WEIGHTS IN GENERALIZATIONS OF SERRE'S CONJECTURE AND THE MOD p LOCAL LANGLANDS CORRESPONDENCE

MICHAEL M. SCHEIN

Abstract. In this mostly expository article we give a survey of some of the generalizations of Serre's conjecture and results towards them that have been obtained in recent years. We also discuss recent progress towards a mod p local Langlands correspondence for p-adic fields and its connections with Serre's conjecture. A theorem describing the structure of some mod p Hecke algebras for GL_n is proved.

1. Introduction

1.1. **The classical Serre conjecture.** Algebraic number theory is, in some sense, the study of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular, we are interested in the continuous representations of this group and its finite index subgroups. One source of such representations is the cohomology of algebraic varieties. For instance, if X is a variety defined over a number field F and $\mathcal{F}$ is an étale sheaf on X, then the étale cohomology $H^*_{ét}(X \otimes \overline{\mathbb{Q}}, \mathcal{F})$ carries an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Serre's conjectures and its generalizations tell us which representations arise in this way.

In this section, we will briefly review the original conjecture of Serre. The reader is directed to the excellent expository article [RS] for details. For each prime l, fix a decomposition subgroup $G_l \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at l, let $I_l \subset G_l$ be the corresponding inertia subgroup, and let $P_l \subset I_l$ be the pro-l-Sylow wild inertia subgroup. Let $N \geq 1$ and recall that $\Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$ is the subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a - 1 \equiv c \equiv d - 1 \equiv 0 \mod N \right\}.$$

A modular form of weight k and level N is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, where $\mathcal{H}$ is the complex upper half plane, satisfying the following automorphy condition, as well as some growth conditions at infinity:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \ \forall z \in \mathcal{H}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N).$$

Note that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1(N)$, whence $f(z+1) = f(z)$ for all $z \in \mathcal{H}$, so that f has a Fourier expansion. If f is, moreover, cuspidal and an eigenform for the Hecke operators,

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then the expansion has only positive terms: $f(z) = \sum_{n \geq 1} a_n q^n$, where $q = e^{2\pi i z}$. The classical construction of Eichler and Shimura associates to such an f a two-dimensional Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ which is unramified at all primes $l \nmid Np$. In other words, $I_l \subset \ker \rho_f$; note that I_l is defined up to conjugation, but this condition is well-defined. It follows that if Frob_l is an arithmetic Frobenius element for l, the characteristic polynomial of $\rho_f(\mathrm{Frob}_l)$ is well-defined for $l \nmid Np$. It is $x^2 - a_l x + l^{k-1}$, and by the Chebotarev density theorem the facts stated here determine ρ_f up to isomorphism. The Eichler-Shimura construction essentially comes down to finding ρ_f inside the cohomology of a suitable modular curve. We say that a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is *modular* if $\rho \simeq \rho_f$ for some modular form f.

Suppose we are given $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, and let $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the element induced by complex conjugation. Since c^2 is the identity, we must have $\det \rho(c) = \pm 1$. We say that ρ is *odd* if $\det \rho(c) = -1$ and *even* otherwise. In the early 1970's, J.-P. Serre conjectured that

Conjecture 1.1. Suppose that $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, irreducible, and odd. Then ρ is modular.

Moreover, Serre gave a combinatorial recipe for the weights of the modular forms giving rise to such ρ. The quantitative statement that a continuous, irreducible, odd two-dimensional Galois representation is modular *of the specified weights* is called the strong Serre conjecture. Conjecture 1.1 was recently proved by Khare and Wintenberger [Kha], [KW1], [KW2], relying on work of Kisin [Kis2], [Kis1]. However, the implication that the strong Serre conjecture follows from Conjecture 1.1 (i.e. that if ρ is modular, then it is modular of precisely the predicted weights) was known much earlier, except for a few cases with $p = 2$. It was established by work of Mazur, Ribet, Deligne, Fontaine, Carayol, Gross, Coleman, Voloch, and Edixhoven, among many others. An example of a statement towards this result is the following theorem of Fontaine. It was proved in a letter to Serre in 1979, and a somewhat different proof eventually appeared in print in [Edi]. Recall that $I_p \simeq \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{nr})$, where $\mathbb{Q}_p^{nr}$ is the maximal unramified extension of $\mathbb{Q}_p$. A character $\psi : I_p \to \overline{\mathbb{F}}_p^*$ is said to be of level two if it factors through the quotient $\mathrm{Gal}(K_2/\mathbb{Q}_p^{nr}) \simeq \mathbb{F}_{p^2}^*$ of I_p, where $\mathbb{Q}_p^{nr} \subset K_2 \subset \overline{\mathbb{Q}_p}$ is the unique subextension with $[K_2 : \mathbb{Q}_p^{nr}] = p^2 - 1$.

Theorem 1.2 (Fontaine)**.** *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be modular of weight k and level N, with $2 \leq k \leq p+1$. Suppose that $\rho_{|G_p}$ is irreducible. Let $\psi, \psi' : I_p \to \overline{\mathbb{F}}_p^*$ be the characters of level two induced by the two field embeddings $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$. Then,*

$$\rho_{|I_p} \sim \begin{pmatrix} \psi^{k-1} & 0 \\ 0 & (\psi')^{k-1} \end{pmatrix}.$$

1.2. **Generalizations of Serre's conjecture.** How can this picture be generalized? For any number field F and any $n \geq 1$, we need to have a notion of a Galois representation ρ :

$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ being modular; roughly this should mean that ρ arises "geometrically." In most cases it is not known how to associate Galois representations to automorphic objects, making it difficult to motivate a natural definition of modularity. But given a notion of modularity, one can seek to formulate analogues of the Serre and strong Serre conjectures.

If F is a totally real number field, then it is known how to construct compatible families of Galois representations associated to Hilbert modular forms over F (see [BR], [Tay]). The first generalizations of Serre's conjecture dealt with this case. When p is unramified in F, a conjecture was formulated by Buzzard, Diamond, and Jarvis [BDJ]. It was extended by the author to totally real fields F where p ramifies arbitrarily, but only when ρ is tamely ramified at all places above p. This conjecture is discussed in the next section.

Serre's conjecture has also been generalized in another direction, to Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ for arbitrary n. Ash, Doud, D. Pollack, and Sinnott [AS], [ADP] conjectured a combinatorial recipe for some modular weights in this case, but never claimed to have found all the modular weights. When ρ is tamely ramifed at p, their work was improved by Herzig [Her1] who conjectured a complete list of regular modular weights that is defined more conceptually. A more general conjecture (without the assumption of tame ramification) was made by Gee [Gee1], but it specifies the modular weights in terms of the existence of certain crystalline lifts and does not usually allow them to be written down explicitly. In fact, the conjecture of Buzzard, Diamond, and Jarvis was already phrased in terms of crystalline lifts, and hence non-explicit, when ρ had wild ramification at p.

The framework exists for stating Serre's conjecture in an even more general context. This work was begun by Gross in [Gro1] and [Gro2]. Let $G/\mathbb{Q}$ be a reductive group such that all the arithmetic subgroups of $G(\mathbb{Q})$ are finite, and suppose that G is an inner form of a split group over $\mathbb{Q}$. Let $\hat{G}$ be the split dual group over $\mathbb{Z}$. To each weight and level (a weight in this general context is an irreducible $\overline{\mathbb{F}}_p$-representation of $G(\mathbb{F}_p)$) Gross associated a space of modular forms with a Hecke-algebra action and conjectured ([Gro1], Conj. 1.1) that to any Hecke eigenform one can associate a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \hat{G}(\overline{\mathbb{F}}_p)$ satisfying certain properties. This provides a notion of modularity. One can now ask when a representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \hat{G}(\overline{\mathbb{F}}_p)$ is modular and what the weights are of the modular forms giving rise to it.

2. Serre's conjecture for Hilbert modular forms

2.1. **Weights and modularity.** Let F be a number field and $n \geq 1$. For any place v of F, let $\mathcal{O}_v$ denote the ring of integers of the completion F_v, and let k_v be the residue field. A *Serre weight* is an irreducible $\overline{\mathbb{F}}_p$-representation of the finite group $\mathrm{GL}_n(\mathcal{O}_F/p)$. Any Serre weight factors through the quotient map

$$\mathrm{GL}_n(\mathcal{O}_F/p) \to \Delta = \prod_{v|p} \mathrm{GL}_n(k_v),$$

since the kernel is a p-group. Therefore Serre weights have the form $\sigma = \bigotimes_{v|p} \sigma_v$, where σ_v is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(k_v)$. We call such σ_v *local Serre weights at* v.

