Commutative Algebra 88-813

5770 Semester A

Question Sheet 1

1. Modules

(1) Prove the Five Lemma: Let R be a ring and consider the following commutative diagram of R-modules:

$$
\begin{array}{ccccccccc}
A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow[h_1]{} & B_2 & \xrightarrow[h_2]{} & B_3 & \xrightarrow[h_3]{} & B_4 & \xrightarrow[h_4]{} & B_5.
\end{array}
$$

Suppose that the two rows are exact and that f_1, f_2, f_4, and f_5 are isomorphisms. Then f_3 is also an isomorphism.

(2) Let M be a finitely generated R-module. Prove that M has a maximal submodule.

(3) Recall that $l(M)$ is the length of a module M. If $M = M_1 \oplus M_2$, prove that $l(M) = l(M_1) + l(M_2)$.

(4) Let R be a commutative ring and let M be an R-module generated by m elements. Suppose that there is a surjective map $\varphi : M \to R^{(n)}$. Prove that $m \geq n$.

(5) Let K be a finite extension of $\mathbb{Q}$ and let $\mathcal{O}_K$ be its ring of integers. Prove that an $\mathcal{O}_K$-module M has a composition series if and only if M is finite.

(6) Classify the finitely generated modules over the ring $\mathbb{Z}[X]/(X^2)$.

(7) Let R be a commutative ring. An element $s \in R$ is called *regular* if $sr \neq 0$ for all $0 \neq r \in R$. Let M be an R-module, and define $\operatorname{tor}(M) = \{m \in M : sm = 0 \text{ for some regular } s \in R\}$. Prove the following:

   (a) $\operatorname{tor}(M)$ is a submodule of M, and $\operatorname{tor}(R) = 0$.

   (b) $\operatorname{tor}(M_1 \oplus M_2) \simeq \operatorname{tor}(M_1) \oplus \operatorname{tor}(M_2)$.

   (c) If $A = M/\operatorname{tor}(M)$ is a free R-module, then $M \simeq A \oplus \operatorname{tor}(M)$.

(8) Let R be a commutative ring and $r, s \in R$. Prove that a ring homomorphism $R/Rr \to R/Rs$ exists if and only if $Rr \subseteq Rs$. In particular, $R/Rr \simeq R/Rs$ if and only if $Rr = Rs$.

(9) Let N_1, N_2, and K be submodules of M such that $N_1 \supset N_2$ and $K \cap N_1 = 0$. Then $(K + N_1)/(K + N_2) \simeq N_1/N_2$.

(10) Let $T_n(F)$ be the ring of all upper triangular $n \times n$ matrices with entries in a field F. Consider F^n as a $T_n(F)$-module in the obvious way. Find a composition series.

2. Noetherian and Artinian rings and modules

(1) Consider the Z-module $\mathbb{Q}/\mathbb{Z}$. Is it Noetherian? Is it Artinian?

(2) Let R be a commutative Artinian ring. Recall that R has finitely many maximal ideals $P_1, \dots, P_r$. Let $J(R)$ be the Jacobson radical of R. Prove that $J(R) = P_1 P_2 \cdots P_r$.

(3) Let R be a Noetherian ring, and let $\varphi : R \to R$ be a ring homomorphism. Prove that if φ is surjective, then it is injective.

(4) Let R be a Noetherian ring and I an ideal. Show that the Rees ring $\tilde{R} = R + \sum_{i \geq 1} I^i x^i \subset R[x]$ is Noetherian. Hint: $\tilde{R} = R[a_1 x, \dots, a_m x]$, where a_i are generators of I.

(5) Show that Nakayama's lemma is false for non-finitely generated modules. Here is one counter-example:

   (a) Consider the ring $S = \left\{ \frac{p}{q} \in \mathbb{Q} : q \text{ is odd} \right\}$. (In a few weeks we will study such rings and see that this is the localization of $\mathbb{Z}$ at the prime ideal $2\mathbb{Z}$). Take the S-module $M = \mathbb{Q}$. Show that it is not finitely generated.

   (b) Show that the Jacobson radical of S is the ideal $J(S) = 2S$.

   (c) Show that $J(S)M = M$.

(6) Let R be a commutative ring and $I \subset J(R)$ an ideal. Let M be a finitely generated R-module and $N \subset M$ a submodule. Prove that if $M = IM + N$, then $N = M$.

3. Affine algebras

(1) Let F be a field and let R be an affine F-algebra. Prove that if M is a simple R-module then M has finite rank over F.

   If you also assume that F is algebraically closed, then what can you say about the rank of M over F?

(2) Let R be an affine domain over a field F. Prove that R is algebraic if and only if it is Artinian.

(3) Let F be a field. The polynomial F-algebra $F[X_1, X_2]$ is clearly affine. But prove that the subalgebra generated by 1, $X_1 X_2$, $X_1 X_2^2$, $X_1 X_2^3, \dots$ is not affine. Hence a subalgebra of an affine algebra is not necessarily affine.

(4) Prove that any affine F-algebra has countable dimension as an F-vector space.

(5) Let I be an ideal of a commutative ring R. Prove that the following conditions are equivalent:

   (a) I is a prime ideal (for any elements $x, y \in R$, we have $xy \in I$ if and only if $x \in I$ or $y \in I$).

   (b) The quotient ring R/I is an integral domain.

   (c) If J_1, J_2 are ideals of R with $J_1 J_2 \subseteq I$, then $J_1 \subseteq I$ or $J_2 \subseteq I$.

   (d) If $J_1, J_2, \dots, J_n$ are ideal of R such that $J_1 J_2 \cdots J_n \subseteq I$, then $J_i \subseteq I$ for some $1 \leq i \leq n$.

(e) The complement $R\setminus I$ is a multiplicative submonoid (תת-מונויד כפלי) of R.

(6) Let F be a field that is not algebraically closed. Find a counterexample to the Nullstellensatz for F.

4. Dedekind domains

(1) Is the element $\frac{\sqrt{2}}{2} \in \mathbb{R}$ integral over $\mathbb{Z}$?

(2) Let A be an integral domain that is not a field. If each non-zero ideal of A can be written uniquely as a product of prime ideals, prove that A is a Dedekind domain.

(3) Let A be a Dedekind domain and let $I, J \subset A$ be two ideals. We say that $I|J$ if there exists an ideal $I' \subset A$ such that $II' = J$. Show that $I|J$ if and only if $J \subset I$.

(4) Prove that a Dedekind domain is a principal ideal domain if and only if it is a unique factorization domain.

(5) Let K be a finite extension of $\mathbb{Q}$ and $\mathcal{O}_K$ its ring of integers. For every non-zero ideal $I \subset \mathcal{O}_K$, prove that the quotient $\mathcal{O}_K/I$ is finite.

(6) In view of the previous exercise, we can define the norm of an ideal $I \subset \mathcal{O}_K$ by $N(I) = |\mathcal{O}_K/I|$. Prove that the norm is multiplicative: $N(IJ) = N(I)N(J)$.

(7) Show that the ring $R = \mathbb{Z}[\sqrt{-1}]$ is integral over $\mathbb{Z}$, and that it has two different maximal ideals lying over $5\mathbb{Z}$. In general, prove that for any odd prime $p \in \mathbb{Z}$, there are two different prime ideals lying over $p\mathbb{Z}$ if p can be written in the form $p = m^2 + n^2$ for $m, n \in Z$ and one prime ideal lying over $p\mathbb{Z}$ otherwise. (In fact, $p = m^2 + n^2$ if and only if $p \equiv 1 \mod 4$).