Commutative Algebra 88-813

5769 Semester A

Question Sheet 4, due 1/2/2009

Please feel free to e-mail me at `mschein@math.biu.ac.il` with any questions.

(1) Let $A \subset R$ be rings, and suppose that R is integral (שלם) over A. Let $I \subset A$ be an ideal, and let $r \in IR$. Prove that there exists a monic polynomial (פולינום מתוקן)
$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$$
such that $f(r) = 0$ and $a_i \in I$ for all $0 \leq i \leq n-1$.

Hint: Write $r = \sum_{i=1}^t e_i r_i$, where $e_i \in I$ and $r_i \in R$. Let $M = A[r_1, \ldots, r_t]$. Show that this is a finitely generated A-module. Note that $r \in M$. The existence of a faithful $A[r]$-module that is finitely generated as an A-module (for instance, M) implies that r is integral over A. We proved this in class by constructing a monic polynomial in $f \in A[x]$ satisfied by r. Review the proof and show that this f has the properties we want.

(2) Prove that any ring R contains a prime ideal P such that $\mathrm{ht}(P) = 0$.

(3) Let F be a field and let $R = F[x_1, x_2, \ldots]$ be the commutative polynomial ring in infinitely many variables. Consider the ideal
$$P_i = \langle x_{i(i-1)/2+1}, \ldots, x_{i(i+1)/2-1}, x_{i(i+1)/2} \rangle.$$
Show that P_i is a prime ideal with i generators. Let $S = R \backslash (\bigcup_{i=1}^\infty P_i)$. Show that the ring $S^{-1}R$ is Noetherian and that if $M \subset S^{-1}R$ is a maximal ideal, then $M = S^{-1}P_i$ for some i. Show that $\mathrm{ht}_{S^{-1}R}(S^{-1}P_i) = i$. Why is $S^{-1}R$ an example of a ring of infinite Krull dimension in which every prime ideal has finite height?

(4) Let R be a local ring with maximal ideal J. Let $F = R/J$.
  (a) Prove that J is not generated by fewer than $\mathrm{Kdim}(R)$ elements.
  (b) Let M be a finitely generated R-module. Prove that the minimal number of generators needed to generate M over R is equal to the dimension of M/JM as an F-vector space. Conclude that $\mathrm{Kdim}(R) \leq \dim_F(J/J^2)$.

(5) Let R be a local ring and J the maximal ideal. Suppose that J is a principal ideal (אידאל ראשי) and that $\bigcap_{n=1}^\infty J^n = 0$.

Prove that R is Noetherian and that if $0 \neq I \subset R$ is a proper ideal (אידאל אמיתי) then $I = J^n$ for some n.