ON THE UNIVERSAL SUPERSINGULAR MOD p REPRESENTATIONS OF $\mathrm{GL}_2(F)$

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Abstract. The irreducible supersingular mod p representations of $G = \mathrm{GL}_2(F)$, where F is a finite extension of $\mathbb{Q}_p$, are the building blocks of the mod p representation theory of G. They all arise as irreducible quotients of certain universal supersingular representations. We investigate the structure of these universal modules in the case when $F/\mathbb{Q}_p$ is totally ramified.

1. Introduction

The smooth representation theory of the group $\mathrm{GL}_2(F)$ acting on vector spaces over fields of characteristic p, where F is a finite extension of $\mathbb{Q}_p$, is an essential ingredient in the recent developments towards the mod p and p-adic local Langlands correspondences and therefore has attracted a great deal of interest. The irreducible supersingular representations, which will be defined below, are the building blocks of this theory. They are poorly understood; it was remarked in [Bre2] that not a single explicit example of an irreducible supersingular representation was known for $F \neq \mathbb{Q}_p$, and the situation has not improved since. The irreducible supersingular representations arise as the irreducible quotients of certain universal supersingular representations, which are non-admissible and of infinite length when $F \neq \mathbb{Q}_p$. This paper analyzes the universal representations in the case when $F/\mathbb{Q}_p$ is a totally ramified extension.

In their article [BL], which inaugurated the field, Barthel and Livné classified all the irreducible smooth non-supersingular mod p representations of $G = \mathrm{GL}_2(F)$ with central character, for arbitrary finite extensions $F/\mathbb{Q}_p$. By Frobenius reciprocity [BL, Proposition 5] any such representation arises as a quotient of a compact induction $\mathrm{ind}_{KF^*}^G \sigma$, where $K = \mathrm{GL}_2(\mathcal{O}_F)$ and σ is a *Serre weight*, namely an irreducible $\overline{\mathbb{F}}_p$-representation of K. Barthel and Livné proved that the endomorphism algebra of $\mathrm{ind}_{KF^*}^G \sigma$ is generated over $\overline{\mathbb{F}}_p$ by a single canonical Hecke operator T. The commutativity of the Hecke algebra implies that for every irreducible representation Π of G there exist a Serre weight σ and a scalar $\lambda \in \overline{\mathbb{F}}_p$ such that Π arises as a quotient

$$\mathrm{ind}_{KF^*}^G \sigma / (T - \lambda)(\mathrm{ind}_{KF^*}^G \sigma) \twoheadrightarrow \Pi. \tag{1}$$

Keeping to the terminology of [BL], we say that Π is *supersingular* if for some (hence any) map as in (1) we have $\lambda = 0$. The G-modules $\mathrm{ind}_{KF^*}^G \sigma / T(\mathrm{ind}_{KF^*}^G \sigma)$ are the *universal supersingular representations*. They are irreducible when $F = \mathbb{Q}_p$ by work of Breuil [Bre1], but are G-modules of infinite length otherwise. Morra [Mor1, Mor2] has described their K-socle filtrations when $F = \mathbb{Q}_p$ and their I-socle filtrations when $F/\mathbb{Q}_p$ is unramified, where $I \subset K$ is the Iwahori subgroup. One cannot consider the G-socle filtrations of the universal

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supersingular representations, since they have no non-zero irreducible G-submodules and hence have trivial socle (see Remark 3.10). As a coarse substitute, this paper studies some filtrations $\{\mathcal{F}_i\}$ such that $\mathcal{F}_0 = 0$ and $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the G-module generated by an infinite collection of irreducible components of $\mathrm{soc}_K((\mathrm{ind}_{KF^*}^G \sigma/T(\mathrm{ind}_{KF^*}^G \sigma))/\mathcal{F}_{i-1})$.

Vignéras [Vig] constructed a functor from the category of smooth mod p G-modules of finite presentation to that of mod p representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. It provided an approach towards an analogue of Colmez's functor [Col] that is crucial to his construction of the mod p local Langlands correspondence. Vignéras' functor was the original motivation for this work, which aimed to construct explicit irreducible supersingular representations of finite presentation. Indeed, if such representations existed, then the filtrations $\{\mathcal{F}_i\}$ might be expected to reach them. However, Schraen [Sch4] proved that, when $[F:\mathbb{Q}_p] = 2$, no irreducible supersingular representations of $\mathrm{GL}_2(F)$ have finite presentation. It is expected that the same holds for all $F \neq \mathbb{Q}_p$.

An explicit construction of irreducible supersingular representations as direct limits of representations of finite presentation can likely be achieved by extending the methods of this paper. Note that large families of irreducible supersingular representations were constructed in [BP] for $F/\mathbb{Q}_p$ unramified. That construction works for arbitrary $F/\mathbb{Q}_p$, although it has not been proven that the resulting representations are irreducible. However, the construction involves taking injective envelopes and therefore is inexplicit. We now present a brief summary of our results.

1.1. **Summary of results.** Let $F/\mathbb{Q}_p$ be a finite totally ramified extension. In the context of this paper, a *Serre weight* is an inflation to K of an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathbb{F}_p)$. Every Serre weight is isomorphic to $\det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ for exactly one pair (r, w) such that $0 \leq r \leq p-1$ and $0 \leq w < p-1$. We define an involution on the set of Serre weights as follows. Given $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$, put $\sigma' = \det^{w+r} \otimes \mathrm{Sym}^{p-1-r}$. The following result is Corollary 2.17. It generalizes Breuil's intertwining isomorphisms for $F = \mathbb{Q}_p$ [Bre1, Corollaire 4.1.3].