Suppose for the rest of this section that $n = 2$ and F is totally real. We now introduce a notion of modularity of a mod p Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Let D/F be a quaternion algebra that splits at exactly one infinite place and at all places dividing p. Consider the reductive group $G = \mathrm{Res}_{F/\mathbb{Q}}(D^*)$ and let $U \subset G(\mathbb{A}^\infty)$ be an open compact subgroup. Let X_U/F be the associated Shimura curve; its complex points are $X_U(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}^\infty) \times (\mathbb{C} - \mathbb{R})/U$. Recall that the relative Picard scheme of X_U parametrizes line bundles locally of degree zero, and let $\mathrm{Pic}^0(X_U)/F$ be its component containing the identity. This is an abelian variety.

Let $U'_p = \ker((D \otimes \hat{\mathbb{Z}})^*_p = \prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v) \to \mathrm{GL}_2(\mathcal{O}_F/p))$, and let $U''_p = \ker(\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v) \to \prod_{v|p} \mathrm{GL}_2(k_v))$. Clearly $U'_p \subset U''_p$. We say that an open compact $U \subset G(\mathbb{A}^\infty)$ is *of type* $(*)$ if $U = U'_p \times U^p$, where $U^p \subset G(\mathbb{A}^{\infty,p})$. Let $V = \prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v) \times U^p$. If U^p is sufficiently small in the sense of section 3.1 of [Sch3], then $X_U \to X_V$ is a Galois cover with group $V/U = \mathrm{GL}_2(\mathcal{O}_F/p)$. Hence we have an action of V/U on $\mathrm{Pic}^0(X_U)$.

Definition 2.1. Let σ be a Serre weight. An irreducible Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is *modular of weight* σ if there exists a quaternion algebra D/F as above and an open compact $U \subset (D \otimes \hat{\mathbb{Z}})^* \subset G(\mathbb{A}^\infty)$ of type $(*)$, such that $(\mathrm{Pic}^0(X_U)[p] \otimes_{\overline{\mathbb{F}}_p} \sigma)^{\mathrm{GL}_2(\mathcal{O}_F/p)} = (\mathrm{Pic}^0(X_{U''_p \times U^p})[p] \otimes_{\overline{\mathbb{F}}_p} \sigma)^{\Delta}$ has ρ as a Jordan-Hölder constituent.

If $[F : \mathbb{Q}] = d$, then each of the d embeddings of F into $\mathbb{R} \subset \mathbb{C}$ induces a "complex conjugation" in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Denote these complex conjugations $c_1, \ldots, c_d$. We say that ρ is *totally odd* if $\det \rho(c_i) = -1$ for all $1 \leq i \leq d$. The qualitative Serre conjecture generalizes to our situation as follows.

Conjecture 2.2. Suppose that $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous, irreducible, and totally odd. Then ρ is modular.

To formulate an analogue of the strong Serre conjecture we must, given a ρ, specify its modular weights. The Langlands philosophy suggests that the modular weights should be determined by local information, as was indeed the case for $F = \mathbb{Q}$. We fix a decomposition subgroup $G_v \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for each place $v|p$ and will define a set $W_v(\rho)$ of local Serre weights at v that depends only on the restriction $\rho_{|G_v}$. Then we will conjecture that

Conjecture 2.3. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a Galois representation. Its set of modular weights is

$$W(\rho) = \left\{ \sigma = \bigotimes_{v|p} \sigma_v : \forall v|p, \sigma_v \in W_v(\rho) \right\}.$$

Fix a place $\mathfrak{p}$ of F dividing p, let the cardinality of $k_{\mathfrak{p}}$ be $q = p^f$, and let e be the ramification index of $F_{\mathfrak{p}}$ over $\mathbb{Q}_p$. When $n = 2$, it is easy to give explicit models for the local Serre weights at $\mathfrak{p}$. Let I be the set of field embeddings $\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p$. Let $I = \{\tau_0, \tau_1, \ldots, \tau_{f-1}\}$ be a labeling of the elements of I such that $\tau_{i-1} = \tau_i^p$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. The irreducible $\overline{\mathbb{F}}_p$-representations have the form

$$\sigma_v = \bigotimes_{\tau \in I} (\det{}^{w_\tau} \operatorname{Sym}^{k_\tau - 2} k_{\mathfrak{p}}^2) \otimes_{k_{\mathfrak{p}}, \tau} \overline{\mathbb{F}}_p,$$

where $2 \leq k_\tau \leq p+1$ and $0 \leq w_\tau \leq p-1$, and not all the w_τ are $p-1$. An explicit model is given by the space of polynomials $P \in \overline{\mathbb{F}}_p[X_0, Y_0, \ldots, X_{f-1}, Y_{f-1}]$ in $2f$ variables that are homogeneous of degree $k_{\tau_i} - 2$ in each pair of variables X_i, Y_i. The $\mathrm{GL}_2(k_{\mathfrak{p}})$-action is given as follows. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k_{\mathfrak{p}})$, let $I(\gamma) \in \mathrm{GL}_{2f}(\overline{\mathbb{F}}_p)$ be the matrix

$$I(\gamma) = \begin{pmatrix} \tau_0(a) & \tau_0(b) & & & & & \\ \tau_0(c) & \tau_0(d) & & & & & \\ & & \tau_1(a) & \tau_1(b) & & & \\ & & \tau_1(c) & \tau_1(d) & & & \\ & & & & \ddots & & \\ & & & & & \tau_{f-1}(a) & \tau_{f-1}(b) \\ & & & & & \tau_{f-1}(c) & \tau_{f-1}(d) \end{pmatrix}.$$

Then we have

$$(\gamma P)(X_0, \ldots, Y_{f-1}) = \prod_{\tau \in I} \tau(ad - bc)^{w_\tau} P((X_0, Y_0, \ldots, X_{f-1}, Y_{f-1}) I(\gamma)).$$

When $n \geq 3$ there are no longer such nice models of Serre weights and it is often impossible to do explicit computations; this is one of the many difficulties of $n \geq 3$ relative to $n = 2$.

Let $I_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ be the inertia subgroup, and let $P_{\mathfrak{p}} \subset I_{\mathfrak{p}}$ be the wild inertia; $P_{\mathfrak{p}}$ is the pro-p-Sylow subgroup of $I_{\mathfrak{p}}$. The quotient $I_{\mathfrak{p}}/P_{\mathfrak{p}}$ is called the tame inertia and is isomorphic to $\varprojlim \mathbb{F}_{p^m}^*$. A character $\varphi : I_{\mathfrak{p}} \to I_{\mathfrak{p}}/P_{\mathfrak{p}} \to \overline{\mathbb{F}}_p^*$ is said to be of niveau m if it factors through the quotient $\mathbb{F}_{p^m}^*$. Such a character is called *fundamental* if the resulting map $\mathbb{F}_{p^m}^* \to \overline{\mathbb{F}}_p^*$ is the restriction of an embedding $\mathbb{F}_{p^m} \to \overline{\mathbb{F}}_p$ of fields.

Let $k'_{\mathfrak{p}}$ be the quadratic extension of $k_{\mathfrak{p}}$. Given an embedding of fields $\tau : k_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p$ (resp. $\tilde{\tau} : k'_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{F}}_p$), let ψ_τ (resp. $\psi_{\tilde{\tau}}$) be the corresponding fundamental character of niveau f (resp. $2f$).

The semisimplification $\rho^{ss}_{|I_{\mathfrak{p}}}$ of $\rho_{|I_{\mathfrak{p}}}$ factors through the tame inertia, which is abelian. Hence $\rho^{ss}_{|I_{\mathfrak{p}}}$ is a sum of characters $\varphi \oplus \varphi'$. Moreover, the quotient $G_{\mathfrak{p}}/I_{\mathfrak{p}}$, which is topologically generated by a Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$, acts on the tame inertia by conjugation. It follows that $\{\varphi^q, (\varphi')^q\} = \{\varphi, \varphi'\}$, so that two possible cases arise:

1. The characters φ, φ' have niveau $2f$, so that $\varphi' = \varphi^q$ and $(\varphi')^q = \varphi$. This implies the irreducibility of $\rho_{|G_{\mathfrak{p}}}$.
2. The characters φ and φ' are of niveau f, and $\rho_{|G_{\mathfrak{p}}}$ is reducible.

We are now ready to provide a recipe for $W_{\mathfrak{p}}(\rho)$ when ρ is tamely ramified at $\mathfrak{p}$.

Definition 2.4. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a Galois representation.

