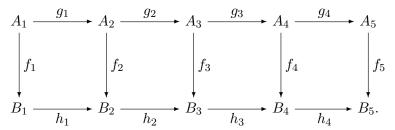
## Commutative Algebra 88-813

## 5772 Semester A

## Question Sheet 1

Due 15/11/2011, יח בחשון תשע"ב

(1) Prove the Five Lemma: Let R be a ring and consider the following commutative diagram of R-modules:



Suppose that the two rows are exact and that  $f_1$ ,  $f_2$ ,  $f_4$ , and  $f_5$  are isomorphisms. Prove that  $f_3$  is also an isomorphism.

- (2) Let M be a finitely generated R-module. Prove that M has a maximal submodule.
- (3) Recall that l(M) is the composition length of a module M. If  $M = M_1 \oplus M_2$ , prove that  $l(M) = l(M_1) + l(M_2)$ .
- (4) Let R be a commutative ring and let M be an R-module generated by m elements. Suppose that there is a surjective map  $\varphi: M \to R^{(n)}$ . Prove that  $m \ge n$ .
- (5) Let R be a commutative ring. An element  $s \in R$  is called regular if  $sr \neq 0$  for all  $0 \neq r \in R$ . Let M be an R-module, and define  $tor(M) = \{m \in M : sm = 0 \text{ for some regular } s \in R\}$ . Prove the following:
  - (a) tor(M) is a submodule of M, and tor(R) = 0.
  - (b)  $tor(M_1 \oplus M_2) \simeq tor(M_1) \oplus tor(M_2)$ .
  - (c) If A = M/tor(M) is a free R-module, then  $M \simeq A \oplus \text{tor}(M)$ .
- (6) Let  $N_1, N_2$ , and K be submodules of M such that  $N_1 \supset N_2$  and  $K \cap N_1 = 0$ . Then  $(K + N_1)/(K + N_2) \simeq N_1/N_2$ .
- (7) Let  $0 \to K \to M \to N \to 0$  be an exact sequence of R-modules. If K is free of rank k and N is free of rank n, prove that M is free of rank k + n.