Proposition 1.1. *Let σ be any Serre weight. There is an isomorphism of G-modules*

$$\mathrm{ind}_{KF^*}^G \sigma/T(\mathrm{ind}_{KF^*}^G \sigma) \simeq \mathrm{ind}_{KF^*}^G \sigma'/T(\mathrm{ind}_{KF^*}^G \sigma').$$

This result fails when $F/\mathbb{Q}_p$ is not totally ramified; see Remark 2.21. One application of this result is to the computation of the K-socle of $\mathrm{ind}_{KF^*}^G \sigma/T(\mathrm{ind}_{KZ}^G \sigma)$. If σ is a Serre weight, we denote by χ_σ the character by which I acts on the one-dimensional space $\sigma^{I(1)}$, where $I(1) \subset I$ is the pro-p-Sylow subgroup. If $\chi : I \to \overline{\mathbb{F}}_p^*$ is a character, then let $\mathcal{S}_\chi$ be the set of all Serre weights σ such that $\chi_\sigma = \chi$. This is a singleton unless χ factors through the determinant, in which case $\mathcal{S}_\chi$ has two elements. Let $\varepsilon : I \to \overline{\mathbb{F}}_p^*$ be the character satisfying $\varepsilon(\mathrm{diag}(a,d)) = \bar{a}^{-1}\bar{d}$, where the bars denote reduction modulo the maximal ideal of $\mathcal{O}_F$. Let χ^s be the conjugate character defined in the following section. It satisfies $\chi^s(\mathrm{diag}(a,d)) = \chi(\mathrm{diag}(d,a))$. The following is Corollary 3.9.

Theorem 1.2. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight. The K-socle of $\mathrm{ind}_{KF^*}^G \sigma/T(\mathrm{ind}_{KF^*}^G \sigma)$ is isomorphic to:*

$$\begin{cases} \mathcal{S}_{\chi_\sigma} \oplus \mathcal{S}_{\chi_\sigma^s} \oplus \bigoplus_{\mathbb{N}} \left(\mathcal{S}_{\chi_\sigma \varepsilon} \oplus \mathcal{S}_{\chi_\sigma^s \varepsilon} \right) & : 1 < r < p-2 \\ \mathcal{S}_{\chi_\sigma} \oplus \bigoplus_{\mathbb{N}} \mathcal{S}_{\chi_\sigma \varepsilon} & : r \in \{0, p-1\}. \end{cases}$$

Moreover, in Theorem 3.8 we explicitly determine the $I(1)$-invariants of each component of the socle. We now briefly explain the connection between this result and the mod p local Langlands correspondence. If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is an irreducible Galois representation, then the mod p local Langlands correspondence should associate to it an irreducible supersingular $\overline{\mathbb{F}}_p$-representation $\pi(\rho)$ of $\mathrm{GL}_2(F)$. One expects that $\mathrm{soc}_K(\pi(\rho)) = \bigoplus_{\sigma\in\mathcal{D}(\rho)} \sigma$, where $\mathcal{D}(\rho)$ is a multiset of Serre weights associated to ρ by generalizations of Serre's modularity conjecture. See [Sch1, Theorem 2.4] for the definition of $\mathcal{D}(\rho)$ for irreducible ρ, and note that the weight part of Serre's conjecture formulated there has been proven for totally ramified $F/\mathbb{Q}$ [GS, GLS2], which is the case considered in this paper.

To take the simplest example, suppose that $[F : \mathbb{Q}_p] = 2$ and $F/\mathbb{Q}_p$ is totally ramified. Then, for generic ρ, the set of modular weights will have the following form, up to twist:

$$\mathcal{D}(\rho) = \left\{\mathrm{Sym}^r\overline{\mathbb{F}}_p^2,\ \det{}^r \otimes \mathrm{Sym}^{p-1-r}\overline{\mathbb{F}}_p^2,\ \det\otimes \mathrm{Sym}^{r-2}\overline{\mathbb{F}}_p^2,\ \det{}^{r-1}\otimes \mathrm{Sym}^{p+1-r}\overline{\mathbb{F}}_p^2\right\}.$$

for some r such that $2 < r < p-1$. Observe that no matter which of its four elements we take to be σ, the set $\mathcal{D}(\rho)$ does not contain Serre weights corresponding to both of the characters $\chi_\sigma\varepsilon$ and $\chi_\sigma^s\varepsilon$. We construct explicit elements $J_0^0(\sigma), Q_0^0(\sigma) \in \mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ (see (11)) that generate irreducible K-submodules isomorphic to the unique elements of $\mathcal{S}_{\chi_\sigma\varepsilon}$ and $\mathcal{S}_{\chi_\sigma^s\varepsilon}$, respectively. Moreover, the G-submodules generated by $J_0^0(\sigma)$ and $Q_0^0(\sigma)$ contain *all* the components of the socle isomorphic to these weights. In addition, the irreducible K-submodule isomorphic to σ (note that $\mathcal{S}_{\chi_\sigma} = \{\sigma\}$ since we are considering generic ρ) generates $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ as a G-module. Therefore, we have a surjective map

$$\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma) \twoheadrightarrow \pi(\rho), \tag{2}$$

and at least one of $J_0^0(\sigma)$ and $Q_0^0(\sigma)$ must lie in its kernel. We may assume without loss of generality that $J_0^0(\sigma)$ lies in the kernel, since the isomorphism of Proposition 1.1 maps $J_0^0(\sigma)$ to $Q_0^0(\sigma')$. This motivates us to study the quotient $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))/\langle J_0^0(\sigma)\rangle_G$. The following statement is essentially Proposition 2.10.

Proposition 1.3. *Let $i < \frac{r}{2}$ be a positive integer. There are explicit elements $J_1, J_2, \ldots, J_i \in \mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ such that $J_1 = J_0^0(\sigma)$ and for each $j \le i$, the element J_j generates an irreducible K-submodule of $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))/\langle J_1, \ldots, J_{j-1}\rangle_G$.*

In the last section of the paper we dig deeper into the quotients studied in Proposition 1.3; see Corollary 4.4 and Proposition 4.7. Note that the work done here complements the results of [Sch3]. Indeed, the image of $Q_0^0(\sigma)$ in $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ generates an irreducible K-module isomorphic, say, to the Serre weight τ. Consider the composition $\mathrm{ind}_{KF^*}^G\tau \xrightarrow{\Psi} \mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma) \to \pi(\rho)$, where Ψ comes from Frobenius reciprocity; the image of Ψ is the G-submodule generated by $Q_0^0(\sigma)$. If $Q_0^0(\sigma)$ is not in the kernel of (2), then this composition must factor through $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))/\Psi(T(\mathrm{ind}_{KF^*}^G\tau))$ since $\pi(\rho)$ is supersingular. Quotients of this kind are the object of study in [Sch3]. Remarks 2.20 and 2.21 contain further discussion of the relation between this paper and [Sch3]. We hope that the methods developed in the two papers will ultimately lead to an explicit understanding of some irreducible supersingular representations.

