A FAMILY OF IRREDUCIBLE SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(F)$ FOR SOME RAMIFIED p-ADIC FIELDS

MICHAEL M. SCHEIN

ABSTRACT. We construct infinite families of irreducible supersingular mod p representations of $\mathrm{GL}_2(F)$ with $\mathrm{GL}_2(\mathcal{O}_F)$-socle compatible with Serre's modularity conjecture, where $F/\mathbb{Q}_p$ is any finite extension with residue field $\mathbb{F}_{p^2}$ and ramification degree $e \leq (p-1)/2$. These are the first such examples for ramified $F/\mathbb{Q}_p$.

1. INTRODUCTION

1.1. **Supersingular representations.** The irreducible smooth mod p representations of $\mathrm{GL}_2(F)$ admitting a central character, where $F/\mathbb{Q}_p$ is a finite extension, were classified by Barthel and Livné [2, Theorems 33-34], except for the *supersingular* representations. The irreducible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ were determined by Breuil [4, Théorème 1.1]. By contrast, little is known about the irreducible supersingular representations of $\mathrm{GL}_2(F)$ when $F \neq \mathbb{Q}_p$. Wu [21, Theorem 1.1] has shown that they do not have finite presentation; this was proved earlier by Schraen [20, Théorème 2.23] in the case $[F : \mathbb{Q}_p] = 2$. Thus a direct construction of these representations appears to be difficult. However, the method of diagrams introduced by Paškūnas [16] can be used to show the existence of supersingular representations with certain properties; indeed, such representations of $\mathrm{GL}_2(F)$, for arbitrary F, were produced in [16, Theorem 6.25].

For applications to the mod p local Langlands correspondence, one is most interested in supersingular representations of $\mathrm{GL}_2(F)$ whose $\mathrm{GL}_2(\mathcal{O}_F)$-socle is compatible with Serre's modularity conjecture; see condition (1) of [8]. We shall refer to such representations as having *good socle*. While the supersingular representations constructed in [16] only have good socle when $F = \mathbb{Q}_p$, Breuil and Paškūnas constructed infinite families of diagrams giving rise to supersingular irreducible representations of $\mathrm{GL}_2(F)$ with good socle for all unramified $F/\mathbb{Q}_p$. These families are parametrized by choices of a collection of isomorphisms between one-dimensional $\overline{\mathbb{F}}_p$-vector spaces. In particular, there are far more irreducible mod p representations of $\mathrm{GL}_2(F)$ than there are mod p representations of the absolute Galois group of F. While the construction of [8] is not exhaustive, the supersingular representations considered there play an important role in ongoing investigations towards the mod p Langlands correspondence; see [6, 7, 10] and references therein.

1.2. **Results.** In this note we construct families of diagrams giving rise to supersingular representations of $\mathrm{GL}_2(F)$ with good socle for arbitrary finite extensions $F/\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^2}$ and ramification degree $e \leq (p-1)/2$. To the author's knowledge this is the first construction of supersingular represenations of good socle for ramified p-adic fields. The representations associated to our diagrams are exactly those of [8] when $F/\mathbb{Q}_p$ is the quadratic unramified extension. However, they appear unlikely to contribute to the mod p local Langlands correspondence when $F/\mathbb{Q}_p$ is ramified, for reasons discussed below. Indeed, the results of the present paper show that the collection of irreducible supersingular representations of $\mathrm{GL}_2(F)$ with good socle has, in general, an even more complex structure than was discovered by Breuil and Paškūnas.

A curious and rather unexpected feature of our diagrams is that, besides depending on choices of isomorphisms analogous to those in [8], they depend on the choice of an element in a large finite set, namely the set of Hamiltonian walks on an $e \times e$ square lattice. The number of such Hamiltonian walks grows exponentially in e^2 and has been studied mostly by physicists; see [3, §9] and numerical computations, as well as predictions based on conformal field-theoretic models, in [14, §XD] and [13].

1.3. **Overview of the construction.** In order to motivate our construction, we briefly review some aspects of the work of Breuil and Paškūnas. Let $\mathcal{O}$ be the ring of integers of F. Fix a uniformizer $\pi \in \mathcal{O}$ and let $k = \mathcal{O}/(\pi)$ be the residue field. Denote $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O})$, let $Z \leq G$ be the center, let $I \leq K$ be the Iwahori subgroup of residually upper triangular matrices, and let $N = N_G(I)$ be its normalizer in G. Recall [16, Definition 5.14] that a diagram for F is a triple (D_0, D_1, ι), where D_0 is a smooth mod p representation of KZ, whereas D_1 is a smooth mod p representation of N and $\iota : D_1 \to D_0$ is an IZ-equivariant homomorphism. If D_0 is admissible, the element $\pi \mathrm{Id}_2 \in Z$ acts trivially, and ι is injective, then (under a further technical hypothesis in the case $p = 2$) the injective envelope Ω of $\mathrm{soc}_K D_0$ in the category of K-modules may be endowed with an action of G such that $D_0 \subset \Omega_{|KZ}$ and $D_1 \subset \Omega_{|N}$ [8, Theorem 9.8].