(1) Suppose that $\rho_{|G_\mathfrak{p}}$ is irreducible. Then the local Serre weight

$$\sigma_\mathfrak{p} = \bigotimes_{\tau \in I} (\det{}^{w_\tau} \mathrm{Sym}^{k_\tau - 2} k_\mathfrak{p}^2) \otimes_{k_\mathfrak{p}, \tau} \overline{\mathbb{F}}_p \tag{1}$$

is contained in $W_\mathfrak{p}(\rho)$ if and only if for each $\tau \in I$ there exists a labeling $\{\tilde{\tau}, \tilde{\tau}'\}$ of the two lifts of τ to $k'_\mathfrak{p}$ and an integer $0 \le \delta_\tau \le e-1$ such that

$$\rho_{|I_\mathfrak{p}} \sim \prod_{\tau \in I} \psi_\tau^{w_\tau} \begin{pmatrix} \prod_\tau \psi_{\tilde{\tau}}^{k_\tau - 1 + \delta_\tau} \psi_{\tilde{\tau}'}^{e-1-\delta_\tau} & 0 \\ 0 & \prod_\tau \psi_{\tilde{\tau}}^{e-1-\delta_\tau} \psi_{\tilde{\tau}'}^{k_\tau - 1 + \delta_\tau} \end{pmatrix}.$$

(2) Suppose that $\rho_{|G_\mathfrak{p}}$ is reducible and that ρ is tamely ramified at $\mathfrak{p}$ (i.e. $P_\mathfrak{p} \subset \ker \rho$). Then $W_\mathfrak{p}(\rho)$ consists precisely of the Serre weights as in (1) for which there exists a subset $J \subset I$ and an integer $0 \le \delta_\tau \le e-1$ for each $\tau \in I$ such that

$$\rho_{|I_\mathfrak{p}} \sim \prod_{\tau \in I} \psi_\tau^{w_\tau} \begin{pmatrix} \prod_{\tau \in J} \psi_\tau^{k_\tau - 1 + \delta_\tau} \prod_{\tau \notin J} \psi_\tau^{e-1-\delta_\tau} & 0 \\ 0 & \prod_{\tau \in J} \psi_\tau^{e-1-\delta_\tau} \prod_{\tau \notin J} \psi_\tau^{k_\tau - 1 + \delta_\tau} \end{pmatrix}.$$

In particular, if $e \ge p-1$, then all Serre weights of suitable central character should be modular.

2.2. **Evidence.** There are some results available towards this conjecture. Dembélé's computations of modular weights of Hilbert modular forms over $\mathbb{Q}(\sqrt{5})$ for $p = 5$ agree with Conjecture 2.3 (see section 4 of [Sch2]). The following theoretical results have also been established.

Theorem 2.5 ([Sch2], Theorem 3.4)**.** *Suppose that $e < p-1$ and let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be such that $\rho_{|G_\mathfrak{p}}$ is irreducible and ρ is modular of weight $\sigma = \otimes_{v|p} \sigma_v$, where $\sigma_\mathfrak{p}$, written as in (1), satisfies $k_\tau - 2 + e \le p-1$ for all $\tau \in I$. Then $\sigma_\mathfrak{p} \in W_\mathfrak{p}(\rho)$.*

If p is unramified in F, then a stronger result has been proved by Gee. We say that a local Serre weight at $\mathfrak{p}$, written as in (1), is *strongly regular* if $3 \le k_\tau \le p-1$ for all $\tau \in I$ and *regular* if $2 \le k_\tau \le p$ for all $\tau \in I$. Gee uses a variant definition of modularity, working with definite quaternion algebras, but his results should be translatable to our setting.

Theorem 2.6 ([Gee2], Thms. 5.1.2 and 5.1.3)**.** *Suppose that p is unramified in F, that $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular of weight σ, and that σ is strongly regular. Then $\sigma \in W(\rho)$.*

If $\sigma \in W(\rho)$ is strongly regular and non-ordinary, then it is a modular weight of ρ.

We refer the reader to Gee's paper for the definition of "non-ordinary," which is a technical condition; "generically" weights are non-ordinary. We will briefly explain why the hypothesis of $k_\tau - 2 + e \le p-1$ in Theorem 2.5 and that of strong regularity in Gee's theorem ultimately stem from the same source.

A representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that arises from a modular form of weight $2 \leq k \leq p+1$ and nebentype w is modular of weight $\det^w \otimes \mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2$. It is now clear that Theorem 2.5 is a generalization of Fontaine's Theorem 1.2 above, and its proof follows the same method, although it works with Shimura curves rather than modular curves and deals with complications introduced by the extra ramification. Let $B_2(k_\mathfrak{p}) \subset \mathrm{GL}_2(k_\mathfrak{p})$ be the upper triangular Borel subgroup, and choose a character $\theta : B_2(k_\mathfrak{p}) \to \overline{\mathbb{F}}_p^*$ such that $\sigma_\mathfrak{p}$ is a subquotient of $\mathrm{Ind}_{B_2(k_\mathfrak{p})}^{\mathrm{GL}_2(k_\mathfrak{p})}\theta$. Then we find ρ inside a suitable piece of $\mathrm{Jac}(X_{U_1^{bal}(\mathfrak{p})\times U^\mathfrak{p}})[p^\infty]$, where

$$U_1^{bal}(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}) : a-1, c, d-1 \in \mathfrak{p} \right\}$$

and $U^\mathfrak{p}$ is the part of U away from $\mathfrak{p}$. This suitable piece is a vector space scheme to which we can apply Raynaud's theory [Ray] to obtain combinatorial restrictions on ρ. Effectively we have lifted a mod p Hilbert modular form giving rise to ρ to trivial weight, but at the price of raising its level to $U_1^{bal}(\mathfrak{p}) \times U^\mathfrak{p}$. In the process we have lost information about $\sigma_\mathfrak{p}$ and retained only θ. In fact, at this step of the proof ([Sch2], Prop. 3.3) ρ is restricted to being precisely one of the Galois representations for which one of the subquotients of $\mathrm{Ind}_{B_2(k_\mathfrak{p})}^{\mathrm{GL}_2(k_\mathfrak{p})}\theta$ is conjectured to be a modular weight, which is the best possible result. Then we consider all characters θ such that $\sigma_\mathfrak{p}$ appears in $\mathrm{Ind}_{B_2(k_\mathfrak{p})}^{\mathrm{GL}_2(k_\mathfrak{p})}\theta$ and intersect the sets of permitted ρ's. If the hypothesis that $k_\tau - 2 + e \leq p-1$ for all $\tau \in I$ does not hold, then this intersection is too large and contains ρ's for which σ is not conjectured to be modular.

Gee's proof follows a completely different method and hinges on a variant of one of Kisin's modularity lifting results. Kisin's functor from crystalline Galois representations to a certain category of $\mathfrak{S}$-modules is only essentially surjective if the Hodge-Tate numbers on both sides are restricted to $\{0,1\}$, and the needed modularity lifting theorem is only available in this case. This forces Gee also to lift to weight 2, work in weight 2, and deal with the same combinatorics at the end. The point is that Theorems 2.5 and 2.6 are the best results obtainable by their methods of proof with the available technology, except for ad hoc tricks such as the result of [Sch1], which allows one to prove a bit more when the residue field $k_\mathfrak{p}$ is small. In order to move forward, it appears that one will need more general modularity lifting theorems.

2.3. **A more conceptual formulation of the sets of modular weights.**

In this section we will briefly discuss the structure of Herzig's conjecture and how his ideas can be used to restate Conjecture 2.3 more conceptually.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a Galois representation that is tamely ramified at p. Then $\rho_{|I_p}$ factors through the abelian group I_p/P_p, so it is a sum of characters. The action of G_p on I_p/P_p by conjugation implies that if a character $\psi : \mathbb{F}_{p^m}^* \to \overline{\mathbb{F}}_p^*$ appears in $\rho_{|I_p}$, then so do all of its Galois conjugates. Hence there is a partition $n_1 + \cdots + n_r = n$ such that

$$\rho_{|I_p} \sim \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix},$$

where each A_i is an $n_i \times n_i$ diagonal matrix whose diagonal entries are a Galois conjugacy class $[\psi_i]$ of characters $\mathbb{F}_{p^{n_i}}^* \to \overline{\mathbb{F}}_p^*$. This conjugacy class defines a cuspidal characteristic zero representation $\Theta([\psi_i])$ of $\mathrm{GL}_{n_i}(\mathbb{F}_p)$. Now let $P \subset \mathrm{GL}_n(\mathbb{F}_p)$ be the standard parabolic subgroup whose Levi subgroup is $\mathrm{GL}_{n_1}(\mathbb{F}_p) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{F}_p)$. We define a characteristic zero representation $V(\rho_{|I_p})$ of $\mathrm{GL}_n(\mathbb{F}_p)$ by

$$V(\rho_{|I_p}) = \mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{F}_p)}(\Theta([\psi_1]) \otimes \cdots \otimes \Theta([\psi_r])).$$

Let $JH(\overline{V(\rho_{|I_p})})$ be the set of Jordan-Hölder constituents of the reduction modulo p of $V(\rho_{|I_p})$. Its elements are irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\mathbb{F}_p)$ – in other words, weights. Herzig defines a class of *regular* weights analogous to that for GL_2. Roughly, a weight is regular if it does not lie on some boundaries of alcoves; for $n = 2$ the notion of regularity coincides with the one defined above. Then he defines an operator $\mathcal{R}$ sending the set of Serre weights into that of regular Serre weights and conjectures ([Her1], Conj. 6.9) that the set of regular modular Serre weights of ρ is $\mathcal{R}(JH(\overline{V(\rho_{|I_p})}))$.