The reader who carefully examines both these papers will observe that Theorem 1.2 contradicts Proposition 2.10 of [Sch3]. We take this opportunity to point out that the proof of Proposition 2.10 of [Sch3] contains an error at the top of page 6280; the conclusion made there that $d = 0$ is false. As a result, $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ has more $I(1)$-invariants than are identified there. This affects Corollary 2.11 and Proposition 3.5 of [Sch3], but all the other statements and proofs in the paper, including the main results, do not rely on that argument and are true as stated.

1.2. **Notation.** Let F be a totally ramified finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be the its ring of integers. Assume $p > 3$; this hypothesis is used several times in our calculations, for instance in the proofs of Proposition 2.11 and Theorem 2.24. The same methods apply to $p \in \{2, 3\}$, but these small primes would have to be treated separately. Fix a uniformizer $\pi \in \mathcal{O}$. Denote by k the residue field $\mathcal{O}/(\pi) = \mathbb{F}_p$, and let $e = [F : \mathbb{Q}_p]$ be the ramification index of F. Throughout the paper, we denote $G = \mathrm{GL}_2(F)$ and let $Z \simeq F^*$ be the center of G. We consider the maximal compact subgroup $K = \mathrm{GL}_2(\mathcal{O}) \subset G$. Let $I \subset K$ be the Iwahori subgroup of elements whose reductions modulo π are upper triangular matrices in $\mathrm{GL}_2(k)$. Write $I(1)$ for the pro-p-Sylow subgroup of I; it consists of matrices whose reductions modulo π are of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. Set $\Delta \subset I$ to be the subgroup of diagonal matrices. Finally, let $K(1)$ be the kernel of the natural projection $K \to \mathrm{GL}_2(k)$.

If $H \subset G$ is an open subgroup and $\tau : H \to \mathrm{Aut}(V_\tau)$ is an H-module, we recall the compact induction $\mathrm{ind}_H^G \tau$. A model for it is given by the space of locally constant functions $f : G \to V_\tau$ which are compactly supported modulo Z and satisfy $f(hg) = \tau(h) \cdot f(g)$ for all $h \in H$ and all $g \in G$. An element $g' \in G$ acts, as usual, by $(g'f)(g) = f(gg')$. In this paper, H will generally be either KZ or IZ. The usual induction of representations from a subgroup $H' \subset H$ to H is written $\mathrm{Ind}_{H'}^H$.

We will often refer to elements of $\mathrm{ind}_{KZ}^G \sigma$ when we implicitly mean their images in a quotient of $\mathrm{ind}_{KZ}^G \sigma$. This should cause no confusion.

If $g \in G$ and $v \in V_\tau$, then as in [Sch3] we denote by $g \otimes v$ the element of $\mathrm{ind}_H^G \tau$ that is supported on the right coset Hg^{-1} and satisfies $(g \otimes v)(g^{-1}) = v$. For $g' \in G$ and $h \in H$ we observe that

$$\begin{aligned} g'(g \otimes v) &= g'g \otimes v \\ gh \otimes v &= g \otimes \tau(h)v. \end{aligned}$$

Clearly, an $\overline{\mathbb{F}}_p$-basis of $\mathrm{ind}_H^G \tau$ is given by $\{g \otimes v : g \in \mathcal{C}, v \in \mathcal{B}\}$, where $\mathcal{C}$ is a set of coset representatives of G/H and $\mathcal{B}$ is an $\overline{\mathbb{F}}_p$-basis of V_τ. If $x \in k$, let $[x] \in \mathcal{O}$ denote the canonical (Teichmüller) lift. As in [Bre1], define the sets $I_0 = \{0\}$ and $I_n = \{\mu = [\mu_0] + \pi[\mu_1] + \cdots + \pi^{n-1}[\mu_{n-1}] : \mu_0, \ldots, \mu_{n-1} \in k\}$ for $n \geq 1$. It is convenient also to define $I_n = \varnothing$ for $n < 0$. Given $n \geq 0$ and $\mu \in I_n$, define the matrices

$$\begin{aligned} g^0_{n,\mu} &= \begin{pmatrix} \pi^n & \mu \\ 0 & 1 \end{pmatrix} \\ g^1_{n,\mu} &= \begin{pmatrix} 1 & 0 \\ \pi\mu & \pi^{n+1} \end{pmatrix}. \end{aligned}$$

Note that $\mathcal{C} = \{g^0_{n,\mu} : n \geq 0, \mu \in I_n\} \cup \{g^1_{n,\mu} : n \geq 0, \mu \in I_n\}$ is a set of coset representatives of G/KZ. If τ is a KZ-module, then for $i \in \{0, 1\}$ and $n \geq 0$ we define the subspaces $S^i_n \subset \mathrm{ind}_{KZ}^G \tau$ consisting of functions supported on $\coprod_{\mu \in I_n} KZ(g^i_{n,\mu})^{-1}$. Define $S_n = S^0_n \coprod S^1_n$ and $B_n = \coprod_{m \leq n} S_n$.

We introduce the notation $\alpha = g^1_{0,0} = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$, and also we write $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\beta = \alpha w = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Observe that β normalizes $I(1)$.

Every element $a \in \mathcal{O}$ can be written in the form $a = \sum_{i=0}^{\infty}[a_i]\pi^i$ for a unique sequence of elements $a_i \in k$. We use this decomposition in the sequel without further comment.