Any irreducible $\overline{\mathbb{F}}_p$-representation of K necessarily factors through the finite quotient $\mathrm{GL}_2(k)$ and is thus a Serre weight; see §2 below. Let G_F denote the absolute Galois group of F. If $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is an irreducible Galois representation, let $\mathcal{D}(\rho)$ be the explicit set of Serre weights associated to it in [17, §2]; see [11, §7] for a more modern perspective on the weight part of Serre's modularity conjecture. When $F/\mathbb{Q}_p$ is unramified, Breuil and Paškūnas define a family of diagrams for which D_0 is the maximal representation of $\mathrm{GL}_2(k)$ such that $\mathrm{soc}_K D_0 = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$ and that no element of $\mathcal{D}(\rho)$ appears as a subquotient of $D_0/\mathrm{soc}_K D_0$. It turns out that the KZ-module D_0 decomposes as $D_0 = \bigoplus_{\sigma \in \mathcal{D}(\rho)} D_{0,\sigma}(\rho)$, where $\mathrm{soc}_K D_{0,\sigma}(\rho) = \sigma$. The key step in the proof that an associated G-module Ω is irreducible is the claim that if $W \subseteq \Omega$ is a G-submodule such that $W \cap D_{0,\sigma}(\rho) \neq 0$ for some $\sigma \in \mathcal{D}(\rho)$, then $D_{0,\sigma}(\rho) \subset W$; this follows from a rather technical analysis of the structure of $D_{0,\sigma}(\rho)$ and of the map $\mathrm{ind}_{KZ}^G \sigma \to W$ arising from Frobenius reciprocity. The analogous claim fails, for an analogous definition of D_0 when $F/\mathbb{Q}_p$ is ramified, since $\mathrm{ind}_{KZ}^G \sigma$ does not contain enough non-trivial extensions of Serre weights factoring through $\mathrm{GL}_2(k)$.

We now briefly describe the present work; the details are found in the body of the paper. Let $\mathbb{Q}_p \subseteq F_0 \subseteq F$ be the maximal unramified subextension, and let $f = [F_0 : \mathbb{Q}_p]$. Then there is a set $\mathcal{S}$ of irreducible Galois representations $\rho' : G_{F_0} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ such that $|\mathcal{S}| = e^f$ and $\mathcal{D}(\rho) = \bigcup_{\rho' \in \mathcal{S}} \mathcal{D}(\rho')$; under our hypotheses on F, this union is disjoint. We define $\widetilde{D}_{0,\sigma} = D_{0,\sigma}(\rho')^{I(1)}$, where $I(1) \leq I$ is the pro-p-Sylow subgroup and $D_{0,\sigma}(\rho')$

is as in [8]. In the case $f = 2$, this is a KZ-module of length two, factoring through $\mathrm{GL}_2(k)$. Furthermore, we identify $\mathcal{S}$ with the set of vertices of an $e \times e$ square lattice and choose a Hamiltonian walk γ through this lattice. Definition 3.2 modifies some of the $\widetilde{D}_{0,\sigma}$, in a way depending on γ, to obtain KZ-modules $D^{\gamma}_{0,\sigma}$ of length two with socle σ. Set $D^{\gamma}_0(\rho) = \bigoplus_{\sigma \in \mathcal{D}(\rho)} D^{\gamma}_{0,\sigma}$ and $D^{\gamma}_1(\rho) = (D^{\gamma}_0(\rho))^{I(1)}$, with the natural inclusion $\iota : D^{\gamma}_1(\rho) \hookrightarrow D^{\gamma}_0(\rho)$. In contract to the situation in [8], Serre weights may appear as subquotients of $D^{\gamma}_0(\rho)$ with multiplicity greater than one. However, there is a unique way to extend the IZ-action on $D^{\gamma}_1(\rho)$ to an action of N, modulo the choices of isomorphisms mentioned above, if we require that no N-orbit be contained in $\mathrm{soc}_K D^{\gamma}_0(\rho)$. This N-action interweaves the KZ-modules $D^{\gamma}_{0,\sigma}$ for Serre weights σ occurring in $\mathcal{D}(\rho')$ for different Galois representations $\rho' \in \mathcal{S}$ and enables us to prove, in Theorem 3.7, that G-modules Ω arising from the diagrams $(D^{\gamma}_0(\rho), D^{\gamma}_1(\rho), \iota)$ are irreducible and supersingular. Since our $D^{\gamma}_{0,\sigma}$ have such a simple structure, in the course of the proof we can show directly that either $D^{\gamma}_{0,\sigma} \cap W = 0$ or $D^{\gamma}_{0,\sigma} \subset W$ for any G-submodule $W \subseteq \Omega$, resolving the difficulty discussed at the end of the previous paragraph.

1.4. **Relation to the mod p local Langlands correspondence.** As mentioned above, some of the supersingular representations of $\mathrm{GL}_2(F)$ constructed in [8] for unramified $F/\mathbb{Q}_p$ appear in local-global compatibility results for the mod p Langlands correspondence. For ramified $F/\mathbb{Q}_p$, by contrast, a consequence of the "breakage of symmetry" in $D^{\gamma}_0(\rho)$ imposed by our choice of a Hamiltonian walk γ is that the associated irreducible supersingular representations of $\mathrm{GL}_2(F)$ seem unlikely to appear in the mod p local Langlands correspondence. One indication of this is the following. Breuil [5] proposed an extension to certain diagrams for unramified $F/\mathbb{Q}_p$ of the Colmez functor from $\mathrm{GL}_2(\mathbb{Q}_p)$-modules to (φ, Γ)-modules, and hence Galois representations, realizing the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. If D is a diagram associated to the irreducible representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, for $F/\mathbb{Q}_p$ unramified, we let $M(D)$ denote the (φ, Γ)-module for $\mathbb{Q}_p$ defined in [5], and $V(M(D))$ the associated representation of $G_{\mathbb{Q}_p}$. Then Breuil [5, Corollaire 5.4] proved that $V(M(D))_{|I_{\mathbb{Q}_p}} \simeq (\mathrm{Ind}^{\otimes G_{\mathbb{Q}_p}}_{G_F} \rho^{\vee})_{|I_{\mathbb{Q}_p}}$, where $I_{\mathbb{Q}_p} \leq G_{\mathbb{Q}_p}$ is the inertia subgroup. Moreover, for certain choices of the isomorphisms defining D, one has that $V(M(D))$ is isomorphic to the tensor induction $\mathrm{Ind}^{\otimes G_{\mathbb{Q}_p}}_{G_F} \rho^{\vee}$ as representations of $G_{\mathbb{Q}_p}$.