Herzig also reformulates the conjecture of Buzzard, Diamond, and Jarvis in this language. Given a tamely ramified $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, for F a totally real field in which p does not ramify, he defines a representation $V(\rho_{|I_\mathfrak{p}})$ of $\mathrm{GL}_2(k_\mathfrak{p})$ such that $W_\mathfrak{p}(\rho) = \mathcal{R}(JH(\overline{V(\rho_{|I_p})}))$ for an operator $\mathcal{R}$ acting on the appropriate set of local Serre weights. Herzig's restatement of the conjecture accounts for the non-regular weights as well, at the price of $\mathcal{R}$ being multi-valued.

If $F_\mathfrak{p}$ has ramification index e over $\mathbb{Q}_p$, then given a tamely ramified Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ it is shown in [Sch4] that one can define a collection $V(\rho)$ of characteristic zero representations of $\mathrm{GL}_2(k_\mathfrak{p})$ such that

$$W_\mathfrak{p}(\rho) = \bigcup_{V \in V(\rho)} \mathcal{R}(JH(\overline{V})).$$

The collection $V(\rho)$ has at most e^f elements, where $f = [k_\mathfrak{p} : \mathbb{F}_p]$.

3. Representations of $\mathrm{GL}_n(M)$, for M a p-adic field

3.1. **Notation.** Let $M/\mathbb{Q}_p$ be a finite extension, and denote by $\mathcal{O}$ its ring of integers, by π a uniformizer, and by k the residue field. Set $G = \mathrm{GL}_n(M)$, and let K denote the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O})$. Let $Z \simeq M^*$ be the center of G. Write $\Gamma \subset K$ for subgroup of matrices congruent to the identity modulo π. Let N be the collection of non-decreasing

n-tuples of integers $\nu = (\nu_1, \dots, \nu_n)$, where $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$. For $\nu \in N$, let $\alpha_\nu \in G$ be the matrix

$$\alpha_\nu = \begin{pmatrix} 1 & & & \\ & \pi^{\nu_2} & & \\ & & \ddots & \\ & & & \pi^{\nu_n} \end{pmatrix}.$$

For $0 \leq i \leq n$, let $\overline{P}_i \subset \mathrm{GL}_n$ be the parabolic subgroup

$$\overline{P}_i = \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : A \in \mathrm{GL}_i, D \in \mathrm{GL}_{n-i} \right\}.$$

In particular, $\overline{P}_0 = \overline{P}_n = \mathrm{GL}_n$.

Recall the Cartan decomposition of G into a disjoint union of double cosets of KZ:

$$G = \coprod_{\nu \in N} KZ\alpha_\nu^{-1}K. \tag{2}$$

Let $B \subset G$ be the upper triangular Borel subgroup, and let $T \subset B$ and $U \subset B$ be the diagonal torus and the unipotent radical, respectively. Let $S = \{\beta_i : 1 \leq i \leq n-1\}$ be the standard set of simple roots of GL_n; here $\beta_i = e_i - e_{i+1}$, where X_{ij} is the dual basis of $M_n(M)^*$ and $e_i = X_{ii}|_T$.

Let $W \simeq S_n$ be the Weyl group of GL_n. We consider W as a subgroup of K via the usual realization by permutation matrices. Let $\hat{w} \in W$ be the element of greatest length. Let $\omega : K \to \mathrm{GL}_n(k)$ be the natural surjection. The Iwahori subgroup $I \subset K$ is the preimage of $B(k)$ under ω, and we denote by $I(1) \subset I$ the pro-p-Sylow subgroup.

If $H \subset G$ is an open subgroup and (τ, V_τ) is a smooth representation of H, then we recall that the compact induction $\mathrm{ind}_H^G \tau$ is a smooth representation of G. A model for it is the space $S(G, \tau)$ of compactly supported locally constant functions $f : G \to V_\tau$ such that $f(hg) = \tau(h)f(g)$ for all $g \in G$ and $h \in H$. The G-action is given by $(g'f)(g) = f(gg')$. For each $v' \in V_\tau$ we can define an element $\hat{f}_{v'} \in \mathrm{ind}_H^G \tau$ by

$$\hat{f}_{v'}(g) = \begin{cases} \tau(g)v' & : g \in H \\ 0 & : g \notin H. \end{cases} \tag{3}$$

We observe that the set $\{\hat{f}_{v'} : v' \in V_\tau\}$ generates $\mathrm{ind}_H^G \tau$ as a G-module.

3.2. **Hecke algebras.** In this section we recall some basic facts about Hecke algebras. Let $H \subset G$ be an open subgroup, and let (τ, V_τ) be a smooth representation of H. Then we define

$$\mathcal{H}(H, \tau) = \mathrm{End}_G(\mathrm{ind}_H^G(\tau)).$$

Recall from section 2.2 of [BL] that $\mathcal{H}(H, \tau)$ can be interpreted as a convolution algebra as follows. Let $\mathcal{H}_H(\tau)$ be the space of functions $\varphi : G \to \mathrm{End}_{\overline{\mathbb{F}}_p}(V_\tau)$ such that

(1) For each $v \in V_\tau$, the function $\varphi_v : G \to V_\tau$ given by $\varphi_v(g) = \varphi(g)v$ is locally constant on G and supported on a set of the form HC, where $C \subset G$ is compact.
(2) For all $g \in G$ and all $h_1, h_2 \in H$, we have $\varphi(h_1 g h_2) = \tau(h_1)\varphi(g)\tau(h_2)$.

Given $\varphi_1, \varphi_2 \in \mathcal{H}_H(\tau)$, we define their convolution $\varphi_2 * \varphi_1$ to be

$$(\varphi_2 * \varphi_1)(g)(v) = \sum_{y \in G/H} \varphi_2(y)\varphi_1(y^{-1}g)(v)$$

for all $g \in G$ and $v \in V_\tau$. We recall the following result, which states that $\mathcal{H}(H, \tau)$ and $\mathcal{H}_H(\tau)$ are isomorphic and that composition in the former corresponds to convolution in the latter.

Proposition 3.1 ([BL], Proposition 5)**.** *Given $\varphi \in \mathcal{H}_H(\tau)$, define $T_\varphi \in \mathcal{H}(H, \tau)$ by*

$$T_\varphi(f)(g) = \sum_{Hx \in H\backslash G} \varphi(gx^{-1})f(x) = \sum_{yH \in G/H} \varphi(y)f(y^{-1}g) \tag{4}$$

for all $g \in G$ and $f \in S(G, \tau)$. Then the map

$$\begin{array}{rcl} \mathcal{H}_H(\tau) & \to & \mathcal{H}(H, \tau) \\ \varphi & \mapsto & T_\varphi \end{array}$$

is an isomorphism of $\overline{\mathbb{F}}_p$-modules. Moreover, if $\varphi_1, \varphi_2 \in \mathcal{H}_H(\tau)$, then $T_{\varphi_2} \circ T_{\varphi_1} = T_{\varphi_2 * \varphi_1}$.

Let σ be an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(k)$. We can view σ as a representation of KZ after inflating via ω and letting π act trivially. Let V_σ be the representation space, and choose a highest weight vector v. Then $v' = \hat{w} \cdot v$ is a lowest weight vector.

3.3. **The structure of $\mathcal{H}(KZ, \sigma)$.** Our immediate aim is to prove Proposition 3.5, which states that the Hecke algebra $\mathcal{H}(KZ, \sigma)$ is isomorphic to a polynomial algebra $\overline{\mathbb{F}}_p[T_1, \dots, T_{n-1}]$ in $n-1$ canonical generators. This was proved independently by Herzig, who recently extended his result for GL_n to a Satake isomorphism determining the structure of Hecke algebras $\mathcal{H}(KZ, \sigma)$, where G is any connected split reductive group, $K \subset G$ is a maximal compact subgroup, and σ is an irreducible mod p representation of KZ (see [Her2]). Nevertheless, we believe it is useful to write down an explicit proof for GL_n; in particular, readers who are unfamiliar with the representation theory of reductive groups may benefit from comparing the two arguments.

