PRO-ISOMORPHIC ZETA FUNCTIONS OF SOME D^* LIE LATTICES OF EVEN RANK

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Abstract. We compute the local pro-isomorphic zeta functions at all but finitely many primes for a certain family of class-two-nilpotent Lie lattices of even rank, parametrized by irreducible non-linear polynomials $f(x) \in \mathbb{Z}[x]$, that corresponds to a family of groups introduced by Grunewald and Segal. The result is expressed in terms of a combinatorially defined family of rational functions.

1. Introduction

1.1. **Pro-isomorphic zeta functions.** Let G be a finitely generated group. The pro-isomorphic zeta function of G, which was originally introduced by Grunewald, Segal, and Smith [12], is the Dirichlet series $\zeta_G^{\wedge}(s) = \sum_{m=0}^{\infty} a_m^{\wedge}(G) m^{-s}$, where s is a complex variable and $a_m^{\wedge}(G)$ is the (necessarily finite) number of subgroups $H \leq G$ of index m such that the profinite completion of H is isomorphic to that of G. In practice it is more convenient to consider a linearized version of this problem. Let $\mathcal{L}$ be a Lie lattice, namely a free $\mathbb{Z}$-module of finite rank endowed with a Lie bracket. Its pro-isomorphic zeta function is the Dirichlet series $\zeta_{\mathcal{L}}^{\wedge}(s) = \sum_{m=0}^{\infty} b_m^{\wedge}(\mathcal{L}) m^{-s}$, where $b_m^{\wedge}(\mathcal{L})$ is the number of sublattices $\mathcal{M} \leq \mathcal{L}$ of index n such that $\mathcal{M} \otimes \mathbb{Z}_p \simeq \mathcal{L} \otimes \mathbb{Z}_p$ for all primes p. A fundamental result [12, Proposition 4] is the Euler decomposition $\zeta_{\mathcal{L}}^{\wedge}(s) = \prod_p \zeta_{\mathcal{L},p}^{\wedge}(s)$, where $\zeta_{\mathcal{L},p}^{\wedge}(s)$ counts only sublattices of p-power index or, equivalently, $\mathbb{Z}_p$-sublattices of $\mathcal{L} \otimes \mathbb{Z}_p$ that are isomorphic to $\mathcal{L} \otimes \mathbb{Z}_p$. An analogous decomposition holds for groups. If the finitely presented group G is nilpotent and torsion-free, then there exists a Lie lattice $\mathcal{L}(G)$ such that $\zeta_{\mathcal{L}(G),p}^{\wedge}(s) = \zeta_{G,p}^{\wedge}(s)$ for all but finitely many p. If G is of class two, then this equality holds for all primes p; see, for instance, [12, §4] and [2, §2.1].

The present work computes the pro-isomorphic zeta functions of many members of a certain family of class-two-nilpotent Lie lattices of even rank considered by Berman, Klopsch, and Onn [4]. It corresponds to the family of D^*-groups of even Hirsch length constructed by Grunewald and Segal [11, §6] as representatives of commensurability classes that they defined in the course their classification of torsion-free radicable class-two-nilpotent groups with finite rank and center of rank two. The pro-isomorphic zeta functions of (a generalization of) the similarly arising family of Lie lattices of odd rank was computed in [6, Theorem 1.4]; this result was generalized in [2, Theorem 5.17] to the restriction of scalars to $\mathbb{Z}$ of the base extension of such Lie lattices to the ring of integers of an arbitrary number field.

1.2. **Statement of results.** We now present our results more precisely. Let $\Delta(x) \in \mathbb{Z}[x]$ be a primitive polynomial, i.e. $\Delta(x) = f(x)^{\ell}$ for an irreducible monic polynomial

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$f(x)$ and $\ell \in \mathbb{N}$. If $\Delta(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for $a_i \in \mathbb{Z}$, recall its companion matrix

$$C_\Delta = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \in M_n(\mathbb{Z}).$$

Let $\mathcal{L}_\Delta$ be the Lie lattice of rank $2n+2$ with basis $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, z_2$ and Lie bracket determined by the following:

- $[x_i, x_j] = [y_i, y_j] = 0$ for all $1 \leq i, j \leq n$;
- $[x_i, y_j] = \delta_{ij} z_1 + (C_\Delta)_{ij} z_2$ for all $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta;
- z_1 and z_2 lie in (and indeed span) the center of $\mathcal{L}_\Delta$.

We consider the case where $\Delta(x) = f(x)$ is an *irreducible* polynomial of degree $n \geq 2$ and determine $\zeta^\wedge_{\mathcal{L}_f,p}(s)$ for all but finitely many p. Indeed, let β be a root of $f(x)$ and consider the number field $K_f = \mathbb{Q}(\beta)$. Recall that the conductor $\mathcal{F}_f$ is the largest ideal of the ring of integers $\mathcal{O}_{K_f}$ that is contained in $\mathbb{Z}[\beta]$. For all primes p coprime to $\mathcal{F}_f$, we obtain an explicit expression for $\zeta^\wedge_{\mathcal{L}_f,p}(s)$ in (7) below for $n \geq 3$. Theorem 1.3 treats the case $n = 2$, which was actually computed 35 years ago, under a different name, by Grunewald, Segal, and Smith.

Moreover, we prove the following finite uniformity statement. Suppose that K is a number field, p is a prime, and $\mathbf{e} = (e_1, \ldots, e_r)$ and $\mathbf{f} = (f_1, \ldots, f_r)$ are vectors of natural numbers. We say that p has decomposition type $(\mathbf{e}, \mathbf{f})$ in K if $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, where the $\mathfrak{p}_i \lhd \mathcal{O}_K$ are distinct prime ideals with residue fields of cardinality $|\mathcal{O}_K/\mathfrak{p}_i| = p^{f_i}$ for every $1 \leq i \leq r$. This implies that $n = \sum_{i=1}^r e_i f_i$. Let $\mathbf{1}$ denote the vector $(1, \ldots, 1)$.