As we observe in §3.3 below, one can naturally associate a (φ, Γ)-module $M(D)$ for $\mathbb{Q}_p$ to the diagrams D constructed in this paper. In Proposition 3.11 we apply the computations of [5] to give an explicit description of the associated Galois representation $V(M(D))$. It turns out that even the restriction of $V(M(D))$ to the inertia subgroup $I_F \leq G_F \leq G_{\mathbb{Q}_p}$ depends on the Hamiltonian walk γ chosen in the construction of D, and that in general there is no choice of γ for which we would obtain an isomorphism $V(M(D))_{|I_F} \simeq (\mathrm{Ind}^{\otimes G_{\mathbb{Q}_p}}_{G_{F_0}} (\mathrm{Ind}^{G_{F_0}}_{G_F} \rho^{\vee}))_{|I_F}$.

The present work suggests that to obtain a generalization of the construction of [8] that would be defined more canonically and would admit generalizations of the more recent local-global compatibility results, one should consider diagrams for which the KZ-module D_0 factors not through $\mathrm{GL}_2(k)$ but only through a larger finite quotient. There is work in progress in this direction.

2. Preliminaries

As in the introduction, let $F/\mathbb{Q}_p$ be a finite extension, and let F_0 be the maximal unramified subextension. Let $\mathcal{O}$ be the ring of integers of F, fix a uniformizer $\pi \in \mathcal{O}$, and let $k = \mathcal{O}/(\pi)$ denote the residue field. Let $q = p^f$ be the cardinality of k. A Serre weight is an irreducible $\overline{\mathbb{F}}_p$-representation of the finite group $\Gamma = \mathrm{GL}_2(k)$, which can be viewed as a representation of $K = \mathrm{GL}_2(\mathcal{O})$ or of $\mathrm{GL}_2(\mathcal{O}_{F_0})$ by inflation. Every irreducible smooth $\overline{\mathbb{F}}_p$-representation of these two profinite groups arises from a Serre weight in this way. Let $B \leq \Gamma$ be the subgroup of upper triangular matrices, and let $U \leq B$ be the subgroup of upper triangular matrices all of whose eigenvalues are 1. Let $K(1)$ be the kernel of the reduction map $K \twoheadrightarrow \Gamma$, and let I and $I(1)$ be the preimages of B and U, respectively.

Given a Serre weight σ, write $\chi(\sigma)$ for the character by which the diagonal torus $H \leq B$ acts on σ^U; we also view $\chi(\sigma)$ as a character of B by inflation. For $v \in \sigma$ and $g \in G = \mathrm{GL}_2(F)$, we denote by $g \otimes v$ the element of the compact induction $\mathrm{ind}_{KZ}^G \sigma$ supported on the right coset KZg^{-1} and sending g^{-1} to v. Set the notations $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and denote $\Pi = \alpha w = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Let Z be the center of G and Id_2 the identity matrix. The normalizer $N = N_G(I(1))$ is generated by IZ and Π. Write $\chi^w : H \to \overline{\mathbb{F}}_p^\times$ for the character $\chi^w(h) = \chi(whw)$; this is denoted χ^s in [8]. Let $\sigma^{[w]}$ be the unique Serre weight distinct from σ such that $\chi(\sigma^{[w]}) = \chi(\sigma)^w$. We view Serre weights as representations of KZ by letting $\pi\mathrm{Id}_2$ act trivially.

Fix an embedding $\varepsilon_0 : k \hookrightarrow \overline{\mathbb{F}}_p$ and define embeddings ε_i for every $i \in \mathbb{N}$ by means of the recursion $\varepsilon_i = \varepsilon_{i-1}^p$. Then every Serre weight has the form

$$\sigma = \bigotimes_{i=0}^{f-1} \left(\mathrm{Sym}^{r_i} k^2 \otimes_{k,\varepsilon_i} \overline{\mathbb{F}}_p\right) \otimes \eta,$$

where $0 \leq r_i \leq p-1$ for every i and $\eta : k^\times \to \overline{\mathbb{F}}_p^\times$ is a character. We say that a Serre weight σ is regular if $1 \leq r_i \leq p-2$ for every i. Observe that a regular Serre weight σ as above is determined by the f-tuple $(r_0, \dots, r_{f-1})$ and its central character $\xi : Z(\Gamma) \simeq k^\times \to \overline{\mathbb{F}}_p^\times$. If ξ has been fixed, then we will write $\sigma = (r_0, \dots, r_{f-1})$ for short; then $\sigma^{[w]} = (p-1-r_0, \dots, p-1-r_{f-1})$ An irreducible $\overline{\mathbb{F}}_p$-representation V of G is called *supersingular* if $\lambda = 0$ for every surjective map (equivalently, for one such map) of the form $\mathrm{ind}_{KZ}^G \sigma/(T-\lambda)\,\mathrm{ind}_{KZ}^G \sigma \to V$, where $\lambda \in \overline{\mathbb{F}}_p$ and $T \in \mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is the operator of [2, Proposition 8].

Lemma 2.1. *Let σ be a Serre weight and let $0 \neq v \in \sigma^{I(1)}$. Then $\langle \alpha \otimes wv \rangle_K \subset (\mathrm{ind}_{KZ}^G \sigma)^{K(1)}$. Moreover, $\langle \alpha \otimes wv \rangle_K \simeq \mathrm{Ind}_I^K \chi^w$ and $\mathrm{soc}_K(\langle \alpha \otimes wv \rangle_K) = T(\langle \mathrm{id} \otimes v \rangle_K)$.*

Proof. This follows from simple computations using [8, Lemma 2.7] and an explicit description of the operator T; see, for instance, [19, Lemma 2.1]. □

Let G_F be the absolute Galois group of F and $I_F \leq G_F$ the inertia subgroup. Let k' be the quadratic extension of k, let F^{nr} be the maximal unramified extension of F, and let $L'/L/F^{\mathrm{nr}}$ be a tower of totally ramified extensions such that $\mathrm{Gal}(L/F^{\mathrm{nr}}) \simeq k^\times$ and $\mathrm{Gal}(L'/F^{\mathrm{nr}}) \simeq (k')^\times$. Consider the natural projections $\nu : I_F \to \mathrm{Gal}(L/F^{\mathrm{nr}})$ and