As mentioned above, a *Serre weight* is an irreducible $\overline{\mathbb{F}}_p$-representation of K. It is well-known that the action of a pro-p group on a vector space over $\overline{\mathbb{F}}_p$ always has non-zero invariants. Since the pro-p group $K(1)$ is normal in K, it follows that every Serre weight of K factors through $\mathrm{GL}_2(k)$. We extend the action of K on a Serre weight to an action of KZ by decreeing that the matrix $\pi \cdot \mathrm{Id}_2$ act trivially; here $\mathrm{Id}_2 \in G$ is the identity matrix. Consider the representation $\det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$; a model is given by the $(r+1)$-dimensional space of homogeneous polynomials $P(x, y) \in \overline{\mathbb{F}}_p[x, y]$ of degree r. The action of K is given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)(x, y) = (\overline{a}\overline{d} - \overline{b}\overline{c})^w P(\overline{a}x + \overline{c}y, \overline{b}x + \overline{d}y).$$

The representations $\det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ such that $0 \le r \le p-1$ and $0 \le w < p-1$ are all irreducible and non-isomorphic, and every Serre weight is isomorphic to one of these. Note that the reduction of the determinant is a character of order $p-1$. For every Serre weight σ, we set $\chi_\sigma : I \to \overline{\mathbb{F}}_p^*$ to be the character satisfying $ix^r = \chi_\sigma(i)x^r$ for all $i \in I$.

Let $\varepsilon : I \to \overline{\mathbb{F}}_p$ be the character

$$\varepsilon : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \bar{a}^{-1}\bar{d}.$$

We assume throughout the paper that $e > 1$. Indeed, in the case of $F = \mathbb{Q}_p$ Breuil [Bre1] has proven that the universal quotients $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ are all irreducible G-modules, and therefore their G-socle filtration is very simple and is fully understood. A consequence of the assumption $e > 1$ that is crucial to the computations in this paper is that

$$[\lambda] + [\mu] \equiv [\lambda + \mu] \mod \pi^2 \tag{3}$$

for all $\lambda, \mu \in k$; see [Sch3], Lemma 2.2. We will use this fact on numerous occasions throughout the article. Note that the failure of (3) is essential to the argument in [Bre1] that proves the irreducibility of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ in the case $F = \mathbb{Q}_p$.

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2. Invariants

2.1. **Action of the Hecke operator.** Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight. For the reader's convenience, we recall an explicit description of the subspace $T(\mathrm{ind}_{KZ}^G\sigma) \subset \mathrm{ind}_{KZ}^G\sigma$ which will be used many times in the computations below. If $m < n$ and $\mu \in I_n$, then we define $[\mu]_m$ to be the truncation $[\mu]_m = \sum_{i=0}^{m-1}[\mu_i]\pi^i$. The following is [Sch3, Lemma 1.4]:

Lemma 2.1. *Let $v = \sum_{i=0}^{r} c_i x^{r-i} y^i \in V_\sigma$. If $n \geq 1$ and $\mu \in I_n$, then the action of T is given by:*

$$T(g_{n,\mu}^0 \otimes v) = \sum_{\lambda \in I_1} g_{n+1,\mu+\pi^n\lambda}^0 \otimes \left(\sum_{i=0}^{r} c_i(-\lambda)^i \right) x^r + g_{n-1,[\mu]_{n-1}}^0 \otimes c_r(\mu_{n-1}x + y)^r,$$

$$T(g_{n,\mu}^1 \otimes v) = \sum_{\lambda \in I_1} g_{n+1,\mu+\pi^n\lambda}^1 \otimes \left(\sum_{i=0}^{r} c_{r-i}(-\lambda)^i \right) y^r + g_{n-1,[\mu]_{n-1}}^1 \otimes c_0(x + \mu_{n-1}y)^r.$$

In the remaining cases the action of T is given by:

$$T(\mathrm{id} \otimes v) = \sum_{\lambda \in I_1} g_{1,\lambda}^0 \otimes \left(\sum_{i=0}^{r} c_i(-\lambda)^i \right) x^r + \alpha \otimes c_r y^r,$$

$$T(\alpha \otimes v) = \sum_{\lambda \in I_1} g_{1,\lambda}^1 \otimes \left(\sum_{i=0}^{r} c_{r-i}(-\lambda)^i \right) y^r + \mathrm{id} \otimes c_0 x^r.$$

2.2. **Pro-p-Iwahori invariants of $\mathrm{ind}_{KZ}^G \sigma$.** Recall that F is an arbitrary totally ramified finite extension of $\mathbb{Q}_p$, and $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ is any Serre weight.

Lemma 2.2. *Let $n \geq 1$, and let $P(\mu)$ be a homogeneous polynomial of degree s in the variables $\mu_0, \ldots, \mu_{n-1}$. Let $0 \leq j \leq r$. Let $a, d \in \mathbb{F}_p^*$, and let $\tilde{a}, \tilde{d} \in \mathcal{O}$ be arbitrary lifts of a and d, respectively. Set $X^0 = \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes P(\mu) x^{r-j} y^j$ and $X^1 = \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes P(\mu) x^j y^{r-j}$. Then the following identities hold in $\mathrm{ind}_{KZ}^G \sigma$:*

$$\begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix} X^0 = (ad)^w (a^{-1}d)^{s+j} a^r X^0$$

$$\begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix} X^1 = (ad)^w (ad^{-1})^{s+j} d^r X^1.$$

Proof. Computation. □

Note that any character $\chi : I \to \overline{\mathbb{F}}_p^*$ factors through $I/I(1) \simeq \Delta$, since $I(1)$ is a pro-p group. We denote by χ^s the conjugate character given by $\chi^s(i) = \chi(\beta i \beta^{-1})$ for all $i \in I$.