Given $\nu \in N$, we will define a linear transformation U_ν of the vector space V_σ. Let $\omega : K \to \mathrm{GL}_n(k)$ be the natural projection, and set $\overline{P}_\nu = \omega(K \cap \alpha_\nu^{-1} K \alpha_\nu)$. This is a parabolic subgroup of $\mathrm{GL}_n(k)$. Let $\overline{N}_\nu$ be its unipotent radical, let $L_\nu \in \overline{N}_\nu$ be the Levi subgroup, and let $P_\nu = L_\nu N_\nu$ be the opposite parabolic. Let $J \subset S$ be the set of simple roots corresponding to L_ν.

Let $\lambda : T(k) \to \overline{\mathbb{F}}_p^*$ be the highest weight of $\sigma = F(\lambda)$. For a weight μ, let $(V_\sigma)_\mu \subset V_\sigma$ be the corresponding weight subspace. Then [Jan], II.2.11a states that

$$V_\sigma^{N_\nu} = \bigoplus_{\alpha \in \mathbb{Z}J} (V_\sigma)_{\lambda - \alpha}.$$

It is easy to derive the analogous result for the negative standard parabolic subgroups. Recall that $\hat{w} \in W$ is the longest element of the Weyl group, and define $J' \subset S$ by $J' = \{\beta_i : \beta_{n-i} \in J\}$. If $P_{J'} \subset \mathrm{GL}_n(k)$ is the corresponding parabolic subgroup, then $\overline{P}_\nu = \hat{w} P_{J'} \hat{w}^{-1}$. Therefore, if $\hat{\lambda} = \hat{w}\lambda$ is the lowest weight of V_σ, we have

$$V_\sigma^{\overline{N}_\nu} = \hat{w}\left(\bigoplus_{\alpha' \in \mathbb{Z}J'} (V_\sigma)_{\lambda - \alpha'}\right) = \bigoplus_{\alpha \in \mathbb{Z}J} (V_\sigma)_{\hat{\lambda}+\alpha}. \tag{5}$$

We define an endomorphism U_ν of V_σ as follows. Let $\mathcal{B}$ be a basis of V_σ consisting of weight vectors. Then for $v \in \mathcal{B}$ we set

$$U_\nu(v) = \begin{cases} v & : v \in V_\sigma^{\overline{N}_\nu} \\ 0 & : v \notin V_\sigma^{\overline{N}_\nu}. \end{cases}$$

By (5) this definition is independent of the choice of $\mathcal{B}$. For each $\nu \in N$, let $\varphi_\nu \in \mathcal{H}_{KZ}(\sigma)$ be the function supported on the double coset $KZ\alpha_\nu^{-1}K$ that is determined by $\varphi_\nu(\alpha_\nu^{-1}) = U_\nu \in \mathrm{End}_{\overline{\mathbb{F}}_p}(V_\sigma)$.

Lemma 3.2. *The set $\{\varphi_\nu : \nu \in N\}$ is a basis of the $\overline{\mathbb{F}}_p$-vector space $\mathcal{H}_{KZ}(\sigma)$.*

Proof. The φ_ν are linearly independent, so we only need to show that they span $\mathcal{H}_{KZ}(\sigma)$. Clearly it suffices to show that any non-zero element $\varphi \in \mathcal{H}_{KZ}(\sigma)$ whose support consists of a sole double coset $KZ\alpha_\nu^{-1}K$ is a scalar multiple of φ_ν.

As above, let L_ν be the Levi subgroup of $\overline{P}_\nu \subset \mathrm{GL}_n(k)$. It is easy to see that $V_\sigma^{\overline{N}_\nu}$ is preserved by L_ν. It is an irreducible L_ν-module by the result of [Smi].

The definition of $\mathcal{H}_{KZ}(\sigma)$ implies that for all $k_1, k_2 \in K$ such that $k_1\alpha_\nu^{-1} = \alpha_\nu^{-1}k_2$, we have

$$\sigma(k_1)\varphi(\alpha_\nu^{-1}) = \varphi(\alpha_\nu^{-1})\sigma(k_2). \tag{6}$$

In particular, for every $\bar{k} \in \overline{N}_\nu$ we may choose $k_1 \in \omega^{-1}(\bar{k})$ and $k_2 \in \Gamma$ such that $k_1\alpha_\nu^{-1} = \alpha_\nu^{-1}k_2$. Hence the image of $\varphi(\alpha_\nu^{-1})$ lies in $V_\sigma^{\overline{N}_\nu}$. Similarly, we can take k_1 and k_2 to lie in $\omega^{-1}(l)$ for any $l \in L_\nu$. Thus $\varphi(\alpha_\nu^{-1})|_{V_\sigma^{\overline{N}_\nu}} : V_\sigma^{\overline{N}_\nu} \to V_\sigma^{\overline{N}_\nu}$ is an L_ν-module homomorphism, so by Schur's lemma it is a scalar $c \in \overline{\mathbb{F}}_p$.

Now suppose that $v \in V_\sigma$ is a weight vector that is not contained in $V_\sigma^{\overline{N}_\nu}$, and let μ be the weight of v. Any $t \in T(\mathcal{O}) \subset K$ commutes with α_ν^{-1}, and hence we find that

$$\sigma(t)\varphi(\alpha_\nu^{-1})(v) = \varphi(\alpha_\nu^{-1})(\sigma(t)v) = \mu(t)\varphi(\alpha_\nu^{-1})(v).$$

Hence $\varphi(\alpha_\nu^{-1})(v) \in (V_\sigma)_\mu$. But $(V_\sigma)_\mu \cap V_\sigma^{\overline{N}_\nu} = 0$, whence $v \in \ker\varphi(\alpha_\nu^{-1})$. We have shown that $\varphi(\alpha_\nu^{-1}) = cU_\nu$ and hence that $\varphi = c\varphi_\nu$. Moreover, it is easy to check that φ_ν is indeed an element of $\mathcal{H}_{KZ}(\sigma)$. □

Lemma 3.3. *The Hecke algebra $\mathcal{H}(KZ, \sigma)$ is commutative.*

Proof. We will show that $\mathcal{H}_{KZ}(\sigma)$ is commutative using a Gelfand-type argument suggested by Florian Herzig. Denote the transpose of a matrix g by g^T. Recall that V_σ is the representation space of σ. Following [Jan] II.2.12, we define V_σ^T as follows: as a vector space it is the dual space $\mathrm{Hom}(V_\sigma, \overline{\mathbb{F}}_p)$, but the $\mathrm{GL}_n(k)$-action is defined by $(gf)(v) = f(g^T v)$ for all $g \in \mathrm{GL}_n(k)$ and $f \in V_\sigma^T$. It is easy to see that V_σ^T is an irreducible $\mathrm{GL}_n(k)$-module. Since V_σ and V_σ^T have the same highest weight, they are isomorphic. Let $\alpha : V_\sigma \to V_\sigma^T$ be an isomorphism and let $\langle , \rangle : V_\sigma \times V_\sigma \to \overline{\mathbb{F}}_p$ be the corresponding non-degenerate bilinear form; for every $v, v' \in V_\sigma$, we have $(\alpha(v))(v') = \langle v, v' \rangle$. It is easy to see that for all $g \in \mathrm{GL}_n(k)$ we must have

$$\langle v, v' \rangle = \langle gv, (g^{-1})^T v' \rangle. \tag{7}$$

As usual, we view V_σ^T as a KZ-module. If $A \in \mathrm{End}_{\overline{\mathbb{F}}_p}(V_\sigma)$, let $A^T : V_\sigma \to V_\sigma$ be given by $A^T(v) = \alpha^{-1}(\alpha(v) \circ A)$. Equivalently, A^T is characterized by the relation $\langle A^T v, v' \rangle = \langle v, Av' \rangle$ for all $v, v' \in V_\sigma$. From this it is easy to check that $(AB)^T = B^T A^T$ for $A, B \in \mathrm{End}_{\overline{\mathbb{F}}_p}(V_\sigma)$. Moreover, if $\rho : KZ \to \mathrm{Aut}(V_\sigma)$ is the action of KZ on V_σ and $A = \rho(h)$ for $h \in KZ$, then $\langle A^T v, v' \rangle = \langle v, hv' \rangle = \langle h^T v, v' \rangle$ by (7), and hence $A^T = \rho(h^T)$.