$\nu' : I_F \to \mathrm{Gal}(L'/F^{\mathrm{nr}})$. Let $\omega_{2f} = \varepsilon_0' \circ \nu' : I_F \to \overline{\mathbb{F}}_p^\times$ be a fundamental character of level $2f$ corresponding to an embedding $\varepsilon_0' : k' \hookrightarrow \overline{\mathbb{F}}_p$ that restricts to ε_0 on k. We also denote the analogous character of I_{F_0} by ω_{2f}. Let $\rho' : G_{F_0} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible Galois representation such that $\rho'_{|I_{F_0}}$ is isomorphic to a twist of

$$\omega_{2f}^{\sum_{i=0}^{f-1} p^i(r_i+1)} \oplus \omega_{2f}^{q\sum_{i=0}^{f-1} p^i(r_i+1)} \tag{1}$$

and $\det \rho' = \xi \circ \nu$. Let $\mathcal{D}(r_0, \ldots, r_{f-1})$ denote the set of modular Serre weights associated to ρ' in [9, §3.1]; see [8, §11] for an explicit description. If $1 \le r_0 \le p-2$ and $0 \le r_i \le p-3$ for $i > 0$, then $\mathcal{D}(r_0, \ldots, r_{f-1})$ consists of 2^f distinct Serre weights.

Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be irreducible and let $\mathcal{D}(\rho)$ be the set of Serre weights associated to it in [17, §2]. It was observed in [18, Proposition 3.1] that $\mathcal{D}(\rho)$ is a union of sets of Serre weights associated to irreducible representations of G_{F_0}.

Assume from now on that ρ is generic, namely that $\rho_{|I_F}$ is isomorphic to a twist of (1) with $2e-1 \le r_0 \le p-2$ and $2e-2 \le r_i \le p-3$ for $i > 0$. Suppose that $\det \rho = \xi \circ \nu$. Consider the set $\Delta = \{(\delta_0, \ldots, \delta_{f-1}) : \forall\, i,\, 0 \le \delta_i \le e-1\}$. Then

$$\mathcal{D}(\rho) = \coprod_{\underline{\delta} \in \Delta} \mathcal{D}(r_0 - 2\delta_0, \ldots, r_{f-1} - 2\delta_{f-1}).$$

We associate to each element σ of this disjoint union a pair $(\underline{\delta}_\sigma, J_\sigma)$, where $\underline{\delta}_\sigma = (\delta_0, \ldots, \delta_{f-1})$ is determined by $\sigma \in \mathcal{D}(r_0 - 2\delta_0, \ldots, r_{f-1} - 2\delta_{f-1})$, and $J_\sigma \subseteq \{0, \ldots f-1\}$ is the set associated to σ in [8, §11] as an element of $\mathcal{D}(r_0 - 2\delta_0, \ldots, r_{f-1} - 2\delta_{f-1})$.

For the rest of this note, assume $f = 2$. Recall that the socle filtration $\{\mathrm{soc}_i(M)\}$ of a C-module M, for any group C, is defined recursively by $\mathrm{soc}_0(M) = \mathrm{soc}_C(M)$ and $\mathrm{soc}_i(M) = \mathrm{soc}_C(M/\mathrm{soc}_{i-1}(M))$.

Lemma 2.2. *Let $\sigma = (r_0, r_1)$ be a regular Serre weight with central character ξ. The socle filtration of $\mathrm{Ind}_B^\Gamma \chi(\sigma)^w$ is as follows, reading from left to right:*

$$(r_0, r_1) \underline{\quad\quad} (p-2-r_0, r_1-1) \oplus (r_0-1, p-2-r_1) \underline{\quad\quad} (p-1-r_0, p-1-r_1).$$

Proof. This is a special case of [8, Theorem 2.4], which itself was originally established by Bardoe and Sin [1, Theorem C]. □

Given a Serre weight $\sigma = (a_0, a_1)$ as above, define $Q_{\{0\}}(\sigma)$ and $Q_{\{1\}}(\sigma)$ to be the quotients of $\mathrm{Ind}_B^\Gamma \chi(p-2-a_0, a_1+1)$ and $\mathrm{Ind}_B^\Gamma \chi(a_0+1, p-2-a_1)$, respectively, having socle σ. It is clear from Lemma 2.2 that these are Γ-modules of length two, and it is easy to see that $\dim_{\overline{\mathbb{F}}_p} Q_{\{0\}}(\sigma)^U = \dim_{\overline{\mathbb{F}}_p} Q_{\{1\}}(\sigma)^U = 2$. We view $Q_{\{0\}}(\sigma)$ and $Q_{\{1\}}(\sigma)$ as K-modules by inflation.

Let $(\delta_0, \delta_1) \in \Delta$, and let $\widetilde{D}_0(\delta_0, \delta_1)$ be the K-module generated by the $I(1)$-invariants of the K-module $D_0(\rho')$ associated to $\mathcal{D}(r_0 - 2\delta_0, r_1 - 2\delta_1)$ in [8, §13]. This is a direct sum of the following four K-modules, each factoring through Γ:

$$
\begin{array}{rcl}
(r_0-2\delta_0, r_1-2\delta_1) & \underline{\hspace{1.5em}} & (r_0-2\delta_0+1, p-r_1+2\delta_1-2) \\
& \oplus & \\
(r_0-2\delta_0-1, p-r_1+2\delta_1-2) & \underline{\hspace{1.5em}} & (p-r_0+2\delta_1-1, p-r_1+2\delta_1-1) \\
& \oplus & \\
(p-r_0+2\delta_0-2, r_1-2\delta_1+1) & \underline{\hspace{1.5em}} & (r_0-2\delta_0, r_1-2\delta_1+2) \\
& \oplus & \\
(p-r_0+2\delta_0-1, p-r_1+2\delta_1-3) & \underline{\hspace{1.5em}} & (p-r_0+2\delta_0, r_1-2\delta_1+1).
\end{array} \tag{2}
$$

Observe that $\widetilde{D}_0(\delta_0,\delta_1) = \bigoplus_{\sigma\in\mathcal{D}(r_0-2\delta_0,r_1-2\delta_1)} \widetilde{D}_{0,\sigma}$, where $\widetilde{D}_{0,\sigma} = Q_{\{0\}}(\sigma)$ if $J_\sigma = \{0\}$ or $J_\sigma = \{1\}$, and $\widetilde{D}_{0,\sigma} = Q_{\{1\}}(\sigma)$ otherwise.