Theorem 1.1. *Let $n \geq 3$, and let $\mathbf{e} = (e_1, \ldots, e_r)$ and $\mathbf{f} = (f_1, \ldots, f_r)$ satisfy $n = \sum_{i=1}^r e_i f_i$. There exists an explicit rational function $W_{\mathbf{e},\mathbf{f}}(X, Y) \in \mathbb{Q}(X, Y)$ with the following property: if $f(x) \in \mathbb{Z}[x]$ is irreducible of degree n, and if the prime p is coprime to $\mathcal{F}_f$ and has decomposition type $(\mathbf{e}, \mathbf{f})$ in K_f, then $\zeta^\wedge_{\mathcal{L}_f,p}(s) = W_{\mathbf{e},\mathbf{f}}(p, p^{-s})$.*

If, moreover, $\mathbf{e} = \mathbf{1}$, i.e. p is unramified in K_f, then $W_{\mathbf{1},\mathbf{f}}(X, Y)$ satisfies the following functional equation:

$$W_{\mathbf{1},\mathbf{f}}(X^{-1}, Y^{-1}) = (-1)^{r+1} p^{9n-(2n+4)s} W_{\mathbf{1},\mathbf{f}}(X, Y). \tag{1}$$

In fact, in (8) we realize the functions $W_{\mathbf{1},\mathbf{f}}$ as specializations of combinatorially defined functions Φ_r in 2^r variables introduced for every $r \in \mathbb{N}$ in Definition 2.1. While these Φ_r are reminiscent of some functions that have appeared recently in the literature in the context of enumerative problems arising from algebra [18, 7, 17, 15], they do not seem to be special cases of them. Then (1) is immediate from Proposition 2.2, which proves a general functional equation for the functions Φ_r.

We illustrate the explicit formulas of Theorem 1.1 in a simple example:

Corollary 1.2. *Let $f(x) = x^3 - 2$.*

(1) *If $p \equiv 1 \bmod 3$ and there exist $a, b \in \mathbb{Z}$ such that $p = a^2 + 27b^2$ (equivalently, if p is totally split in $K_f = \mathbb{Q}(\sqrt[3]{2})$), then*
$$\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \frac{1 + 2p^{13-5s} + 2p^{14-5s} + p^{27-10s}}{(1-p^{12-5s})(1-p^{13-5s})(1-p^{14-5s})(1-p^{15-5s})}.$$

(2) *If $p \equiv 1 \bmod 3$ and there do not exist $a, b \in \mathbb{Z}$ such that $p = a^2 + 27b^2$ (equivalently, if p is inert in K_f), then*
$$\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \frac{1}{(1-p^{12-5s})(1-p^{15-5s})}.$$

(3) *If $p > 2$ and $p \equiv 2 \bmod 3$ (equivalently, if $p\mathcal{O}_{K_f} = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathcal{O}_{K_f}/\mathfrak{p}_1 \simeq \mathbb{F}_p$ and $\mathcal{O}_{K_f}/\mathfrak{p}_2 \simeq \mathbb{F}_{p^2}$) then*
$$\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \frac{1 - p^{27-10s}}{(1-p^{12-5s})(1-p^{13-5s})(1-p^{14-5s})(1-p^{15-5s})}.$$

(4) *If $p \in \{2, 3\}$ (equivalently, if p is totally ramified in K_f), then*
$$\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \frac{1 + p^{13-5s} + p^{14-5s}}{(1-p^{12-5s})(1-p^{15-5s})}.$$

Note that the rational function governing $\zeta^{\wedge}_{\mathcal{L}_f,p}(s)$ for the ramified primes $p \in \{2, 3\}$ does not satisfy a functional equation for any symmetry factor.

Observe in passing that the functional equation (1) satisfies [4, Conjecture 1.5]. Unlike the situation for zeta functions counting subrings, ideals, and some related structures [20, 14], it is not known in general that local pro-isomorphic zeta functions of nilpotent Lie lattices, even of class two, satisfy functional equations. See [3] for an example of a Lie lattice of class four none of whose local pro-isomorphic zeta functions satisfies a functional equation. However, Berman, Klopsch, and Onn have conjectured, based on a study of known examples, that if $\mathcal{L}$ is graded and $\zeta^{\wedge}_{\mathcal{L},p}(s)$ satisfies a functional equation at almost all primes p, then the exponent of p^{-s} in the symmetry factor at almost all primes should be the weight of a minimal grading of $\mathcal{L}$; see [4] for definitions and details. Indeed, the Lie lattices $\mathcal{L}_f$ considered above are naturally graded in the sense of [4], and hence the weight of a minimal grading is $\mathrm{rk}_{\mathbb{Z}}\mathcal{L}_f + \mathrm{rk}_{\mathbb{Z}}[\mathcal{L}_f, \mathcal{L}_f] = (2n+2) + 2 = 2n + 4$.

1.3. **The quadratic case.** For completeness, we state the pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L}_f,p}(s)$ at all but finitely many primes when $f(x) \in \mathbb{Z}[x]$ is an irreducible quadratic polynomial. Note that there are only three decomposition types for a prime in a quadratic number field: inert $((\mathbf{e}, \mathbf{f}) = ((1), (2)))$, totally split $((\mathbf{e}, \mathbf{f}) = ((1, 1), (1, 1)))$ and totally ramified $((\mathbf{e}, \mathbf{f}) = ((2), (1)))$. The following claim is analogous to Theorem 1.1.

