Commutative Algebra 88-813
5772 Semester A
Question Sheet 9
Due 19/1/2012, כ"ד טבת תשע"ב

(1) Let $R$ be a ring and $S$ a sub-monoid such that $S \subset Z(R)$. In class we defined an equivalence relation on $R \times S$ as follows: $(r_1, s_1) \sim (r_2, s_2)$ if there exists $s \in S$ such that $r_1 s_2 s = r_2 s_1 s$. We defined $S^{-1}R$ to be the set of equivalence classes and claimed that it can be given a ring structure as follows:

$$\begin{aligned}(r_1, s_1) + (r_2, s_2) &= (r_1 s_2 + r_2 s_1, s_1 s_2) \\ (r_1, s_1) \cdot (r_2, s_2) &= (r_1 r_2, s_1 s_2).\end{aligned}$$

Show that these operations are well-defined.

(2) Let $R$ be a ring and $I \subset R$ a left ideal. Let $\overline{R} = R/I$, and let $\overline{S} = \{s + I : s \in S\} \subset \overline{R}$. Prove that $\overline{S}^{-1}\overline{R} \simeq S^{-1}R/S^{-1}I$.

(3) Construct an integral domain with exactly 5772 maximal ideals.

(4) Let $F$ be a field and let $R = F[[x]]$ be the ring of power series with coefficients in $F$. Show that $R$ is a local ring.

(5) Let $R$ be a commutative ring and $S$ a sub-monoid (not containing zero). If $M$ is an $R$-module, we define an equivalence relation on $M \times S$ by $(m_1, s_1) \sim (m_2, s_2)$ if there exists $s \in S$ such that $s(s_2 m_1 - s_1 m_2) = 0$. Show that this is an equivalence relation, that the set $S^{-1}M$ of equivalence classes can be made into an abelian group by defining $(m_1, s_1) + (m_2, s_2) = (s_2 m_1 + s_1 m_2, s_1 s_2)$, and that it is a well-defined $S^{-1}R$-module under the scalar multiplication $(r, s) \cdot (m, s') = (rm, ss')$.

(6) Let $R$ be a commutative ring and $M$ an $R$-module. If $P \subset R$ is a prime ideal and $S = R - P$ is its complement, we write $R_P$ for $S^{-1}R$ and $M_P$ for $S^{-1}M$. Prove that the following three statements are equivalent:
   (a) $M_P = 0$ for all prime ideals $P \subset R$.
   (b) $M_Q = 0$ for all maximal ideals $Q \subset R$.
   (c) $M = 0$.