3. Families of diagrams

3.1. **Construction.** We follow the spirit, although not the actual technique, of the constructions of non-admissible irreducible $\overline{\mathbb{F}}_p$-representations of G in [15, 12] to obtain an irreducible diagram from $\widetilde{D}_0(\rho) = \bigoplus_{(\delta_0,\delta_1)\in\Delta} \widetilde{D}_0(\delta_0,\delta_1)$ by defining an action of Π on $\widetilde{D}_0(\rho)^{I(1)}$ that interweaves the direct summands $\widetilde{D}_0(\delta_0,\delta_1)$. We first replace $\widetilde{D}_0(\rho)$ with a new K-module $D_0(\rho)$ by modifying some of the components $\widetilde{D}_{0,\sigma}$.

If $\sigma \in \mathcal{D}(\rho)$, let $\kappa(\sigma)$ denote the K-cosocle of $\widetilde{D}_{0,\sigma}$. Note that $\kappa(\sigma)$ is a Serre weight. For generic ρ the following is obvious by inspection.

Lemma 3.1. *Let $\sigma, \tau \in \mathcal{D}(\rho)$. Then $\kappa(\sigma) \simeq \kappa(\tau)^{[w]}$ if and only if one of the following holds:*

1. *The pairs associated to σ and τ are $((\delta_0,\delta_1),\{0\})$ and $((\delta_0,\delta_1+1),\{1\})$ for some $(\delta_0,\delta_1)\in\Delta$.*
2. *The pairs associated to σ and τ are $((\delta_0,\delta_1),\{0,1\})$ and $((\delta_0+1,\delta_1),\varnothing)$ for some $(\delta_0,\delta_1)\in\Delta$.*

Consider the graph with vertex set Δ, where two vertices (δ_0,δ_1) and (δ_0',δ_1') are adjacent if $(\delta_0',\delta_1') \in \{(\delta_0\pm1,\delta_1),(\delta_0,\delta_1\pm1)\}$; this is an $e\times e$ square lattice. Fix a Hamiltonian walk γ in this graph, namely an undirected path that traverses each vertex exactly once, with no restriction on the starting and ending vertices. It is clear that such paths exist. We say that two adjacent elements of Δ are γ-adjacent if γ contains the edge connecting them.

Definition 3.2. Let $\sigma \in \mathcal{D}(\rho)$ be associated to the pair $((\delta_0,\delta_1),J)$, and let γ be a Hamiltonian walk. Define a Γ-module $D_{0,\sigma}^\gamma$ as follows:

$$
D_{0,\sigma}^\gamma = \begin{cases}
Q_{\{1\}}(\sigma) & : (\delta_0,\delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0,\delta_1+1) \text{ and } J=\{0\} \\
Q_{\{1\}}(\sigma) & : (\delta_0,\delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0,\delta_1-1) \text{ and } J=\{1\} \\
Q_{\{0\}}(\sigma) & : (\delta_0,\delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0+1,\delta_1) \text{ and } J=\{0,1\} \\
Q_{\{0\}}(\sigma) & : (\delta_0,\delta_1) \text{ is } \gamma\text{-adjacent to } (\delta_0-1,\delta_1) \text{ and } J=\varnothing \\
\widetilde{D}_{0,\sigma} & : \text{ otherwise.}
\end{cases}
$$

Set $D_0^\gamma(\rho) = \bigoplus_{\sigma\in\mathcal{D}(\rho)} D_{0,\sigma}^\gamma$. We usually write $D_0(\rho)$ for $D_0^\gamma(\rho)$ to lighten the notation.

Informally, if we view $\widetilde{D}_0(\rho)$ schematically as in (2), then to obtain $D_0(\rho)$ we switch one Serre weight appearing in the cosocle of $\widetilde{D}_0(\delta_0,\delta_1)$ with a Serre weight appearing in the cosocle of $\widetilde{D}_0(\delta_0',\delta_1')$ whenever (δ_0,δ_1) and (δ_0',δ_1') are γ-adjacent. If $e=1$, then Δ has only one vertex and the $D_0(\rho)$ constructed here is the Γ-submodule of the Γ-module $D_0(\rho)$ constructed in [8] generated by its U-invariants.

Remark 3.3. Observe that the Γ-module $D_0(\rho)$ defined above is a direct sum of $(2e)^f$ non-split extensions of two Serre weights. Moreover, since $D_0(\rho)$ is a direct sum of Γ-modules of the form (2), up to permutation of Serre weights appearing in the cosocle, it is clear that a Serre weight σ appears in the socle of $D_0(\rho)$ if and only if $\sigma^{[w]}$ appears in the cosocle. If $e>1$, then there exist Serre weights σ such that $\{\sigma,\sigma^{[w]}\}\subset\mathcal{D}(\rho)$; see Example 3.9 below. Thus, unlike the situation of [8, §14], the irreducible subquotients of $D_0(\rho)$ do not all arise with multiplicity one.

We have the following analogue of [8, Corollary 13.6].

Lemma 3.4. *There is a unique partition of the B-eigencharacters of $D_0(\rho)^U$ into pairs $\{\chi,\chi^w\}$, with $\chi\neq\chi^w$, such that one character of each pair arises from the socle of $D_0(\rho)$ and the other from the cosocle.*

Proof. The claim is true by inspection for the B-eigencharacters of $\widetilde{D}_0(\delta_0,\delta_1)^U$ for every $(\delta_0,\delta_1)\in\Delta$. Since the sets $\mathcal{D}(\delta_0,\delta_1)$ are disjoint, the claim remains true for the B-eigencharacters of $\widetilde{D}_0(\rho)^U$. Now the sets of B-eigencharacters arising from the socle and cosocle of $D_0(\rho)$ are the same as for $\widetilde{D}_0(\rho)$, completing the proof.