Theorem 1.3. *Consider the rational function*
$$W(X, Y) = \frac{1}{(1 - X^4Y^2)(1 - X^5Y^2)}.$$
For each of the three decomposition types $(\mathbf{e}, \mathbf{f})$ above, set $W_{\mathbf{e},\mathbf{f}}(X, Y) = \prod_{i=1}^{r} W(X^{f_i}, Y^{f_i})$. If $f[x] \in \mathbb{Z}[x]$ is an irreducible quadratic polynomial and p is coprime to $\mathcal{F}_f$ and has decomposition type $(\mathbf{e}, \mathbf{f})$ in the quadratic number field K_f, then
$$\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = W_{\mathbf{e},\mathbf{f}}(p, p^{-s}).$$

For all three decomposition types, the following functional equation holds:

$$W_{\mathbf{e},\mathbf{f}}(X^{-1},Y^{-1}) = p^{(\sum_{i=1}^{r} f_i)(9-4s)} W_{\mathbf{e},\mathbf{f}}(X,Y).$$

Proof. Let $\beta = \beta_1$ be a root of $f(x)$, and let β_2 be the other root. Observe that they are both contained in the Galois extension $K_f/\mathbb{Q}$. Let $\mathcal{H}$ be the Heisenberg Lie lattice $\mathcal{H} = \langle x, y, z\rangle_{\mathbb{Z}}$ such that $[x,y] = z$ and the product of any other pair of generators vanishes. Consider $\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}_{K_f}$ as a Lie lattice by restriction of scalars. Since p is coprime to $\mathcal{F}_f$, we have $\mathbb{Z}_p[\beta] = \mathcal{O}_{K_f} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and it is easy to verify that there is an isomorphism $\varphi : \mathcal{L}_f \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} (\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}_{K_f}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ given by

$$(x_1, x_2, y_1, y_2, z_1, z_2) \mapsto (x \otimes 1, x \otimes \beta_1, y \otimes (-\beta_2), y \otimes 1, z \otimes (-\beta_2), z \otimes 1).$$

Thus $\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \zeta^{\wedge}_{\mathcal{H}\otimes_{\mathbb{Z}}\mathcal{O}_{K_f},p}(s)$, and the right-hand side was computed by Grunewald, Segal, and Smith [12, Theorem 7.1]; see Theorem 5.10 and Remark 5.12 of [2] for an alternative derivation of the same result. □

Observe that the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L}_f,p}(s)$ naturally decompose as products of factors parametrized by primes of K_f dividing p. This is a special case of a general phenomenon [2, Proposition 3.14]. The Lie algebras $\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Q}_p$ satisfy a rigidity property [2, Definition 3.8] originally introduced by Segal [19]; as a consequence, the pro-isomorphic zeta function $\zeta^{\wedge}_{\mathcal{H}\otimes\mathcal{O}_K,p}(s)$ may be computed easily for any number field K. Such rigidity does not hold for the Lie algebras $\mathcal{L}_f \otimes_{\mathbb{Z}} \mathbb{Q}_p$ appearing in our work; this is essentially a consequence of the arithmetic of the number field K_f, which is larger than $\mathbb{Q}$, controlling the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L}_f,p}(s)$.

1.4. **Overview.** It is a simple but fundamental observation that computations of local factors of pro-isomorphic zeta functions can be reduced to p-adic integrals of a certain form. Consider the $\mathbb{Q}$-Lie algebra $L_\Delta = \mathcal{L}_\Delta \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $\mathbf{G}_\Delta$ be its algebraic automorphism group. This is the algebraic group defined over $\mathbb{Q}$ characterized by the property that $\mathbf{G}_\Delta(E) \simeq \mathrm{Aut}_E(L_\Delta \otimes_{\mathbb{Q}} E)$ for every field E of characteristic zero. Fixing the ordered basis $(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, z_2)$ of $\mathcal{L}_\Delta$ gives an embedding $\mathbf{G}_\Delta \hookrightarrow \mathrm{GL}_{2n+2}$. Now set $G^+_\Delta(\mathbb{Q}_p) = \mathbf{G}_\Delta(\mathbb{Q}_p) \cap M_{2n+2}(\mathbb{Z}_p)$, and let $G_\Delta(\mathbb{Z}_p) = \mathbf{G}_\Delta(\mathbb{Q}_p) \cap \mathrm{GL}_{2n+2}(\mathbb{Z}_p)$. Let μ be the right Haar measure on the group $\mathbf{G}_\Delta(\mathbb{Q}_p)$, normalized so that $\mu(G_\Delta(\mathbb{Z}_p)) = 1$. Then [12, Proposition 3.4] we have

$$\zeta^{\wedge}_{\mathcal{L}_\Delta,p}(s) = \int_{G^+_\Delta(\mathbb{Q}_p)} |\det g|_p^s \, d\mu, \tag{2}$$

where $|\cdot|_p$ is the normalized valuation on $\mathbb{Q}_p$. The structure of $\mathbf{G}_\Delta$, for all primitive polynomials $\Delta(x) = f(x)^\ell$, was determined by Berman, Klopsch, and Onn; see Proposition 3.1 below for the case $\deg f(x) \geq 3$. When $\Delta(x) = f(x)$, the domain of integration of (2) is sufficiently simple that the integral may be computed directly using the Cartan decomposition of $\mathrm{SL}_2(F)$ for p-adic fields $F/\mathbb{Q}_p$. See Remark 3.4 for the reason for the restriction to the case $\ell = 1$. The earlier work cited in the proof of Theorem 1.3 also amounts to the computation of an integral (2). After establishing several preliminary results, we prove Theorem 1.1 and its corollary in Section 3 below.

The algebraic group $\mathbf{G}_\Delta$ has a particularly complicated structure when $\Delta(x)$ is a power of a linear polynomial. The pro-isomorphic zeta functions of $\mathcal{L}_\Delta$ are obtained with in [4] for $\Delta(x) = x^2$ and $\Delta(x) = x^3$ after computations substantially more involved than the ones in Section 3; it is notable that the simplifying assumptions used in [10] to analyze the integrals (2) do not hold in these cases.

2. Preliminaries

This section contains two results that will be used in the computation of pro-isomorphic zeta functions and their functional equations that comprise the core of the paper. We give their proofs here to avoid breaking the flow of the computation later.

2.1. **A combinatorial function.** We introduce a family of combinatorially defined functions in terms of which it will be convenient to express the local pro-isomorphic zeta functions $\zeta^{\wedge}_{\mathcal{L}_f,p}(s)$. For every $r \in \mathbb{N}$, let $[r]$ denote the set $\{1, 2, \ldots, r\}$.

