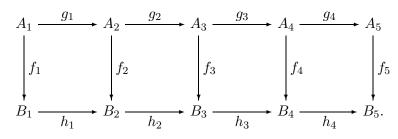
Commutative Algebra 88-813 5769 Semester A Question Sheet 1, due 23/11/2008

Please feel free to e-mail me at mschein@math.biu.ac.il with any questions of translation or otherwise.

(1) Prove the Five Lemma: Let R be a ring and consider the following commutative diagram of R-modules:



Suppose that the two rows are exact and that f_1 , f_2 , f_4 , and f_5 are isomorphisms. Then f_3 is also an isomorphism.

- (2) Let M be a finitely generated R-module. Prove that M has a maximal submodule.
- (3) Recall that l(M) is the composition length of a module M. If $M = M_1 \oplus M_2$, prove that $l(M) = l(M_1) + l(M_2)$.
- (4) Let R be a commutative ring and let M be an R-module generated by m elements. Suppose that there is a surjective map $\varphi: M \to R^{(n)}$. Prove that $m \ge n$.
- (5) Let R be a commutative ring. An element $s \in R$ is called *regular* if $sr \neq 0$ for all $0 \neq r \in R$. Let M be an R-module, and define $tor(M) = \{m \in M : sm = 0 \text{ for some regular } s \in R\}$. Prove the following:
 - (a) tor(M) is a submodule of M, and tor(R) = 0.
 - (b) $\operatorname{tor}(M_1 \oplus M_2) \simeq \operatorname{tor}(M_1) \oplus \operatorname{tor}(M_2).$
 - (c) If A = M/tor(M) is a free *R*-module, then $M \simeq A \oplus tor(M)$.
- (6) Let R be a commutative ring and $r, s \in R$. Prove that a homomorphism $R/Rr \to R/Rs$ exists if and only if $Rr \subseteq Rs$. In particular, $R/Rr \simeq R/Rs$ if and only if Rr = Rs.
- (7) Let N_1, N_2 , and K be submodules of M such that $N_1 \supset N_2$ and $K \cap N_1 = 0$. Then $(K+N_1)/(K+N_2) \simeq N_1/N_2$.
- (8) Complete this alternative proof of the Schreier-Jordan-Hölder Theorem. Suppose that M has a composition series $M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$, denoted C. Let $M = N_0 \supset N_1 \cdots \supset N_k \supset 0$ be an arbitrary chain of submodules, denoted D. We wish to prove that D can be refined (אפשר לעדן) to a composition series equivalent to C.

Consider the quotient module $\overline{M} = M/M_{t-1}$. For a submodule $N \subset M$, we write \overline{N} for its image in \overline{M} . Prove that $\overline{M} = \overline{M_0} \supset \overline{M_1} \supset \cdots \supset \overline{M_{t-1}} = 0$ is a composition series for

 \overline{M} . Show that $\overline{N_i} = (N_i + M_{t-1})/M_{t-1}$. Consider the chain $\overline{M} = \overline{N_0} \supseteq \overline{N_1} \supseteq \cdots \supseteq \overline{N_k}$, which we call $\overline{\mathcal{D}}$.

Let j be the largest integer such that $N_j \supseteq M_{t-1}$. Show that the desired claim is obvious if $j \ge k$. So assume j < k. For all i > j, prove that $N_i/N_{i+1} \simeq (N_i + M_{t-1})/(N_{i+1} + M_{t-1}) \simeq \overline{N_i}/\overline{N_{i+1}}$. Deduce that, for i > j, $\overline{N_i} \supseteq \overline{N_{i+1}}$ is a strict inclusion.

Use induction on t, applied to \overline{M} , to show that $k-1 \leq l(\overline{D}) \leq t-1$, and deduce that D can be refined to a composition series equivalent to C.

To show that all composition series for M are equivalent, suppose that \mathcal{D} was a composition series. Now show that k = t and deduce that $\overline{N_j} = \overline{N_{j+1}}$, so that $N_j = N_{j+1} + M_{t-1}$. Use induction to show that $N_j/N_{j+1} \simeq M_{t-1}$ and complete the proof.