As noted above, if $e>1$, then $\mathcal{D}(\rho)$ contains pairs $\{\sigma,\sigma^{[w]}\}$ of Serre weights. Thus the uniqueness in the claim fails without the requirement that one character of each pair $\{\chi,\chi^w\}$ come from the socle and one from the cosocle. □

View $D_0^\gamma(\rho)$ as a KZ-module, with $\pi\mathrm{Id}_2$ acting trivially. The partition of Lemma 3.4 gives rise to a family of basic 0-diagrams $(D_0^\gamma(\rho),\{\,\})$ in the sense of [8, Definition 13.7], for each choice of Hamiltonian walk γ. In this way we obtain families of diagrams for any extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^2}$ and ramification index at most $(p-1)/2$. For a fixed γ, there is a natural bijection between diagrams for different such extensions arising from $(D_0^\gamma(\rho),\{\,\})$.

3.2. **Irreducible supersingular representations.** It remains to show that the families of diagrams that we have just constructed give rise to irreducible supersingular representations of G. First we show that the families are indecomposable.

Proposition 3.5. *The family $(D_0(\rho),\{\,\})$ cannot be written as a direct sum of two non-zero families of diagrams.*

Proof. For every $\sigma\in\mathcal{D}(\rho)$, let $\beta(\sigma)\in\mathcal{D}(\rho)$ be the Serre weight such that $\chi(\sigma)^w$ arises in the cosocle of $D_{0,\beta(\sigma)}^\gamma$. Note that $\beta(\sigma)$ is well-defined by Lemma 3.4 and that β is a bijection of $\mathcal{D}(\rho)$ onto itself. It suffices to show that the action of $\mathbb{Z}\simeq\langle\beta\rangle$ on the set $\mathcal{D}(\rho)$ is transitive. Choose a direction of the path γ, which amounts to fixing a numbering $(\gamma_1,\dots,\gamma_{e^2})$ of the elements of Δ such that γ_i and γ_{i+1} are γ-adjacent for every $0 \leq i \leq e^2 - 1$. For every $\widetilde{D}_0(\delta_0, \delta_1)$, the analogously defined $\langle \beta \rangle$-action is transitive; this follows by inspection of (2) or by [8, Theorem 15.4], noting that the pairing $\{\ \}$ there matches characters arising from the socle with characters arising from the cosocle. Since γ_1 is γ-adjacent to only one other element of Δ, there is a unique Serre weight $\tau \in \mathcal{D}(\gamma_1)$ such that $D^{\gamma}_{0,\tau} \not\simeq \widetilde{D}_{0,\tau}$. The previous observation implies that $\{\tau, \beta(\tau), \beta^2(\tau), \beta^3(\tau)\} = \mathcal{D}(\gamma_1)$.

We proceed by induction. Suppose it is known that all elements of $\mathcal{D}(\gamma_{k-1})$ lie in the same orbit as τ. The same holds for $\mathcal{D}(\gamma_k)$ by an easy but tedious analysis of cases. For instance, if $\gamma_k = \gamma_{k-1} + (0,1)$ and $\gamma_{k+1} = \gamma_k + (1,0)$, then $\beta(\gamma_{k-1}, \varnothing) = (\gamma_k, \{1\})$ and $\beta(\gamma_k, \{1\}) = (\gamma_k, \varnothing)$ and $\beta(\gamma_k, \varnothing) = (\gamma_k, \{0\})$, whereas $\beta^{-1}(\gamma_{k-1}, \{0\}) = (\gamma_k, \{0,1\})$; the other cases, including the case where γ_k is the terminal vertex of γ, are treated similarly. $\square$

Remark 3.6. The choice of Hamiltonian walk γ in the definition of $D_0(\rho)$ really is necessary. It would be more canonical to start with $\widetilde{D}_0(\rho)$ and switch Serre weights in the cosocle for every pair of adjacent elements of Δ. However, the family of diagrams obtained in this way is easily seen to be decomposable in general.

Observe that if V is any smooth representation of G, then the triple $(V^{K(1)}, V^{I(1)}, \mathrm{can})$, where $\mathrm{can} : V^{I(1)} \hookrightarrow V^{K(1)}$ is the natural inclusion, is a diagram.

Theorem 3.7. *Let $(D_0(\rho), D_1(\rho), r)$ be a basic 0-diagram arising from the family constructed above. Let V be a smooth admissible representation of $G = \mathrm{GL}_2(F)$ satisfying the following conditions:*

1. $\mathrm{soc}_K(V) = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$.
2. $(D_0(\rho), D_1(\rho), r) \hookrightarrow (V^{K(1)}, V^{I(1)}, \mathrm{can})$.
3. *V is generated by $D_0(\rho)$.*

Then V is irreducible and supersingular.

Proof. Note first that smooth admissible representations of G satisfying our hypotheses exist by the proof of [8, Theorem 19.8]. Let $W \subseteq V$ be a non-zero G-submodule, let σ be a Serre weight contained in $\mathrm{soc}_K(W)$, and let $0 \neq v \in \sigma^{I(1)}$. By Frobenius reciprocity, the inclusion $\varphi : \sigma \hookrightarrow W_{|K}$ corresponds to a non-zero map $\psi : \mathrm{ind}_{KZ}^G \sigma \to W$ of G-modules with $\psi(\mathrm{id} \otimes v) = \varphi(v)$. Hence $\psi(\alpha \otimes wv) = \Pi(\varphi(v))$ generates $D^{\gamma}_{0,\beta(\sigma)}$ and in particular $\beta(\sigma) \subseteq \mathrm{soc}_K(W)$. Proposition 3.5 ensures that by iterating this procedure we obtain $D^{\gamma}_{0,\sigma} \subset W$ for all $\sigma \in \mathcal{D}(\rho)$. Since $D_0(\rho)$ generates V, this implies $W = V$. Thus V is irreducible.