Definition 2.1. Let $r \in \mathbb{N}$. Let $\{X_I\}_{I\subseteq[r]}$ be a collection of 2^r variables, one for each subset $I \subseteq [r]$. We consider the following function in these variables:

$$\Phi_r(\{X_I\}_{I\subseteq[r]}) = \sum_{I\subseteq[r]} (-1)^{|I|} \frac{X_I}{1 - X_I}.$$

Proposition 2.2. *The function Φ_r satisfies the following self-reciprocity upon inversion of the variables:*

$$\Phi_r(\{X_I^{-1}\}) = -\Phi_r(\{X_I\}).$$

Proof. Let $\mathcal{P}[r]$ denote the power set of $[r]$. Writing the rational function over a common denominator, we find that

$$\Phi_r(\{X_I^{-1}\}) = \sum_{I\in\mathcal{P}[r]} (-1)^{|I|} \frac{X_I^{-1}}{1 - X_I^{-1}} = \sum_{I\in\mathcal{P}[r]} \frac{(-1)^{|I|+1}}{1 - X_I} = \frac{\sum_{I\in\mathcal{P}[r]} (-1)^{|I|+1} \prod_{\substack{J\in\mathcal{P}[r]\\ J\neq I}} (1 - X_J)}{\prod_{J\in\mathcal{P}[r]} (1 - X_J)}. \tag{3}$$

Similarly,

$$-\Phi_r(\{X_I\}) = \frac{\sum_{I\in\mathcal{P}[r]} (-1)^{|I|+1} X_I \prod_{\substack{J\in\mathcal{P}[r]\\ J\neq I}} (1 - X_J)}{\prod_{J\in\mathcal{P}[r]} (1 - X_J)}. \tag{4}$$

Thus it suffices to show that the numerators of the two expressions are the same. Multiplying out the parentheses and computing the coefficient of the monomial $\prod_{J\in T} X_J$ for each $T \subseteq \mathcal{P}[r]$, we find that the numerator of (3) is

$$\sum_{I\in\mathcal{P}[r]} (-1)^{|I|+1} \prod_{\substack{J\in\mathcal{P}[r]\\ J\neq I}} (1 - X_J) = \sum_{I\in\mathcal{P}[r]} (-1)^{|I|+1} \sum_{\substack{T'\subseteq\mathcal{P}[r]\\ I\notin T'}} (-1)^{|T'|} \prod_{J\in T'} X_J =$$

$$\sum_{T\subseteq\mathcal{P}[r]} (-1)^{|T|+1} \left(\sum_{I\in\mathcal{P}[r]\setminus T} (-1)^{|I|} \right) \prod_{J\in T} X_J = \sum_{T\subseteq\mathcal{P}[r]} (-1)^{|T|} \left(\sum_{I\in T} (-1)^{|I|} \right) \prod_{J\in T} X_J,$$

where the last equality follows from the elementary observation that $\sum_{I\in\mathcal{P}[r]}(-1)^{|I|} = \prod_{i=1}^{r}(1-1) = 0$. Analogously, the numerator of (4) is

$$\sum_{I\in\mathcal{P}[r]}(-1)^{|I|+1}X_I \sum_{\substack{T'\subseteq\mathcal{P}[r]\\ I\not\in T'}}(-1)^{|T'|}\prod_{J\in T'}X_J = \sum_{I\in\mathcal{P}[r]}(-1)^{|I|+1}\sum_{\substack{U\subseteq\mathcal{P}[r]\\ I\in U}}(-1)^{|U|-1}\prod_{J\in U}X_J =$$
$$\sum_{T\subseteq\mathcal{P}[r]}\left(\sum_{I\in T}(-1)^{|T|+|I|}\right)\prod_{J\in T}X_J,$$

where the first equality is obtained by setting $U = T'\cup\{I\}$. This completes the proof of our claim. $\square$

2.2. **The Cartan decomposition of $\mathrm{SL}_2(F)$, for a p-adic field F.** Let p be a prime, and let v_p be the normalized additive valuation on $\mathbb{Q}_p$. Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_F$. Fix a uniformizer $\pi\in\mathcal{O}_F$. Let $k_F = \mathcal{O}_F/(\pi)$ be the residue field, and let q denote its cardinality. Given $\lambda\in k_F^\times$, let $[\lambda]\in\mathcal{O}_F$ denote the $(q-1)$-st root of unity lifting λ, and set $[0]=0$. Let $I_0=\{0\}$, and for every $m\in\mathbb{N}$ define the set

$$I_m = \left\{[\lambda_0]+\pi[\lambda_1]+\cdots+\pi^{m-1}[\lambda_{m-1}] : (\lambda_0,\ldots,\lambda_{m-1})\in k_F^m\right\}\subset\mathcal{O}_F.$$

Lemma 2.3. *Let $F/\mathbb{Q}_p$ be a finite extension, and let $\pi\in\mathcal{O}_F$ be a uniformizer. A list of representatives of right cosets of $\mathrm{SL}_2(\mathcal{O}_F)$ in $\mathrm{SL}_2(F)$ is given by*

$$\coprod_{m\geq 0}\left\{\begin{pmatrix}\pi^m & 0\\ \pi^{-m}\kappa & \pi^{-m}\end{pmatrix} : \kappa\in I_{2m}\right\}\coprod\coprod_{m\geq 1}\left\{\begin{pmatrix}0 & -\pi^m\\ \pi^{-m} & -\pi^{-m+1}\kappa\end{pmatrix} : \kappa\in I_{2m-1}\right\}.$$