Lemma 2.3. *Let $\chi : I \to \overline{\mathbb{F}}_p^*$ be a character. Then the principal series $\mathrm{Ind}_I^K \chi$ is a $(p+1)$-dimensional representation of K and has length two as a K-module. The space $(\mathrm{Ind}_I^K \chi)^{I(1)}$ of $I(1)$-invariants is two-dimensional and is spanned by the elements $f_0(\chi)$ and $f_1(\chi)$ defined as follows:*

$$f_0(\chi) = \mathrm{id} \otimes 1$$

$$f_1(\chi) = \sum_{\lambda \in k^*} \begin{pmatrix} 1 & 0 \\ [\lambda] & 1 \end{pmatrix} \otimes \chi(\mathrm{diag}(\lambda^{-1}, -\lambda)) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1.$$

The K-submodule of $\mathrm{Ind}_I^K \chi$ generated by $f_1(\chi)$ is irreducible. If $\chi \neq \chi^s$, then this is the unique irreducible submodule of $\mathrm{Ind}_I^K \chi$, it is isomorphic to the unique Serre weight σ satisfying $\chi_\sigma = \chi^s$, and $\mathrm{Ind}_I^K \chi / \sigma \simeq \sigma'$. If $\chi = \chi^s = (\det)^w$, then the K-submodule generated by $f_1(\chi)$ is isomorphic to $\det^w \otimes \mathrm{Sym}^{p-1}$, the K-submodule generated by $f_0(\chi) + f_1(\chi)$ is isomorphic to $\det^w$, and $\mathrm{Ind}_I^K \chi$ is the direct sum of these two irreducible submodules.

Proof. The socle filtration of $\mathrm{Ind}_I^K \chi$ is well known; see, for instance, [BP, Theorem 2.4]. It is easy to verify that $f_0(\chi)$ and $f_1(\chi)$ have the properties claimed. $\square$

Note that a Serre weight σ is uniquely determined by the character χ_σ provided that $\chi_\sigma \neq \chi_\sigma^s$. If this condition fails (i.e. if $\chi_\sigma = \det^w_{|\Delta}$ for a suitable exponent w) then the weights $\det^w$ and $\det^w \otimes \mathrm{Sym}^{p-1}\overline{\mathbb{F}}_p^2$ both correspond to this character. The following simple condition will be used to distinguish them.

Lemma 2.4. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight, and let $0 \neq v \in \sigma^{I(1)}$. Then*

$$\sum_{\lambda \in k} \lambda^{p-1-r} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} v + (-1)^w \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v = 0.$$

In particular, consider the two statements

$$\sum_{\lambda \in k} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} v = 0 \tag{4}$$

$$\sum_{\lambda \in k} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} v + (-1)^w \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v = 0. \tag{5}$$

If $r = 0$, then (4) *holds and* (5) *is false. If $r = p-1$, then* (4) *is false and* (5) *is true.*

Proof. Computation, using the standard model of the Serre weights. $\square$

For each $n \geq 0$ we define the following elements of $\mathrm{ind}_{KZ}^G \sigma$:

$$\begin{aligned} A_n^0(\sigma) &= \sum_{\mu \in I_n} g_{n,\mu}^0 \otimes x^r \\ A_n^1(\sigma) &= \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes y^r. \end{aligned}$$

We will occasionally suppress the Serre weight σ from the notation when it will not cause confusion. Recalling that we defined $I_{-1} = \varnothing$, observe that $A_{-1}^0(\sigma) = A_{-1}^1(\sigma) = 0$.

Proposition 2.5. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight. Then the set $\{A_n^0(\sigma) : n \in \mathbb{N}\} \cup \{A_n^1(\sigma) : n \in \mathbb{N}\}$ is a basis for the space $(\mathrm{ind}_{KZ}^G \sigma)^{I(1)}$ of $I(1)$-invariants.*

If $r > 0$, then the KZ-submodule of $(\mathrm{ind}_{KZ}^G \sigma)$ generated by each $A_n^0(\sigma)$ is irreducible and isomorphic to σ, whereas the KZ-submodule generated by $A_n^1(\sigma)$ is isomorphic to a reducible principal series. Moreover, if $r = p-1$, then for each $n \geq 1$ the element $A_n^0(\sigma) + A_{n-1}^1(\sigma)$ generates an irreducible KZ-submodule isomorphic to σ'.

If $r = 0$, then for all $n \geq 0$ the elements $A_n^0(\sigma) + A_{n-1}^1(\sigma)$ generate irreducible KZ-modules isomorphic to σ.

Proof. The first statement is [BL, Proposition 14(2)]. It is easy to check that A_n^0 and A_n^1 are the elements denoted ψ_n and ψ_{-n-1}, respectively, in [BL]. Moreover, by [BL, Proposition 15(1)] the A_n^0 and A_n^1 are eigenvectors of the I-action. From Lemma 2.2 we see that I acts on the A_n^0 and A_n^1 via the characters χ_σ and χ'_σ, respectively.

It is obvious that $A_0^0(\sigma) = \mathrm{id} \otimes x^r$ generates an irreducible KZ-submodule isomorphic to σ. Moreover, a simple computation shows us that:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ [\lambda] & 1 \end{pmatrix} (\alpha \otimes y^r) &= g^0_{1,[\lambda^{-1}]} \otimes (-\lambda^{-1})^r x^r \text{ if } \lambda \in k^* \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\alpha \otimes y^r) &= g^0_{1,0} \otimes (-1)^w x^r. \end{aligned} \tag{6}$$

where $\lambda \in k^*$ in the first displayed line. It is clear that the p elements above, together with $A_0^1 = \alpha \otimes y^r$ itself, span the KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma$ generated by $\alpha \otimes y^r$, since $\left\{ \begin{pmatrix} 1 & 0 \\ [\lambda] & 1 \end{pmatrix} : \lambda \in k \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is a set of coset representatives of K/I. All these $p+1$ elements are obviously linearly independent.

By Frobenius reciprocity, we obtain a map $a : \mathrm{Ind}_I^K \chi'_\sigma \to \mathrm{ind}_{KZ}^G \sigma$ of K-modules. From the explicit description of the Frobenius reciprocity map given in [BL, Section 2.1], we see that $a(f_0(\chi'_\sigma)) = \alpha \otimes y^r$ and that the image of a is the KZ-submodule generated by $\alpha \otimes y^r$. By dimension considerations, a must be an isomorphism onto its image.