If v and v' are weight vectors with weights $\lambda : T(k) \to \overline{\mathbb{F}}_p^*$ and $\lambda' : T(k) \to \overline{\mathbb{F}}_p^*$, respectively, with $\lambda \neq \lambda'$, then by (7) for all $t \in T(k)$ we have $\langle v, v' \rangle = \langle tv, t^{-1} v' \rangle = \lambda(t)(\lambda')^{-1}(t) \langle v, v' \rangle$, and hence $\langle v, v' \rangle = 0$. Therefore, if B is a basis for V_σ consisting of weight vectors and $v, v' \in B$, then for any $\nu \in N$ we have

$$\langle U_\nu v, v' \rangle = \langle v, U_\nu v' \rangle = \begin{cases} \langle v, v' \rangle & : v, v' \in V^{\overline{N}_\nu} \\ 0 & : \text{otherwise.} \end{cases}$$

It follows that $U_\nu^T = U_\nu$. We now define an involution $\varphi \mapsto \tilde{\varphi}$ of $\mathcal{H}_{KZ}(\sigma)$. For all $\varphi \in \mathcal{H}_{KZ}(\sigma)$ and $g \in G$ we put $\tilde{\varphi}(g) = (\varphi(g^T))^T$. If $h_1, h_2 \in KZ$ and $g \in G$, then we see that $\tilde{\varphi}(h_1 g h_2) = (\varphi(h_2^T g^T h_1^T))^T = \sigma(h_1^T)^T \varphi(g^T)^T \sigma(h_2^T)^T = \sigma(h_1)\tilde{\varphi}(g)\sigma(h_2)$, so that indeed $\tilde{\varphi} \in \mathcal{H}_{KZ}(\sigma)$. Moreover, the map $\varphi \mapsto \tilde{\varphi}$ is an anti-automorphism of $\mathcal{H}_{KZ}(\sigma)$. Indeed, as y runs over a set of coset representatives of G/KZ, so does $z = g(y^T)^{-1}$, and hence we have:

$$\begin{aligned}(\widetilde{\varphi_1 * \varphi_2})(g) = \left(\sum_{y \in G/KZ} \varphi_1(y)\varphi_2(y^{-1}g^T) \right)^T &= \sum_{y \in G/KZ} \varphi_2(y^{-1}g^T)^T \varphi_1(y)^T = \\ \sum_{z \in G/KZ} \varphi_2(z^T)^T \varphi_1(g^T(z^T)^{-1})^T = \sum_{z \in G/KZ} \tilde{\varphi_2}(z)\tilde{\varphi_1}(z^{-1}g) &= (\tilde{\varphi_2} * \tilde{\varphi_1})(g).\end{aligned} \tag{8}$$

We claim that $\tilde{\varphi} = \varphi$ for all $\varphi \in \mathcal{H}_{KZ}(\sigma)$. Indeed, for any $\nu, \mu \in N$ we have $(\alpha_\nu^{-1})^T = \alpha_\nu^{-1}$ and hence

$$\tilde{\varphi_\mu}(\alpha_\nu^{-1}) = (\varphi_\mu(\alpha_\nu^{-1}))^T = \begin{cases} U_\mu & : \nu = \mu \\ 0 & : \nu \neq \mu. \end{cases}$$

Therefore $\tilde{\varphi_\mu} = \varphi_\mu$. Now (8) implies the commutativity of $\mathcal{H}_{KZ}(\sigma)$. $\square$

If $1 \leq i \leq n-1$, define $\nu(i) \in N$ to be the vector $(0,\dots,0,1,\dots,1)$ consisting of i zeroes followed by $n-i$ ones. We will write α_i and φ_i for $\alpha_{\nu(i)}$ and $\varphi_{\nu(i)}$, respectively. Let $T_\nu \in \mathcal{H}(KZ,\sigma)$ be the operator corresponding to φ_ν under the isomorphism of Proposition 3.1, and write T_i for $T_{\nu(i)}$. Note that $\omega(K \cap \alpha_i^{-1} K \alpha_i)$ is the parabolic subgroup $\overline{P}_i \subset \mathrm{GL}_n(k)$ that was defined in the introduction.

Let $W_i \subset W$ be the subgroup of permutations that preserve the subsets $\{1,\dots,i\}$ and $\{i+1,\dots,n\}$, and let $\mathcal{W}_i$ be a set of coset representatives of W/W_i. Then by the Bruhat decomposition in $\mathrm{GL}_n(k)$ we see that

$$\mathrm{GL}_n(k) = \coprod_{w\in\mathcal{W}_i} \overline{U}(k) w \overline{P}_i,$$

where $\overline{U}$ is the unipotent radical of the lower triangular Borel subgroup. Lifting to K, we can obtain a set of coset representatives for $K/(K\cap\alpha_i^{-1}K\alpha_i)$ of the form $\bigcup_{w\in\mathcal{W}_i}\{uw : u \in \Lambda_w\}$, where Λ_w is a subset of the lower triangular pro-p-Iwahori subgroup $I(1) \subset K$.

Consider the reverse lexicographical ordering on N defined as follows. If $\nu = (\nu_1,\dots,\nu_n)$ and $\nu' = (\nu'_1,\dots,\nu'_n)$, then $\nu' \prec \nu$ if and only if there is some $1 \leq r \leq n$ such that $\nu'_r < \nu_r$ and $\nu_i = \nu'_i$ for $i > r$. This gives a complete linear ordering of N. Observe that for every $\nu \in N$ there exist only finitely many $\nu' \in N$ such that $\nu' \prec \nu$. Note that this ordering is not the opposite of the usual lexicographical ordering; perhaps it would be clearer to call it the Hebrew lexicographical ordering.

Lemma 3.4. *Let $\nu = \nu(i)$ for some $1 \leq i \leq n-1$, and suppose that $\mu, \lambda \in N$ are such that $(\varphi_i * \varphi_\mu)(\alpha_\lambda^{-1}) \neq 0$. Then $\lambda \preceq \mu + \nu(i)$, with respect to the ordering defined above. Moreover, for each $\lambda \prec \mu + \nu(i)$ there exists $c_\lambda \in \overline{\mathbb{F}}_p$ such that*

$$\varphi_i * \varphi_\mu = \varphi_{\mu+\nu(i)} + \sum_{\lambda \prec \mu+\nu(i)} c_\lambda \varphi_\lambda.$$

Proof. By definition of the convolution, we have that

$$(\varphi_i * \varphi_\mu)(\alpha_\lambda^{-1}) = \sum_{y\in G/KZ} \varphi_i(y)\varphi_\mu(y^{-1}\alpha_\lambda^{-1}) = \sum_{y\in K\alpha_i^{-1}K/K} \varphi_i(y)\varphi_\mu(y^{-1}\alpha_\lambda^{-1}). \tag{9}$$

A set of coset representatives of $K\alpha_i^{-1}K/K$ is given by $\{z\alpha_i^{-1}\}$, where z runs over coset representatives of $K/(K\cap\alpha_i^{-1}K\alpha_i) \simeq \mathrm{GL}_n(k)/\overline{P}_i$. We saw above that such a set is given by $\bigcup_{w\in\mathcal{W}_i}\Lambda_w w$, where Λ_w is a set of matrices that may be taken to be lower triangular with ones on the diagonal. Let $x \in \Lambda_w$, and let $u = x^{-1}$. Then $(xw\alpha_i^{-1})^{-1}\alpha_\lambda^{-1} = \alpha_i w^{-1} u \alpha_\lambda^{-1} = w^{-1}(w\alpha_i w^{-1})u\alpha_\lambda^{-1}) = w^{-1}C$, where $C = (c_{ij})$ is a lower triangular matrix whose elements are given by

$$c_{ij} = u_{ij}\pi^{\nu_{w^{-1}(i)}-\lambda_j}.$$

Since $\lambda_1 \leq \cdots \leq \lambda_n$, we see that by adding integer multiples of the rightmost column to the other columns, we can clear out the non-diagonal terms in the bottom row. Proceeding in this way, we find that $C = \mathrm{diag}(c_{11}, c_{22}, \ldots, c_{nn})C'$, where $C' \in K$ is some lower triangular matrix. Now let $s \in W$ be a permutation such that

$$\nu_{w^{-1}(s(1))} - \lambda_{s(1)} \geq \nu_{w^{-1}(s(2))} - \lambda_{s(2)} \geq \cdots \geq \nu_{w^{-1}(s(n))} - \lambda_{s(n)}$$

and let $\kappa_j = \nu_{w^{-1}(s(1))} - \lambda_{s(1)} - (\nu_{w^{-1}(s(j))} - \lambda_{s(j)})$. Then $\kappa = (\kappa_1, \ldots, \kappa_n) \in N$, and we have

$$(xw\alpha_i^{-1})^{-1}\alpha_\lambda^{-1} = w^{-1}s\alpha_\kappa^{-1}s^{-1}C' \in K\alpha_\kappa^{-1}KZ. \tag{10}$$

Observe that $\kappa = \kappa(\lambda, w)$ depends on λ and w, although we did not indicate this in the notation to avoid encumbering it further. It follows from the above that if $(\varphi_i * \varphi_\mu)(\alpha_\lambda^{-1}) \neq 0$, then $\mu = \kappa(\lambda, w)$ for some $w \in \mathcal{W}_i$. It is easy to see that $\nu_{w^{-1}(s(1))} - \lambda_{s(1)} \in \{0, 1\}$ and that we may assume that $\lambda(s(j)) = \lambda_j$ for all $1 \leq j \leq n$. Therefore $\kappa_j = \mu_j \geq \lambda_j - 1$. This proves the first part of the lemma.