Proof. Set $\delta = \begin{pmatrix}\pi & 0\\ 0 & \pi^{-1}\end{pmatrix}\in\mathrm{SL}_2(F)$. From the Cartan decomposition

$$\mathrm{SL}_2(F) = \coprod_{m\geq 0}\mathrm{SL}_2(\mathcal{O}_F)\delta^m\mathrm{SL}_2(\mathcal{O}_F),$$

setting $K_m = \mathrm{SL}_2(\mathcal{O}_F)\cap\delta^{-m}\mathrm{SL}_2(\mathcal{O}_F)\delta^m$ for $m\geq 0$, one deduces the decomposition

$$\mathrm{SL}_2(F) = \coprod_{m\geq 0}\coprod_{K_mk\in K_m\backslash\mathrm{SL}_2(\mathcal{O}_F)}\mathrm{SL}_2(\mathcal{O}_F)\delta^m k$$

of $\mathrm{SL}_2(F)$ into right cosets of $\mathrm{SL}_2(\mathcal{O}_F)$. The claim follows by a straightforward computation. An alternative list of coset representatives may be obtained from [1, Proposition 1.1], noting that *left* cosets of $\mathrm{SL}_2(\mathcal{O}_F)$ correspond to vertices of the Bruhat-Tits tree of $\mathrm{SL}_2(F)$ lying at an even distance from v_0, in the notation of [1]. $\square$

Corollary 2.4. *Let e denote the ramification degree of $F/\mathbb{Q}_p$. Let ν_F be the right Haar measure on $\mathrm{SL}_2(F)$, with the normalization $\mu(\mathrm{SL}_2(\mathcal{O}_F))=1$. For $a\in\mathbb{Z}_p\setminus\{0\}$, set*

$$S_F(a) = \left\{\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\in\mathrm{SL}_2(F) : \begin{pmatrix}a\alpha_{11} & \alpha_{12}\\ a\alpha_{21} & \alpha_{22}\end{pmatrix}\in\mathrm{SL}_2(\mathcal{O}_F)\right\}.$$

Then $\nu_F(S_F(a)) = \frac{1-q^{ev_p(a)+1}}{1-q}$.

Proof. It is easy to see that $S_F(a)$ is invariant under left multiplication by any element of $\mathrm{SL}_2(\mathcal{O}_F)$ and thus consists of a union of right cosets of $\mathrm{GL}_2(\mathcal{O}_F)$. Observe that $a\mathcal{O}_F = \pi^{ev_p(a)}\mathcal{O}_F$. Among the coset representatives listed in Lemma 2.3, the ones contained in $S_F(a)$ are precisely $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -\pi^m \\ \pi^{-m} & -\pi^{-m+1}\kappa \end{pmatrix}$ for $m \in [ev_p(a)]$ and $\kappa = \sum_{i=1}^{2m-2} \pi^i[\lambda_i]$ satisfying $\lambda_0 = \cdots = \lambda_{m-2} = 0$. There are

$$1 + \sum_{m=1}^{ev_p(a)} q^m = \frac{1 - q^{ev_p(a)+1}}{1-q}$$

of these. Since each right coset has measure 1, the claim follows. □

3. Computation

3.1. The algebraic automorphism group. Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial of degree $n \geq 3$. Consider the primitive polynomial $\Delta(x) = f(x)^\ell$ for $\ell \in \mathbb{N}$, and set $K_\Delta = \mathbb{Q}[x]/(\Delta(x))$; this ring has dimension ℓn as a $\mathbb{Q}$-vector space. To describe the algebraic automorphism group of L_Δ we define three algebraic subgroups of $\mathbf{GL}_{2\ell n+2}$. There is a morphism of algebraic groups $\rho_2 : \mathrm{Res}_{K_\Delta/\mathbb{Q}}\mathbf{SL}_2 \to \mathbf{SL}_{2\ell n+2}$ given by

$$\rho_2 \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \iota(\alpha_{11}) & \iota(\alpha_{12}) & \\ \iota(\alpha_{21}) & \iota(\alpha_{22}) & \\ & & I_2 \end{pmatrix},$$

where for any $\mathbb{Q}$-algebra R the map $\iota : K \otimes_{\mathbb{Q}} R \to M_{\ell n}(R)$ is determined by $\iota(\beta^i \otimes r) = rC_\Delta^i$ for any $i \in \mathbb{N} \cup \{0\}$ and $r \in R$, and $\beta = x + (\Delta(x)) \in K_\Delta$. Equivalently, $\iota(\alpha)$ is the matrix of the R-linear endomorphism of $K \otimes_{\mathbb{Q}} R$ corresponding to multiplication by α, with respect to the basis $(\beta^i \otimes 1)_{i=0}^{\ell n-1}$. By a standard exercise in linear algebra, or [13, Theorem 1], the image of ρ_2 is indeed contained in $\mathbf{SL}_{2\ell n+2}$.

Consider the embedding of algebraic groups $\rho_1 : \mathbf{G}_m \to \mathbf{GL}_{2\ell n+2}$ given by

$$\rho_1(a) = \begin{pmatrix} aI_{\ell n} & 0 & 0 \\ 0 & I_{\ell n} & 0 \\ 0 & 0 & aI_2 \end{pmatrix}$$

and the embedding $\rho_3 : \mathbf{G}_a^{4\ell n} \to \mathbf{SL}_{2\ell n+2}$ given by

$$\rho_3(c_1, \ldots, c_{4\ell n}) = \begin{pmatrix} 1 & & & c_1 & c_{2\ell n+1} \\ & \ddots & & \vdots & \vdots \\ & & 1 & c_{2\ell n} & c_{4\ell n} \\ & & & 1 & 0 \\ & & & 0 & 1 \end{pmatrix}.$$

Recall that our chosen $\mathbb{Z}$-basis of $\mathcal{L}_\Delta$ allows us to identify the algebraic automorphism group $\mathbf{G}_\Delta$ of L_Δ with an algebraic subgroup of $\mathbf{GL}_{2\ell n+2}$. Its structure, which was determined by Berman, Klopsch, and Onn, consists essentially of an internal semi-direct product of the three subgroups of $\mathbf{GL}_{2\ell n+2}$ just defined. There exists a symmetric matrix

$\sigma \in \mathrm{GL}_{\ell n}(\mathbb{Z})$ such that $\sigma C_\Delta \sigma^{-1} = C_\Delta^T$; see [4, §2.1]. Set

$$\Sigma = \begin{pmatrix} I_{\ell n} & & \\ & \sigma & \\ & & I_2 \end{pmatrix} \in \mathrm{GL}_{2\ell n+2}(\mathbb{Z}).$$

Proposition 3.1. *Let $\Delta(x) = f(x)^\ell$, where $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial of degree $n \geq 3$. Then*

$$\mathbf{G}_\Delta = (\rho_3(\mathbf{G}_a^{4\ell n}) \rtimes \Sigma(\rho_2(\mathrm{Res}_{K_\Delta/\mathbb{Q}}\mathbf{SL}_2))\Sigma^{-1}) \rtimes \rho_1(\mathbf{G}_m),$$

where the action in each internal semi-direct product is by conjugation.