Observe by Lemma 2.1 that if $r > 0$, then $A_n^0 = T^n(\mathrm{id} \otimes x^r)$ and $A_n^1 = T^n(\alpha \otimes x^r)$ for all $n \geq 0$. Since T is an injective map of G-modules (the injectivity follows from an inspection of the formulae of Lemma 2.1, or is proved in much greater generality in [Her]) our claim about the A_n^0 and A_n^1 follows.

Now consider the case $r = p - 1$. We have already shown that $A_1^0(\sigma) + A_0^1(\sigma)$ is an $I(1)$-invariant, and an explicit computation that is left to the reader verifies that this element is an eigenvector for the actions of the coset representatives of K/I. Since $A_n^0(\sigma) + A_{n-1}^1(\sigma) = T^{n-1}(A_1^0(\sigma) + A_0^1(\sigma))$, the proof is complete.

Finally, suppose $r = 0$. For integers $i, j \in \mathbb{Z}$, we consider the mod 2 Kronecker delta function: $\delta_{i,j} = 1$ if i and j have the same parity, and $\delta_{i,j} = 0$ otherwise. Then one easily deduces by induction from Lemma 2.1 that the following identities hold in $\mathrm{ind}_{KZ}^G \sigma$ for all $m \geq 0$:

$$T^m(\mathrm{id} \otimes 1) = \sum_{i=0}^{m} \left(\sum_{\mu \in I_i} g_{i,\mu}^{1-\delta_{i,m}} \otimes 1 \right). \tag{7}$$

Since $A_0^0(\sigma) = \mathrm{id} \otimes 1$ generates a one-dimensional KZ-submodule isomorphic to σ, and hence so does each $T^m(A_0^0)$, the same is also true of $A_1^0 + A_0^1 = T(A_0^0)$ and of $A_n^0 + A_{n-1}^1 = T^n(A_0^0) - T^{n-2}(A_0^0)$ for each $n \geq 2$. $\square$

The following fact will be useful to us in the sequel.

Corollary 2.6. *Let W be any G-module, and let $X \in W$. Suppose that $X \in W^{I(1)}$ and that the KZ-submodule of W generated by X is irreducible and isomorphic to a Serre weight $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$. Define the elements*

$$\begin{aligned} X'_n &= \sum_{\mu \in I_n} g^0_{n,\mu} X \\ X''_n &= \sum_{\mu \in I_n} g^1_{n,\mu} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X \end{aligned}$$

Then for each $n \geq 0$, the elements X'_n and X''_n are $I(1)$-invariants. Moreover, if $r > 0$ and $X'_n \neq 0$, then X'_n generates a KZ-submodule of W that is irreducible and isomorphic to σ. If $r = 0$, then for each $n \geq 0$ the KZ-submodule of W generated by $X'_n + (-1)^w X''_{n-1}$ is either trivial or isomorphic to σ.

Proof. By assumption we have an embedding of KZ-modules $\sigma \hookrightarrow W$ whose image is the KZ-submodule generated by X. Note that every element of the one-dimensional space $\sigma^{I(1)}$ must map to a scalar multiple of X. By Frobenius reciprocity, we obtain a map of G-modules $\Phi : \mathrm{ind}_{KZ}^G \sigma \to W$ such that $\Phi(\mathrm{id} \otimes x^r) = X$. But now it is clear that $X'_n = \Phi(A_n^0(\sigma))$ and $X''_n = (-1)^w \Phi(A_n^1(\sigma))$ for all $n \geq 0$, and our claim follows from Proposition 2.5. The argument for $r = 0$ is the same. □

2.3. **Pro-p-Iwahori invariants of the universal supersingular quotients.** For each Serre weight $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ we define $r'(\sigma) = \lceil r/2 \rceil - 1$. In other words, $r'(\sigma)$ is the largest integer that is strictly smaller than $r/2$. We write r' for $r'(\sigma)$ when there can be no confusion about the weight σ. For each $0 \leq i \leq r$ we define an element $J_i \in \mathrm{ind}_{KZ}^G \sigma$ as follows:

$$J_i = \sum_{\mu \in I_1} g_{1,\mu}^0 \otimes x^{r-i} y^i. \tag{8}$$

For each $0 \leq i \leq r'(\sigma)$ consider the subspace $\tilde{\mathcal{J}}_i \subset \mathrm{ind}_{KZ}^G \sigma_0$ generated as a G-module by $\{J_0, J_1, \ldots, J_i\}$. Observe that if $r > 0$, then $J_0 = X_1^0(\sigma) = T(\mathrm{id} \otimes x^r)$, so that $\tilde{\mathcal{J}}_0 = T(\mathrm{ind}_{KZ}^G \sigma)$.

Lemma 2.7. *Let $0 \leq i \leq r'(\sigma)$. The following identities hold in $\mathrm{ind}_{KZ}^G \sigma$:*

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{1,\nu}^0 \otimes (1 - \lambda\nu)^{r-2i} x^{r-i} y^i + (-1)^i \alpha \otimes \lambda^{r-2i} x^i y^{r-i} \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{1,\nu}^0 \otimes (-1)^{w+r-i} \nu^{r-2i} x^{r-i} y^i + \alpha \otimes (-1)^w x^i y^{r-i} \\
\alpha \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{1,\nu}^1 \otimes (-1)^{r-i} (\nu - \lambda)^{r-2i} x^i y^{r-i} + \mathrm{id} \otimes x^{r-i} y^i \\
\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{1,\nu}^1 \otimes (-1)^w x^i y^{r-i} \\
g_{n,\mu}^0 \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{n+1,\mu+\pi^n\nu}^0 \otimes (1 - \lambda\nu)^{r-2i} x^{r-i} y^i + (-1)^i g_{n-1,[\mu]_{n-1}}^0 \otimes \lambda^{r-2i} x^i (\mu_{n-1} x + y)^{r-i} \\
g_{n,\mu}^0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{n+1,\mu+\pi^n\nu}^0 \otimes (-1)^{w+r-i} \nu^{r-2i} x^{r-i} y^i + g_{n-1,[\mu]_{n-1}}^0 \otimes (-1)^w x^i (\mu_{n-1} x + y)^{r-i} \\
g_{n,\mu}^1 \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{n+1,\mu+\pi^n\nu}^1 \otimes (-1)^{r-i} (\nu - \lambda)^{r-2i} x^i y^{r-i} + g_{n-1,[\mu]_{n-1}}^1 \otimes (x + \mu_{n-1} y)^{r-i} y^i \\
g_{n,\mu}^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J_i &= \sum_{\nu \in I_1} g_{n+1,\mu+\pi^n\nu}^1 \otimes (-1)^w x^i y^{r-i}
\end{aligned}$$