Moreover, we find that $\kappa(\lambda, w) = \lambda - \nu(i)$ if and only if $w \in \mathcal{W}_i$ belongs to the class of the identity permutation. Then we may take $\Lambda_w = \{I_n\}$ to consist of the identity matrix. Therefore, if $\lambda = \mu + \nu(i)$, then the only non-zero term of the sum in (9) is the one corresponding to $y = \alpha_i^{-1}$, and this term is clearly equal to $U_i U_\mu = U_\lambda$. The second part of our claim follows. □

Proposition 3.5. *As an $\overline{\mathbb{F}}_p$-algebra, $\mathcal{H}(KZ, \sigma)$ is isomorphic to the ring $\overline{\mathbb{F}}_p[T_1, \ldots, T_{n-1}]$ of polynomials in $n - 1$ variables.*

Proof. We will prove the equivalent statement that $\mathcal{H}_{KZ}(\sigma) \simeq \overline{\mathbb{F}}_p[\varphi_1, \ldots, \varphi_{n-1}]$. We saw in Lemma 3.3 that $\mathcal{H}_{KZ}(\sigma)$ is commutative. By Lemma 3.2 the set $\{\varphi_\nu : \nu \in N\}$ spans $\mathcal{H}_{KZ}(\sigma)$, so it suffices to prove that each φ_ν is contained in the subalgebra generated by $\{\varphi_1, \ldots, \varphi_{n-1}\}$.

We will argue by induction on ν with respect to the reverse lexicographical ordering. The claim is trivial for $\nu = (0, \ldots, 0)$. Otherwise, let $i \leq n - 1$ be the largest integer such that $\nu_i = 0$. Then $\mu = \nu - \nu(i) \in N$. By Lemma 3.4, we know that

$$\varphi_\nu = \varphi_i * \varphi_\mu - \sum_{\lambda \prec \nu} c_\lambda \varphi_\lambda$$

for some $c_\lambda \in \overline{\mathbb{F}}_p$. Since $\mu \prec \nu$, our claim follows by induction. □

Remark 3.6. We remark that using the decomposition of $V_\sigma^{\overline{N}_\nu}$ into weight spaces from [Jan], II.2.11, it is possible to obtain an explicit recursive formula expressing T_ν in terms of $T_1, \ldots, T_{n-1}$. Such a formula would be useful for any work involving computations on the Hecke algebra $\mathcal{H}(KZ, \sigma)$.

3.4. **Realization of irreducible admissible representations of G.** As before, if σ is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(k)$, we inflate it to K via the map ω. Letting the uniformizer π act trivially, we obtain a representation of KZ that we continue to denote σ.

Lemma 3.7. *Let (ρ, V_ρ) be a smooth irreducible representation of KZ. Then there exist a unique unramified character $\chi : M^* \to \overline{\mathbb{F}}_p^*$ and irreducible representation σ of $\mathrm{GL}_n(k)$ such that $\rho = (\chi \circ \det) \otimes \sigma$.*

Proof. This is the same proof as in [BL], Proposition 4. Since the uniformizer π lies in the center of KZ, the operator $\rho(\pi)$ is a KZ-module automorphism of V_ρ. Hence, by Schur's Lemma, π acts as a scalar $\lambda \in \overline{\mathbb{F}}_p$. So $\rho|_K$ is still irreducible. Since the congruence subgroup $\Gamma \subset K$ is a pro-p group, the space of invariants V_ρ^Γ is non-trivial. Since it is a normal subgroup, V_ρ^Γ is preserved by K, so by irreducibility it must be all of V_ρ. Thus $\rho|_K$ factors through $K/\Gamma \simeq \mathrm{GL}_n(k)$, so it is the inflation of a unique irreducible representation σ of $\mathrm{GL}_n(k)$. Let $\chi : F^* \to \overline{\mathbb{F}}_p^*$ be the unramified character defined by setting $\chi(\pi) = \lambda$. Clearly, $\rho = (\chi \circ \det) \otimes \sigma$. □

Now let (ρ, V_ρ) be an irreducible $\overline{\mathbb{F}}_p$-representation of $G = \mathrm{GL}_n(M)$ with central character. Since $I(1)$ is a pro-p-group, V_ρ has non-zero $I(1)$-invariants. Clearly I preserves $V_\rho^{I(1)}$ and its action factors through the abelian quotient $I/I(1)$, so there is an eigenvector $v' \in V_\rho^{I(1)}$ on which I acts via a character $\varepsilon : I \to \overline{\mathbb{F}}_p^*$. The map $V_\varepsilon \to V_\rho|_I$ given by $1 \mapsto v'$ corresponds by Frobenius reciprocity to a non-zero element of $\mathrm{Hom}_K(\mathrm{ind}_I^K \varepsilon, V_\rho|_K)$. Let σ be an irreducible subrepresentation of $\mathrm{ind}_I^K \varepsilon$. Since ρ has a central character, there is a KZ-module homomorphism $(\chi \circ \det) \otimes \sigma \to V_\rho|_{KZ}$ for a suitable unramified character χ. Applying Frobenius reciprocity again, we get a non-zero, hence surjective, G-module homomorphism $\Phi : (\chi \circ \det) \otimes \mathrm{ind}_{KZ}^G \sigma \to V_\rho$.

The functor $V_\rho \mapsto V_\rho \otimes (\chi \circ \det)$ is an equivalence of categories from the category of G-modules to itself. Hence $\mathrm{End}_G((\chi \circ \det) \otimes \mathrm{ind}_{KZ}^G \sigma) \simeq \mathcal{H}(KZ, \sigma)$. Recall from Proposition 3.5 that $\mathcal{H}(KZ, \sigma)$ is isomorphic to a polynomial ring $\overline{\mathbb{F}}_p[T_1, \ldots, T_{n-1}]$.

Theorem 3.8. *Let (ρ, V_ρ) be an irreducible admissible $\overline{\mathbb{F}}_p$-representation of $G = \mathrm{GL}_n(M)$ such that there exists a surjective G-module map $\Phi : \mathrm{ind}_{KZ}^G \sigma \to V_\rho$, where σ is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(k)$. Then there exist scalars $\lambda_1, \ldots, \lambda_{n-1} \in \overline{\mathbb{F}}_p$ for which there is a G-module surjection*

$$\mathrm{ind}_{KZ}^G \sigma / (T_1 - \lambda_1, \ldots, T_{n-1} - \lambda_{n-1}) \mathrm{ind}_{KZ}^G \sigma \to V_\rho.$$

Proof. We need to show that the space $\mathrm{Hom}_G(\mathrm{ind}_{KZ}^G \sigma, V_\rho)$, which is non-trivial by assumption, contains an eigenvector for the action of $\mathcal{H}(KZ, \sigma)$. Since this Hecke algebra is commutative by Lemma 3.3, it suffices to show that $\mathrm{Hom}_G(\mathrm{ind}_{KZ}^G \sigma, V_\rho)$ is finite-dimensional.

Recall that $\Gamma \subset K$ acts trivially on σ. By Frobenius reciprocity we have

$$\mathrm{Hom}_G(\mathrm{ind} KZ^G \sigma, V_\rho) \simeq \mathrm{Hom}_{KZ}(\sigma, V_\rho|_{KZ}) = \mathrm{Hom}_{KZ}(\sigma, V_\rho^\Gamma).$$

The subspace $V_\rho^\Gamma \subset V$ of invariants is finite-dimensional by admissibility of ρ, and the theorem follows. □

Remark 3.9. In view of the remark before the statement of Theorem 3.8, the theorem remains true if we replace $\mathrm{ind}_{KZ}^G\sigma$ by $(\chi\circ\det)\otimes\mathrm{ind}_{KZ}^G\sigma$ for an unramified character $\chi : M^* \to \overline{\mathbb{F}}_p^*$.

Remark 3.10. When $n = 2$, the previous result was proved by Barthel and Livné [BL] without assuming admissibility. All irreducible complex representations of $\mathrm{GL}_2(M)$ are admissible, but is not known whether this is true for $\overline{\mathbb{F}}_p$-representations.