Proof. The subgroup $\mathbf{G}_{0,\Delta} \subset \mathbf{G}_\Delta$ of automorphisms acting trivially on the center $\langle z_1, z_2\rangle$ is described by [4, Theorem 2.3] and its proof and is $\rho_3(\mathbf{G}_a^{4\ell n}) \rtimes \Sigma(\rho_2(\mathrm{Res}_{K_\Delta/\mathbb{Q}}\mathbf{SL}_2))\Sigma^{-1}$. Under the assumption $n \geq 3$, every automorphism acts on the center as a scalar by [5, Theorem 1.4]; this case is not treated in the final version of [4], which focuses on $n = 1$. It is easy to check that $\rho_1(\mathbf{G}_a) \subset \mathbf{G}_\Delta$. Thus, for any field extension $E/\mathbb{Q}$, any element of $\mathbf{G}_\Delta(E)$ may be expressed uniquely as a product of an element of $\rho_1(E^\times)$ and one of $\mathbf{G}_{0,\Delta}(E)$, and the claim follows. □

3.2. **Notation.** From now on we assume $\ell = 1$, namely that $\Delta(x) = f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial of degree $n \geq 3$. To simplify the notation, we write $\mathbf{G} \subset \mathbf{GL}_{2n+2}$ for $\mathbf{G}_\Delta$. Similarly, write K for K_Δ; this is the number field $\mathbb{Q}(\beta)$, where β is a root of $f(x)$. Let $\mathcal{O}_K$ denote the ring of integers of K, and recall that the conductor $\mathcal{F}_f$ is the largest ideal of $\mathcal{O}_K$ contained in $\mathbb{Z}[\beta]$.

Now let p be a rational prime that decomposes in K as $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, where the distinct prime ideals $\mathfrak{p}_i \lhd \mathcal{O}_K$ have residue fields $\mathcal{O}_K/\mathfrak{p}_i$ of cardinality $q_i = p^{f_i}$. Then $\mathbb{Q}_p \otimes_\mathbb{Q} K \simeq F_1 \times \cdots \times F_r$, where for every $i \in [r]$ we write F_i for the localization $K_{\mathfrak{p}_i}$. Similarly, $\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_K \simeq \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_r}$.

Assume that p is coprime to $\mathcal{F}_f$. In this case $\mathbb{Z}_p[x]/(f(x)) = \mathbb{Z}_p \otimes_\mathbb{Z} \mathbb{Z}[\beta] = \mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_K$.

3.3. **Setup and evaluation of a p-adic integral.** It is immediate from Proposition 3.1 that

$$\begin{aligned}\mathbf{G}(\mathbb{Q}_p) =& \rho_1(\mathbb{Q}_p^\times) \ltimes (\Sigma(\rho_2(\mathrm{SL}_2(\mathbb{Q}_p \otimes_\mathbb{Q} K)))\Sigma^{-1} \ltimes \rho_3(\mathbb{Q}_p^{4n})) = \\ & \rho_1(\mathbb{Q}_p^\times) \ltimes \left(\Sigma \left(\rho_2 \left(\prod_{i=1}^r \mathrm{SL}_2(F_i) \right) \right) \Sigma^{-1} \ltimes \rho_3(\mathbb{Q}_p^{4n}) \right).\end{aligned}$$

We now explicitly determine the two subsets of $\mathbf{G}(\mathbb{Q}_p)$ necessary for our calculation.

Lemma 3.2. *Suppose that p is coprime to the conductor $\mathcal{F}_f$. Suppose that $a \in \mathbb{Q}_p^\times$, that $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p \otimes_\mathbb{Q} K)$, and that $\mathbf{c} = (c_1, \ldots, c_{4n}) \in \mathbb{Q}_p^{4n}$. Then $\rho_3(\mathbf{c})\Sigma\rho_2(A)\Sigma^{-1}\rho_1(a) \in G(\mathbb{Z}_p)$ if and only if $a \in \mathbb{Z}_p^\times$, whereas $A \in \mathrm{SL}_2(\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_K)$ and $\mathbf{c} \in \mathbb{Z}_p^{4n}$. Given $a \in \mathbb{Z}_p \setminus \{0\}$, define*

$$\begin{aligned} G_2^+(a) = & \quad \left\{ A \in \mathrm{SL}_2(\mathbb{Q}_p \otimes_\mathbb{Q} K) : \begin{pmatrix} a\alpha_{11} & \alpha_{12} \\ a\alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_K) \right\} \\ G_3^+(a) = & \quad \{\mathbf{c} \in \mathbb{Q}_p^{4n} : (ac_1, \ldots, ac_{4n}) \in \mathbb{Z}_p^{4n}\}. \end{aligned}$$

Then $\rho_3(\mathbf{c})\Sigma\rho_2(A)\Sigma^{-1}\rho_1(a) \in G^+(\mathbb{Q}_p)$ *if and only if* $a \in \mathbb{Z}_p \setminus \{0\}$, *while* $A \in G_2^+(a)$ *and* $\mathbf{c} \in G_3^+(a)$.