Moreover, the restriction to $\langle \alpha \otimes wv \rangle_K$ of the map ψ above has image $\langle \Pi(\varphi(v)) \rangle_K = D^{\gamma}_{0,\beta(\sigma)}$, which is a K-module of length two by construction. We have $\langle \alpha \otimes wv \rangle_K \simeq \mathrm{Ind}_I^K \chi^w$ by Lemma 2.1, which is a K-module of length $2^f = 4$ by Lemma 2.2. Hence $T(\mathrm{id} \otimes v) \in \mathrm{soc}_K(\langle \alpha \otimes wv \rangle_K) \subset \ker \psi$, and ψ factors through $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$. Moreover, ψ is surjective since V is irreducible. Hence V is supersingular. $\square$

It is easy to see that different choices of Hamiltonian walks in the construction of $D_0(\rho)$ give disjoint families of irreducible supersingular representations, as we now prove.

Proposition 3.8. *Let γ and γ' be distinct Hamiltonian walks. Let V and V' be smooth admissible representations of $G = \mathrm{GL}_2(F)$ satisfying the hypotheses of Theorem 3.7 for diagrams arising from the families $(D_0^{\gamma}(\rho), \{\,\})$ and $(D_0^{\gamma'}(\rho), \{\,\})$, respectively. Then V and V' are not isomorphic.*

Proof. There exist $\underline{\delta}, \underline{\delta}' \in \Delta$ which are γ-adjacent but not γ'-adjacent. Let β and β' be the permutations of $\mathcal{D}(\rho)$ defined in the proof of Proposition 3.5 for the families $(D_0^{\gamma}(\rho), \{\,\})$ and $(D_0^{\gamma'}(\rho), \{\,\})$, respectively. There is a unique Serre weight $\sigma \in \mathcal{D}(\underline{\delta})$ such that $\beta(\sigma) \in \mathcal{D}(\underline{\delta}')$. Let $0 \neq v \in \sigma^{I(1)} \subset (\mathrm{soc}_K V)^{I(1)}$. As in the proof of Theorem 3.7, we have $\langle \Pi v \rangle_K \simeq D^{\gamma}_{0,\beta(\sigma)}$. If there is a G-isomorphism $f : V \xrightarrow{\sim} V'$, then $f(v) \in \sigma^{I(1)} \subset (\mathrm{soc}_K V')^{I(1)}$ and thus $D^{\gamma}_{0,\beta(\sigma)} \simeq \langle \Pi f(v) \rangle_K \simeq D^{\gamma'}_{0,\beta'(\sigma)}$. Taking K-socles, we find that $\beta(\sigma) = \beta'(\sigma)$, but this is impossible since $\beta'(\sigma) \notin \mathcal{D}(\underline{\delta}')$ and the sets $\mathcal{D}(\underline{\delta})$ are disjoint, under our global hypotheses, for distinct $\underline{\delta} \in \Delta$. $\square$

Example 3.9. We write out $D_0^{\gamma}(\rho)$ in the case $e = 2, f = 2$, where γ is the Hamiltonian walk in Δ consisting of solid edges in the diagram below.

$$\begin{tikzcd}
(0,1) \arrow[r, dash] \arrow[d, dash, dotted] & (1,1) \arrow[d, dash] \\
(0,0) \arrow[r, dash] & (1,0)
\end{tikzcd}$$

The superscript n of each Serre weight $\sigma \in \mathcal{D}(\rho)$ is such that $\sigma = \beta^n((0,0), \varnothing)$, illustrating Proposition 3.5. The superscripts on the cosocles indicate the matching of Lemma 3.4.

$((0,0), \varnothing)$	$(r_0, r_1)^0$	—	$(r_0+1, p-2-r_1)^{15}$
$((0,0), \{1\})$	$(r_0-1, p-2-r_1)^1$	—	$(p-1-r_0, p-1-r_1)^0$
$((0,0), \{0\})$	$(p-2-r_0, r_1+1)^{15}$	—	$(r_0, r_1+2)^{14}$
$((0,0)\{0,1\})$	$(p-1-r_0, p-3-r_1)^{14}$	—	$(r_0-1, p-2-r_1)^{13}$
$((0,1), \varnothing)$	$(r_0, r_1-2)^6$	—	$(r_0+1, p-r_1)^5$
$((0,1), \{1\})$	$(r_0-1, p-r_1)^7$	—	$(p-1-r_0, p+1-r_1)^6$
$((0,1), \{0\})$	$(p-2-r_0, r_1-1)^5$	—	$(r_0, r_1)^4$
$((0,1), \{0,1\})$	$(p-1-r_0, p-1-r_1)^4$	—	$(r_0-1, p-r_1)^3$
$((1,0), \varnothing)$	$(r_0-2, r_1)^2$	—	$(p-r_0, r_1+1)^1$
$((1,0), \{1\})$	$(r_0-3, p-2-r_1)^{11}$	—	$(r_0-2, r_1)^{10}$
$((1,0), \{0\})$	$(p-r_0, r_1+1)^{13}$	—	$(r_0-2, r_1+2)^{12}$
$((1,0), \{0,1\})$	$(p+1-r_0, p-3-r_1)^{12}$	—	$(p+2-r_0, r_1+1)^{11}$
$((1,1), \varnothing)$	$(r_0-2, r_1-2)^8$	—	$(p-r_0, r_1-1)^7$
$((1,1), \{1\})$	$(r_0-3, p-r_1)^9$	—	$(p+1-r_0, p+1-r_1)^8$
$((1,1), \{0\})$	$(p-r_0, r_1-1)^3$	—	$(p+1-r_0, p-1-r_1)^2$
$((1,1), \{0,1\})$	$(p+1-r_0, p-1-r_1)^{10}$	—	$(p+2-r_0, r_1-1)^9$

3.3. **Associated (φ, Γ)-modules.** Breuil [5] associated étale (φ, Γ)-modules for $\mathbb{Q}_p$ to the diagrams of [8] by adapting the Colmez functor realizing the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In this section we observe that his construction applies also to the diagrams arising from the families $(D_0^\gamma(\rho), \{\,\})$ constructed above, for a generic Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$.