4. Towards the mod p local Langlands correspondence

In this section, all representations are over $\overline{\mathbb{F}}_p$. Let $M/\mathbb{Q}_p$ be a finite extension, and maintain all the notation from the previous section. The Langlands philosophy predicts roughly that there is a natural bijection as follows:

$$\left\{\begin{array}{c} n\text{-dimensional reps.} \\ \text{of } \mathrm{Gal}(\overline{M}/M)\end{array}\right\} \xrightarrow{\pi} \left\{\begin{array}{c}\text{certain smooth admissible} \\ \text{irreducible reps. of } \mathrm{GL}_n(M)\end{array}\right\}.$$

For complex representations, such a bijection was established by Harris and Taylor [HT]; on the left hand side in that case one considers Weil-Deligne representations rather than Galois representations. A mod l correspondence for p-adic fields, when $l \neq p$, has been given by work of Vignéras [Vig] and Emerton [Eme]. The $l = p$ case is considerably more involved. The bijection is then understood only when $n = 2$ and $M = \mathbb{Q}_p$. We briefly describe the mod p local Langlands correspondence in this case.

We just showed in Theorem 3.8 above that any irreducible admissible representation V of $\mathrm{GL}_n(M)$ is a quotient of $\mathrm{ind}_{KZ}^G\sigma/(T_1-\lambda_1,\dots,T_{n-1}-\lambda_{n-1})\mathrm{ind}_{KZ}^G\sigma$ for some scalars $\lambda_1,\dots,\lambda_{n-1}\in\overline{\mathbb{F}}_p$. We say that V is *supersingular* if this is true for $\lambda_1=\cdots=\lambda_{n-1}=0$. All $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_1(M)$ are also considered supersingular.

The non-supersingular representations of $\mathrm{GL}_2(M)$ were classified by Barthel and Livné for arbitrary M. When $M=\mathbb{Q}_p$, Breuil [Bre] proved that the $\mathrm{ind}_{KZ}^G\sigma/(T_1)\mathrm{ind}_{KZ}^G\sigma$ are all irreducible, and that the only isomorphisms between them are precisely those that are required to make the following definition well-defined.

Definition 4.1. Given an irreducible representation $\rho_p : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$, set $\pi(\rho_p) = \mathrm{ind}_{KZ}^G\sigma/(T_1)\mathrm{ind}_{KZ}^G\sigma$, where σ is a modular weight of ρ_p.

Recall that we may speak, somewhat abusively, of the modular weights of a local Galois representation, since the modular weights of $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ are determined by the restriction of ρ to $G_p\simeq\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Thus we have obtained a natural bijection between irreducible two-dimensional representations $\rho_p : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In all other cases, rather little is known. In particular, there appear to be far more supersingular representations of $\mathrm{GL}_n(M)$ than there are irreducible n-dimensional representations of $\mathrm{Gal}(\overline{M}/M)$. The supersingular representations are far from being classified; the $\mathrm{ind}_{KZ}^G\sigma/(T_1,\dots,T_{n-1})\mathrm{ind}_{KZ}^G\sigma$ are in general of infinite length, even when $n=2$, and their constituents are not understood. The first construction of supersingular representations of $\mathrm{GL}_2(M)$ was by Paskunas [Pas]. Together with Breuil [BP], they construct some representations, for p unramified in M, of the form that one would expect to find on the right-hand side of the local Langlands correspondence, but the picture is still very murky. In particular, it is not known how to characterize the representations of $\mathrm{GL}_n(M)$ that appear in the image of the mod p local Langlands correspondence.

The strong connection between the mod p local Langlands correspondence and Serre's conjecture that is evident in Definition 4.1 should hold in general. An argument of Emerton strongly suggests that for any irreducible $\rho : \mathrm{Gal}(\overline{M}/M) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, a non-zero surjection

$$\mathrm{ind}_{KZ}^{G}\sigma/(T_1,\ldots,T_{n-1})\mathrm{ind}_{KZ}^{G}\sigma \to \pi(\rho)$$

exists if and only if σ is a modular weight of ρ.

We conclude with some questions for future work on the representation theory of $\mathrm{GL}_n(M)$. Recall that $S = \{\beta_i : 1 \le i \le n-1\}$ is the standard set of simple roots of GL_n; here $\beta_i = e_i - e_{i+1}$, where X_{ij} is the dual basis of $M_n(M)^*$ and $e_i = X_{ii}|_T$.

Given an irreducible representation V of $\mathrm{GL}_n(M)$ and a surjective G-module homomorphism $\Phi : \mathrm{ind}_{KZ}^{G}\sigma/(T_1-\lambda_1,\ldots,T_{n-1}-\lambda_{n-1})\mathrm{ind}_{KZ}^{G}\sigma \to V$, let $I_{V,\Phi} = \{i \in [1,n-1] : \lambda_i = 0\}$. This induces a subset $J_{V,\Phi} \subset S$ consisting of the roots β_i such that $i \in I_{V,\Phi}$. Let $P_{V,\Phi} \subset \mathrm{GL}_n$ be the standard parabolic subgroup associated to the set of roots $J_{V,\Phi}$. We conjecture that V arises from induction to $\mathrm{GL}_n(M)$ of a representation of $P_{V,\Phi}(M)$.

More precisely, suppose that $P_{V,\Phi} \subset \mathrm{GL}_n$ is the parabolic subgroup consisting of elements of the form

$$\begin{pmatrix} A_1 & * & & * \\ 0 & A_2 & & \\ & & \ddots & * \\ 0 & & 0 & A_R \end{pmatrix},$$

where $n = \sum_{r=1}^{R} n_r$ is a partition and $A_r \in \mathrm{GL}_{n_r}$. For each $1 \le r \le R$, we let $G_r \subset G$ be the image of $\mathrm{GL}_{n_r}(M)$ under the embedding

$$\begin{array}{rcl} \mathrm{GL}_{n_r}(M) & \to & P_{V,\Phi} \subset \mathrm{GL}_n(M) \\ A_r & \mapsto & \begin{pmatrix} 1 & & & 0 \\ & A_r & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}. \end{array}$$

Let $Z_r \subset G_r$ be the center, and let $K_r = \mathrm{GL}_{n_r}(\mathcal{O}) \subset G_r$ be a maximal compact subgroup. For convenience, we define $N_r = \sum_{s=1}^{r} n_s$. If V_r is a representation of $\mathrm{GL}_{n_r}(M)$ for each $1 \le r \le R$, then we write $V_1 \otimes \cdots \otimes V_R$ for the inflation to $P_{V,\Phi}$ of the obvious representation of its Levi subgroup $\mathrm{GL}_{n_1}(M) \times \cdots \times \mathrm{GL}_{n_R}(M)$.

For each $1 \leq r \leq R$, we let σ_r be the following $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_{n_r}(k)$, in the standard Weyl-module notation:

$$\sigma_r = F(a_{N_r-(n_r-1)}, a_{N_r-(n_r-2)}, \ldots, a_{N_r}).$$

In other words, σ_r is generated as a $G_r(k)$-module by the highest weight vector $v \in V_\sigma$. Note that if $n_r = 1$, then $\sigma_r : M^* \to \overline{\mathbb{F}}_p^*$ is just the character inflated from the map $k^* \to \overline{\mathbb{F}}_p^*$ defined by $x \mapsto x^{a_{N_r}}$.

Conjecture 4.2. Maintain the notations defined above. For each $1 \leq r \leq R$, there exists a supersingular irreducible $\overline{\mathbb{F}}_p$-representation V_r of $\mathrm{GL}_{n_r}(M)$ such that V is a subquotient of the parabolic induction

$$\mathrm{ind}_{P_{V,\Phi}}^G (V_1 \otimes \cdots \otimes V_R).$$

Moreover, each V_r is a quotient of $\mathrm{ind}_{K_r Z_r}^{G_r} \sigma_r / (T_1^r, \ldots, T_{n_r-1}^r) \mathrm{ind}_{K_r Z_r}^{G_r} \sigma_r$, where $T_1^r, \ldots, T_{n_r-1}^r$ are the canonical generators of $\mathcal{H}(K_r Z_r, \sigma_r)$.

Some progress towards results of this type has been made recently by Florian Herzig. He is able to express the quotient $\mathrm{ind}_{KZ}^G \sigma / (T_1 - \lambda_1, \ldots, T_{n-1} - \lambda_{n-1}) \mathrm{ind}_{KZ}^G \sigma$, for many representations σ of $\mathrm{GL}_n(k)$, as a parabolic induction from the predicted parabolic subgroup of G.

The conjecture above expects the modular representation theory of $\mathrm{GL}_n(M)$ to have the same general structure as the complex representation theory. The basic objects are the supersingular representations, and everything else can be built from them by parabolic induction. At the present time, we have no understanding of either the supersingular representations of $\mathrm{GL}_n(F)$ or the structure of the parabolic inductions (except for the criterion of Rachel Ollivier [Oll] specifying exactly when the induction of a character from a Borel subgroup is irreducible), but these are beautiful and fruitful questions for research in the near future.

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Department of Mathematics, Bar-Ilan University, Ramat Gan 52900, Israel
E-mail address: mschein@math.biu.ac.il