation ring of K and M is its maximal ideal. In particular, $k_{\mathbb{Q}_p} = \mathbb{F}_p$. A finite extension $F/\mathbb{Q}_p$ is called unramified if $[k_F : \mathbb{F}_p] = [F : \mathbb{Q}_p]$. Prove that any unramified extension $F/\mathbb{Q}_p$ of degree n is isomorphic to $\mathbb{Q}_p(\zeta)$, where ζ is a primitive $(p^n - 1)$-th root of unity.