Algebraic Number Theory (88-798)
5779 Semester A
Question Sheet 1

(1) Is $\frac{3+2\sqrt{6}}{1-\sqrt{6}} \in \mathbb{R}$ an algebraic integer, i.e. is it integral over $\mathbb{Z}$?

(2) Let L/K be a finite separable extension of fields. If $\alpha \in L$, define $M_\alpha : L \to L$ to be the K-linear map given by $M_\alpha(y) = \alpha y$ for all $y \in L$. Choose a basis of L as a K-vector space, let Mat_α be the matrix of M_α with respect to this basis, and let $c_\alpha(x) \in K[x]$ be the characteristic polynomial of this matrix. Clearly $c_\alpha(x)$ is independent of the choice of basis. We call it the *characteristic polynomial* of α.

Recall that the *minimal polynomial* $m_\alpha(x) \in K[x]$ of α is the unique monic polynomial in $K[x]$ that generates the ideal $\{f \in K[x] : f(\alpha) = 0\}$.

Suppose that $n = [L : K]$ and $d = [K(\alpha) : K]$. Prove that $c_\alpha(x) = (m_\alpha(x))^{n/d}$.

Hint: First prove the claim in the case $L = K(\alpha)$. In the general case, let A be the matrix of $M_\alpha|_{K(\alpha)}$ with respect to some K-basis of $K(\alpha)$. Show that one can find a K-basis of L such that Mat_α has the form

$$\begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & A \end{pmatrix}.$$

(3) Maintaining the notation of the previous exercise, let $\overline{K}$ be an algebraic closure of K, and recall that Σ is the set of embeddings $\sigma : L \hookrightarrow \overline{K}$ such that $\sigma(y) = y$ for all $y \in K$. Prove that $c_\alpha(x) = \prod_{\sigma \in \Sigma}(x - \sigma(\alpha))$.

Hint: As in the previous exercise, first consider the case $L = K(\alpha)$ and then the general case.

(4) Prove that $N_{L/K}(\alpha)$ and $\mathrm{Tr}_{L/K}(\alpha)$ are, respectively, the determinant and trace of the matrix Mat_α. In particular, they both lie in K.

(5) Let A be an integrally closed integral domain, $K = \mathrm{Frac}\, A$, and let L/K be an algebraic extension. If $\alpha \in L$, let $m_\alpha(x) \in K[x]$ be the minimal polynomial of α over K. Prove that the following are equivalent.
  (a) α is integral over A.
  (b) $m_\alpha(x) \in A[x]$.

Hint: Consider the roots of $m_\alpha(x)$ over an algebraic closure of L.

(6) Let $R \subset S \subset T$ be rings. Show that the following are equivalent:
  (a) T is integral over R. (Recall this means that t is integral over R for all $t \in T$.)
  (b) S in integral over R and T is integral over S.

(7) Let $A \subset L$ be an extension of rings, where L is a field. Let B be the integral closure of A in L. Prove that B is integrally closed.

(8) Let $K/\mathbb{Q}$ be a number field. Let $\{x_1, \dots, x_n\}$ be a $\mathbb{Q}$-basis of K such that $\mathcal{O}_K = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$. Define the discriminant of K to be $d_K = d(x_1, \dots, x_n)$. Prove that this notion is well-defined, i.e. that it is independent of the choice of basis.

(9) Let $d \notin \{0, 1\}$ be a square-free integer, and let $K = \mathbb{Q}(\sqrt{d})$. Prove that

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & : d \equiv 2, 3 \bmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & : d \equiv 1 \bmod 4. \end{cases}$$

Find the discriminant of K.

(10) Let $K = \mathbb{Q}(\sqrt[4]{2}) \subset \mathbb{R}$. Every element $\alpha \in K$ can be written uniquely as $\alpha = b_0 + b_1\theta + b_2\theta^2 + b_3\theta^3$, where $\theta = \sqrt[4]{2}$ and $b_i \in \mathbb{Q}$. Find $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha)$.

(11) Prove that $\sqrt{3} \notin \mathbb{Q}(\sqrt[4]{2})$.

Hint: Assume that $\sqrt{3}$ can be written in the form above and compute the traces of $\sqrt{3}$ and $\theta\sqrt{3}$. The Eisenstein criterion may be useful.