Proof. A simple computation shows that

$$\rho_3(\mathbf{c})\Sigma\rho_2(A)\Sigma^{-1}\rho_1(a) = \begin{pmatrix} a\iota(\alpha_{11}) & \iota(\alpha_{12})\sigma^{-1} & aC_1 \\ a\sigma\iota(\alpha_{21}) & \sigma\iota(\alpha_{22})\sigma^{-1} & aC_2 \\ 0 & 0 & aI_2 \end{pmatrix}, \tag{5}$$

where use ρ_1, ρ_2, ρ_3 to denote the corresponding morphisms on $\mathbb{Q}_p$-points, and where

$$C_1 = \begin{pmatrix} c_1 & c_{2n+1} \\ \vdots & \vdots \\ c_n & c_{3n} \end{pmatrix}, \quad C_2 = \begin{pmatrix} c_{n+1} & c_{3n+1} \\ \vdots & \vdots \\ c_{2n} & c_{4n} \end{pmatrix}.$$

Observe, given $\alpha \in \mathbb{Q}_p \otimes_{\mathbb{Q}} K$, that $\iota(\alpha) \in M_n(\mathbb{Z}_p)$ if and only if $\alpha \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]$, which is equivalent to $\alpha \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K$ by our hypothesis on p. Since $\sigma \in \mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Z}_p)$, the claim is now immediate from (5). □

The previous claim allows us to express the pro-isomorphic zeta function $\zeta^{\wedge}_{\mathcal{L}_f,p}(s)$ as an iterated integral. Indeed, let μ_1 be the right Haar measure on $\mathbb{Q}_p^{\times}$, normalized so that $\mu_1(\mathbb{Z}_p^{\times}) = 1$. Similarly, let μ_2 and μ_3 be the right Haar measures on $\mathrm{SL}_2(\mathbb{Q}_p \otimes_{\mathbb{Q}} K)$ and on $\mathbb{Q}_p^{4n}$, respectively, normalized to $\mu_2(\mathrm{SL}_2(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K)) = 1$ and $\mu_3(\mathbb{Z}_p^{4n}) = 1$. By the first part of Lemma 3.2, these normalizations are compatible with that of the right Haar measure μ on $\mathbf{G}(\mathbb{Q}_p)$. Then

$$\begin{aligned}
\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \quad & \int_{G^+(\mathbb{Q}_p)} |\det g|_p^s d\mu(g) = \\
& \int_{\mathbb{Z}_p\setminus\{0\}} \int_{G_2^+(a)} \int_{G_3^+(a)} |\det \rho_3(\mathbf{c})\Sigma\rho_2(A)\Sigma^{-1}\rho_1(a)|_p^s d\mu_3(\mathbf{c}) d\mu_2(A) d\mu_1(a) = \\
& \int_{\mathbb{Z}_p\setminus\{0\}} \int_{G_2^+(a)} \int_{G_3^+(a)} |a|_p^{(n+2)s} d\mu_3(\mathbf{c}) d\mu_2(A) d\mu_1(a) = \\
& \int_{\mathbb{Z}_p\setminus\{0\}} \int_{G_2^+(a)} |a|_p^{(n+2)s-4n} d\mu_2(A) d\mu_1(a).
\end{aligned}$$

Here the first equality is (2), the second follows from the second part of Lemma 3.2 and [16, Proposition 28], and the last equality holds because the integrand is constant on each set $G_3^+(a)$ and $\mu_3(G_3^+(a)) = |a|_p^{-4n}$ for every $a \in \mathbb{Z}_p \setminus \{0\}$. Since the integrand is also constant on each $G_2^+(a)$, we have

$$\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \int_{\mathbb{Z}_p\setminus\{0\}} |a|_p^{(n+2)s-4n} \mu_2(G_2^+(a)) d\mu_1(a). \tag{6}$$

Recall the notation defined in Section 3.2.

Lemma 3.3. *Suppose that* p *is coprime to* $\mathcal{F}_f$. *Then* $\mu_2(G_2^+(a)) = \prod_{i=1}^r \frac{1-q_i^{e_i v_p(a)+1}}{1-q_i}$ *for all* $a \in \mathbb{Z}_p \setminus \{0\}$.

Proof. The decomposition $\mathrm{SL}_2(\mathbb{Q}_p \otimes_{\mathbb{Q}} K) = \prod_{i=1}^r \mathrm{SL}_2(F_i)$ induces $\mathrm{SL}_2(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K) = \prod_{i=1}^r \mathrm{SL}_2(\mathcal{O}_{F_i})$ and $G_2^+(a) = \prod_{i=1} S_{F_i}(a)$, for the sets $S_{F_i}(a)$ defined in Section 2.2, and the Haar measure μ_2 is the product of the measures ν_{F_i} defined there. Hence the claim follows from Corollary 2.4. $\square$

Remark 3.4. Lemma 3.3 is the step in our computation that obliges us to restrict to the case of irreducible $\Delta(x)$. For a general primitive polynomial $\Delta(x)$, it appears to be difficult to compute the measure of the set

$$\left\{ A \in \mathrm{SL}_2(\mathbb{Q}_p[x]/(\Delta(x))) : \begin{pmatrix} a\alpha_{11} & \alpha_{12} \\ a\alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p[x]/(\Delta(x))) \right\}$$

in the absence of a suitable analogue of the p-adic Cartan decomposition.