Proof. Let $a, b, c, d \in \mathcal{O}$ and $\mu \in I_1$. Then a calculation using (3) shows that

$$\begin{pmatrix} 1+a\pi & b \\ c\pi & 1+d\pi \end{pmatrix} \begin{pmatrix} \pi & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi & [\mu+b_0] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & [b_1 + a_0\mu - (b_0+\mu)(d_0+\mu c_0)] \\ 0 & 1 \end{pmatrix} z, \tag{9}$$

where $z \in K(1)$. Also we observe that if $\lambda, \mu \in I_1$ and if $1+\lambda\mu \in \mathcal{O}^*$, then

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \pi & \mu \\ 0 & 1 \end{pmatrix} = g^0_{1,[\mu(1+\lambda\mu)^{-1}]} \begin{pmatrix} (1+\lambda\mu)^{-1} & 0 \\ 0 & 1+\lambda\mu \end{pmatrix} z,$$

where $z \in K(1)$. On the other hand, if $1+\lambda\mu$ is not invertible in $\mathcal{O}$, then $\mu = -\lambda^{-1}$ and

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \pi & \mu \\ 0 & 1 \end{pmatrix} = \alpha \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z,$$

where again $z \in K(1)$. The lemma follows from these observations by straightforward computation. □

Remark 2.8. We observe from the above that for any $n \geq 1$ and any $0 \leq i \leq r'$, any element of the form

$$\sum_{\nu \in I_n} g^0_{n,\nu} \otimes Q(\nu) x^{r-i} y^i,$$

where $Q(\nu)$ has degree at most $r-2i-1$ in the variable ν_{n-1}, lies in the submodule of $\mathrm{ind}_{KZ}^G \sigma$ generated by J_i.

Lemma 2.9. *Suppose that $0 \leq i \leq r'$. The image of J_i in the quotient $\mathrm{ind}_{KZ}^G \sigma / \tilde{\mathcal{J}}_{i-1}$ is invariant under $I(1)$.*

Proof. In the statement of the lemma we implicitly take $\tilde{\mathcal{J}}_{-1}$ to be trivial. If $i = 0$, then the claim is true by the observation above that $J_0 = T(\mathrm{id} \otimes x^r)$, since $\mathrm{id} \otimes x^r \in (\mathrm{ind}_{KZ}^G \sigma)^{I(1)}$. Thus we may assume that $i \geq 1$ and proceed by induction on i.

Suppose we know that $\gamma J_j - J_j \in \tilde{\mathcal{J}}_{j-1}$ for all $\gamma \in I(1)$ and all $j < i$. Then it follows from Lemma 2.7 that the KZ-submodule of $\mathrm{ind}_{KZ}^G / \tilde{\mathcal{J}}_{j-1}$ generated by the image of J_j consists precisely of the images of the elements of the form

$$\sum_{\nu \in I_1} g^0_{1,\nu} \otimes \left(\sum_{m=0}^{r-2j} c_m \nu_m \right) x^{r-j} y^j + (-1)^{r-j} \alpha \otimes c_{r-2j} x^j y^{r-j},$$

where $c_0, \ldots, c_{r-2j} \in \overline{\mathbb{F}}_p$. Now let $a, b, c, d \in \mathcal{O}$ and consider the element

$$\gamma = \begin{pmatrix} 1+\pi a & b \\ \pi c & 1+\pi d \end{pmatrix} \in I(1). \tag{10}$$

It follows from (9) that the following holds in $\mathrm{ind}_{KZ}^G \sigma_0$:

$$\gamma J_i = \sum_{\nu \in I_1} g^0_{1,\nu} \otimes x^{r-i} (P_1^\gamma(\nu) x + y)^i,$$

where $P_1^\gamma(\nu) = [(b_1 - a_0 b_0) + (a_0 - d_0 + b_0 c_0)\nu - c_0 \nu^2]$. Hence,

$$\gamma J_i - J_i = \sum_{j=0}^{i-1} \binom{i}{j} \sum_{\nu \in I_1} g^0_{1,\nu} \otimes (P_1^\gamma(\nu))^{i-j} x^{r-j} y^j.$$

Since $(P_1^\gamma(\nu))^{i-j}$ is a polynomial of degree $2(i-j)$ in the variable ν and since $2(i-j) < r-2j$, we see that $\sum_{\nu\in I_1} g_{1,\nu}^0 \otimes (P_1^\gamma(\nu))^{i-j}x^{r-j}y^j$ lies already in the KZ-submodule generated by J_j for $1 \leq j \leq i-1$. Finally, $\sum_{\nu\in I_1} g_{1,\nu}^0 \otimes (P_1^\gamma(\nu))^i x^r \in T(\mathrm{ind}_{KZ}^G\sigma_0)$ because $2i < r$. $\square$

We can now start computing the space of $I(1)$-invariants of the universal supersingular quotient $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$. For every $n \geq 0$, define the following elements of $\mathrm{ind}_{KZ}^G\sigma$:

$$\begin{aligned}
J_n^0(\sigma) &= \sum_{\mu\in I_{n+1}} g_{n+1,\mu}^0 \otimes x^{r-1}y \\
J_n^1(\sigma) &= \sum_{\mu\in I_{n+1}} g_{n+1,\mu}^1 \otimes xy^{r-1} \\
Q_n^0(\sigma) &= \sum_{\mu\in I_{n+2}} g_{n+2,\mu}^0 \otimes \mu_{n+1}^{r+1}x^r \\
Q_n^1(\sigma) &= \sum_{\mu\in I_{n+2}} g_{n+2,\mu}^1 \otimes \mu_{n+1}^{r+1}y^r. \qquad (11)
\end{aligned}$$

Here we have implicitly assumed that these definitions make sense. Namely, the $J_n^*(\sigma)$ are defined when $r > 0$ (for $* \in \{0,1\}$) and the $Q_n^*(\sigma)$ are defined when $r < p-1$.