Since we assume throughout that ρ is generic, every $\sigma \in \mathcal{D}(\rho)$ is determined by the character $\chi(\sigma)$. Hence if $0 \neq v \in (\mathrm{soc}_K D_0^\gamma(\rho))^{I(1)}$ is an eigenvector for the action of I, then the K-submodule generated by v is irreducible. For every $\sigma = (a_0, a_1) \in \mathcal{D}(\rho)$ set

$$s(\sigma) = \begin{cases} a_0 + 1 & : D^\gamma_{0,\beta(\sigma)} = Q_{\{1\}}(\beta(\sigma)) \\ p(a_1+1) & : D^\gamma_{0,\beta(\sigma)} = Q_{\{0\}}(\beta(\sigma)) \end{cases}$$

$$|s(\sigma)| = \begin{cases} a_0 + 1 & : D^\gamma_{0,\beta(\sigma)} = Q_{\{1\}}(\beta(\sigma)) \\ a_1+1 & : D^\gamma_{0,\beta(\sigma)} = Q_{\{0\}}(\beta(\sigma)), \end{cases}$$

where $\beta : \mathcal{D}(\rho) \to \mathcal{D}(\rho)$ is the bijection from the proof of Proposition 3.5.

Lemma 3.10. *Let $D = (D_0, D_1, \iota)$ be a diagram arising from the family $(D_0^\gamma(\rho), \{\,\})$, and suppose that $0 \neq v \in (\mathrm{soc}_K D_0^\gamma(\rho))^{I(1)}$ is an I-eigenvector. Let $\sigma_v \in \mathcal{D}(\rho)$ be the Serre weight such that $v \in \sigma \subseteq D_0^\gamma(\rho)$. Then $s = s(\sigma_v)$ is the unique integer $0 \leq s \leq p^2 - 1$ such that*

$$S_s(v) = \sum_{\lambda \in k} \lambda^s \begin{pmatrix} \pi & [\lambda] \\ 0 & 1 \end{pmatrix} v = \sum_{\lambda \in k} \lambda^s \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} \Pi v$$

is a non-zero element of $(\mathrm{soc}_K D_0^\gamma(\rho))^{I(1)}$.

Proof. The claim follows from the structure of $D_0^\gamma(\rho)$ and [8, Lemma 2.7]. □

It is immediate from Lemma 3.10 that the diagram D is strongly principal in the sense of [5, Définition 4.3]. Thus the construction of [5, Lemme 4.5], whose details we do not recall here, provides a (φ, Γ)-module $M(D)$ over $\mathbb{Q}_p$. More precisely, let D' be the diagram for $F_0 = \mathbb{Q}_{p^2}$ corresponding to D (see the remark at the end of Section 3.1) and define $M(D)$ to be the (φ, Γ) module associated to D' in [5].

The representation $V(M(D))$ of $G_{\mathbb{Q}_p}$ corresponding to $M(D)$ has dimension $|\mathcal{D}(\rho)| = (2e)^2$ and is described by [5, Proposition 4.7]. For every $d \in \mathbb{N}$, let $\mathbb{Q}_{p^d}/\mathbb{Q}_p$ be the unramified extension of degree d, and let $\nu_d : G_{\mathbb{Q}_{p^d}} \to \overline{\mathbb{F}}_p^\times$ be a fundamental character of level d given by $\nu_d(g) = \frac{g(\sqrt[p^d-1]{-p})}{\sqrt[p^d-1]{-p}} \in \mathbb{F}_{p^d}^\times \hookrightarrow \overline{\mathbb{F}}_p^\times$. The applications below are independent of the choice of embedding $\mathbb{F}_{p^d} \hookrightarrow \overline{\mathbb{F}}_p$.

Proposition 3.11. *Let D be a diagram arising from the family $(D_0^\gamma(\rho), \{\,\})$. Let $\sigma = (a_0, a_1) \in \mathcal{D}(\rho)$. Then*

$$V(M(D)) \simeq \mathrm{Ind}_{G_{\mathbb{Q}_{p^{(2e)^2}}}}^{G_{\mathbb{Q}_p}} (\nu_{(2e)^2}^A \otimes_{\overline{\mathbb{F}}_p} \kappa) \otimes \nu_1^{-(c+2)},$$

where $A = \frac{1}{p-1} \sum_{i=0}^{(2e)^2-1} p^{(2e)^2-1-i} |s(\beta^i(\sigma))|$ and κ is an unramified character of $G_{\mathbb{Q}_{p^d}}$, while $c \in \mathbb{Z}/(p^2-1)\mathbb{Z}$ is such that $(\chi(\sigma)) \begin{pmatrix} [\lambda] & 0 \\ 0 & 1 \end{pmatrix} = \lambda^c$ for every $\lambda \in k^\times$.

Proof. Immediate from [5, Proposition 4.7]. Note that κ may be made explicit and that c depends on the data (a_0, a_1) and the central character ξ, which we have suppressed. □

One might hope to have $V(M(D))_{|I_F} \simeq \left(\mathrm{Ind}_{G_{F_0}}^{\otimes G_{\mathbb{Q}_p}}(\mathrm{Ind}_{G_F}^{G_{F_0}}\rho^\vee)\right)_{|I_F}$; indeed, in the case where $F = F_0$ is an unramified extension of $\mathbb{Q}_p$, this was proved by Breuil [5, Corollaire 5.4]. However, applying Proposition 3.11 to the diagrams arising from Example 3.9 we can verify that this does not hold in general.

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Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel
E-mail address: mschein@math.biu.ac.il