3.4. **Proof of Theorem 1.1.** We can now easily deduce the main result stated in the introduction. Indeed, let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial of degree $n \geq 3$. Let p be a prime coprime to $\mathcal{F}_f$ having decomposition type $(\mathbf{e}, \mathbf{f})$ in the number field $K_f = \mathbb{Q}(x)/(f(x))$. We deduce from (6) and Lemma 3.3 that

$$\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \int_{\mathbb{Z}_p \setminus \{0\}} p^{(4n-(n+2)s)v_p(a)} \prod_{i=1}^{r} \frac{1 - p^{e_i f_i v_p(a) + f_i}}{1 - p^{f_i}} d\mu_1(a).$$

For any $v \geq 0$, we have $\mu_1(\{a \in \mathbb{Z}_p : v_p(a) = v\}) = \mu_1(p^v \mathbb{Z}_p^\times) = 1$. Hence

$$\begin{aligned} \zeta^{\wedge}_{\mathcal{L}_f,p}(s) = \quad & \sum_{v=0}^{\infty} p^{(4n-(n+2)s)v} \prod_{i=1}^{r} \frac{1 - p^{e_i f_i v + f_i}}{1 - p^{f_i}} = \\ & \frac{\sum_{I \subseteq [r]} (-1)^{|I|} \sum_{v=0}^{\infty} p^{(4n-(n+2)s)v} \cdot p^{(\sum_{i \in I} e_i f_i)v + \sum_{i \in I} f_i}}{\prod_{i=1}^{r} (1 - p^{f_i})}, \end{aligned}$$

and by summing geometric series we find that $\zeta^{\wedge}_{\mathcal{L}_f,p}(s) = W_{\mathbf{e},\mathbf{f}}(p, p^{-s})$ for

$$W_{\mathbf{e},\mathbf{f}}(X, Y) = \prod_{i=1}^{r} \left(\frac{1}{1 - X^{f_i}} \right) \sum_{I \subseteq [r]} (-1)^{|I|} \frac{X^{\sum_{i \in I} f_i}}{1 - X^{4n + \sum_{i \in I} e_i f_i} Y^{n+2}}, \tag{7}$$

which indeed depends only on $\mathbf{e}$ and $\mathbf{f}$. If $\mathbf{e} = \mathbf{1}$, we observe by inspection of (7) that

$$W_{\mathbf{1},\mathbf{f}}(X, Y) = \frac{1}{X^{4n} Y^{n+2} \prod_{i=1}^{r} (1 - X^{f_i})} \Phi_r(\{X_I\}_{I \subseteq [r]}), \tag{8}$$

with Φ_r as in Definition 2.1 and $X_I = X^{4n + \sum_{i \in I} f_i} Y^{n+2}$ for all $I \subseteq [r]$. The claimed functional equation (1) follows from Proposition 2.2 and a simple calculation.

3.5. **Proof of Corollary 1.2.** If $f(x) = x^3 - 2$, then $K = K_f = \mathbb{Q}(\sqrt[3]{2})$, and it is a classical fact (see, for instance, [8, Theorem 6.4.13]) that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}]$. Thus $\mathcal{F}_f = (1)$ and Theorem 1.1 applies to all primes. The discriminant of K is -108, so the only ramified primes are $p \in \{2, 3\}$, and one easily verifies that they are both totally ramified. If $p > 3$, then it follows from [8, Corollary 6.4.15] and the characterization of the totally split primes in e.g. [9, Theorem 9.8] that p is totally split if $p = a^2 + 27b^2$ (which implies $p \equiv 1 \bmod 3$), whereas $p\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$ with $f_1 = 1$ and $f_2 = 2$ if $p \equiv 2 \bmod 3$ and p is inert otherwise. With this classification of primes by decomposition type in hand, the claimed formulas are obtained from Theorem 1.1 by straightforward computation.

References

1. R. Abdellatif, *Classification des représentations modulo p de* $\mathrm{SL}(2, F)$, Bull. Soc. Math. France **142** (2014), 537–589.
2. M. N. Berman, I. Glazer, and M. M. Schein, *Pro-isomorphic zeta functions of nilpotent groups and Lie rings under base extension*, Trans. Amer. Math. Soc. **375** (2022), 1051–1100.
3. M. N. Berman and B. Klopsch, *A nilpotent group without local functional equations for pro-isomorphic subgroups*, J. Group Theory **18** (2015), 489–510.
4. M. N. Berman, B. Klopsch, and U. Onn, *On pro-isomorphic zeta functions of D^*-groups of even Hirsch length*, arXiv/1511.06360v3.
5. ———, *On pro-isomorphic zeta functions of D^*-groups of even Hirsch length*, arXiv/1511.06360v2.
6. ———, *A family of class-2 nilpotent groups, their automorphisms and pro-isomorphic zeta functions*, Math. Z. **290** (2018), 909–935.
7. A. Carnevale, M. M. Schein, and C. Voll, *Generalized Igusa functions and ideal growth of nilpotent Lie rings*, arXiv:1903.03090.
8. H. Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics, vol. 138, Springer-Verlag, Berlin, 1993.
9. D. A. Cox, *Primes of the form $x^2 + ny^2$*, 2nd ed., Pure and Applied Mathematics, John Wiley & Sons, Hoboken, 2013.
10. M. P. F. du Sautoy and A. Lubotzky, *Functional equations and uniformity for local zeta functions of nilpotent groups*, Amer. J. Math. **118** (1996), 39–90.
11. F. J. Grunewald and D. Segal, *Reflections on the classification of torsion-free nilpotent groups*, Group Theory. Essays for Philip Hall, Academic Press, London, 1984, pp. 121–158.
12. F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), 185–223.
13. I. Kovacs, D. S. Silver, and S. G. Williams, *Determinants of commuting-block matrices*, Amer. Math. Monthly **106** (1999), 950–952.
14. S. Lee and C. Voll, *Zeta functions of integral nilpotent quiver representations*, arXiv:2006.12346, to appear in IMRN.
15. J. Maglione and C. Voll, *Flag Hilbert-Hoincaré series of hyperplane arrangements and their Igusa zeta functions*, arXiv:2103.03640.
16. L. Nachbin, *The Haar integral*, Van Nostrand, Princeton-Toronto-London, 1965.
17. T. Rossmann and C. Voll, *Groups, graphs, and hypergraphs: average sizes of kernels of generic matrices with support constraints*, arXiv:1908.09589. To appear in Memoirs Amer. Math. Soc.
18. M. M. Schein and C. Voll, *Normal zeta functions of the Heisenberg groups over number rings I – the unramified case*, J. London Math. Soc. (2) **91** (2015), 19–46.
19. D. Segal, *On the automorphism groups of certain Lie algebras*, Math. Proc. Camb. Phil. Soc. **106** (1989), 67–76.
20. C. Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), 1181–1218.

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