Proposition 2.10. *Suppose that $0 \leq i \leq r'(\sigma)$. The KZ-submodule of $\mathrm{ind}_{KZ}^G\sigma/\tilde{\mathcal{J}}_{i-1}$ generated by the image of J_i is irreducible and isomorphic to the Serre weight $\det^{w+i}\otimes\mathrm{Sym}^{r-2i}\overline{\mathbb{F}}_p^2$.*

Proof. The image of J_i is an $I(1)$-invariant by Lemma 2.9. As in the proof of Proposition 2.5, we observe that the KZ-submodule generated by J_i is spanned by the image in $\mathrm{ind}_{KZ}^G\sigma/\tilde{\mathcal{J}}_{i-1}$ of the set $\left\{\begin{pmatrix}1&0\\ \lambda&1\end{pmatrix}J_i : \lambda\in I_1\right\}\cup\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}J_i\right\}$. However, it follows from the explicit formulae of Lemma 2.7 that

$$\sum_{\lambda\in I_1}\lambda^{p-r+2i-1}\begin{pmatrix}1&0\\ \lambda&1\end{pmatrix}J_i + (-1)^{w+i+1}\begin{pmatrix}0&1\\1&0\end{pmatrix}J_i = 0.$$

This identity holds already in $\mathrm{ind}_{KZ}^G\sigma$ and hence in the quotient $\mathrm{ind}_{KZ}^G\sigma/\tilde{\mathcal{J}}_{i-1}$. Thus the KZ-submodule generated by J_i has dimension strictly smaller than $p+1$. Since I acts on the image of J_i via the character $\chi : \mathrm{diag}(a,d) \mapsto (ad)^w a^{r-i}d^i$ (cf. Lemma 2.2), by Frobenius reciprocity we obtain a map of KZ-modules $\mathrm{ind}_{IZ}^{KZ}\chi \to \mathrm{ind}_{KZ}^G\sigma/\tilde{\mathcal{J}}_{i-1}$ whose image is the KZ-module generated by J_i. Since $\mathrm{ind}_{IZ}^{KZ}\chi$ has length two and dimension $p+1$, it follows that this image must be irreducible. Finally we observe that, in view of the hypothesis on r, the only Serre weight whose $I(1)$-invariants are acted on by I via the character χ is $\det^{w+i}\otimes\mathrm{Sym}^{r-2i}\overline{\mathbb{F}}_p^2$. $\square$

Proposition 2.11. *For any $n \geq 2$ and any Serre weight σ, define the element*

$$J'_n = \sum_{\mu\in I_n} g_{n,\mu}^0 \otimes x^{r-r'(\sigma)-1}y^{r'(\sigma)+1} \in \mathrm{ind}_{KZ}^G\sigma. \qquad (12)$$

Then the image of J'_n in the quotient $\mathrm{ind}_{KZ}^G\sigma/\tilde{\mathcal{J}}_{r'(\sigma)}$ is invariant under the action of $I(1)$. Moreover, the KZ-submodule of $\mathrm{ind}_{KZ}^G\sigma/\tilde{\mathcal{J}}_{r'(\sigma)}$ generated by J'_n is irreducible. It is isomorphic to the Serre weight $\det^{w+r'(\sigma)+1}\otimes\mathrm{Sym}^{p-2}\overline{\mathbb{F}}_p^2$ if r is odd, and to $\det^{w+r'(\sigma)+1}\otimes\mathrm{Sym}^{p-1}\overline{\mathbb{F}}_p^2$ if r is even.

Proof. We first prove that the image of J_2' is an $I(1)$-invaraint. It suffices to show invariance under the action of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1+\pi a & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ \pi c & 1 \end{pmatrix}$, for arbitrary $a, b, c \in \mathcal{O}$, since it is easy to see that such matrices generate $I(1)$ modulo its center. One readily computes that

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} J_2' - J_2' = \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \sum_{i=0}^{r'} \binom{r'+1}{i} (b_2 + P(\mu_0, b_0))^{r'+1-i} x^{r-i} y^i \tag{13}$$

$$\tag{14}$$

$$\begin{pmatrix} 1 & 0 \\ \pi c & 1 \end{pmatrix} J_2' - J_2' = \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \sum_{i=0}^{r'} (-1)^{r'+1-i} \binom{r'+1}{i} (2c_0\mu_0\mu_1 + c_0^2\mu_0^3 + c_1\mu_0^2)^{r'+1-i} x^{r-i} y^i$$

$$\begin{pmatrix} 1+\pi a & 0 \\ 0 & 1 \end{pmatrix} J_2' - J_2' = \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \sum_{i=0}^{r'} \binom{r'+1}{i} (a_0\mu_1 + (a_1 - a_0^2)\mu_0)^{r'+1-i} x^{r-i} y^i, \tag{15}$$

where $P(\mu_0, b_0)$ is a polynomial in the variables μ_0 and b_0 that is uniformly zero unless $e = 2$. Since $b_2 + P(\mu_0, b_0)$ is independent of μ_1, it follows from Remark 2.8 that the right-hand side of (13) is always contained in $\tilde{\mathcal{J}}_{r'}$. Moreover, we note that for all $0 \leq i \leq r'$, the coefficient of $x^{r-i}y^i$ on the right-hand sides of (14) and of (15) has degree $r' + 1 + i$ in the variable μ_1. If $i < r'$, and also if $i = r'$ and r is even, then we have that $r' + 1 - i < r - 2i$, and hence the relevant term is contained in $\tilde{\mathcal{J}}_i$ by Remark 2.8. If r is odd, then $r - 2r' = 1$ and a computation deduces the following from Lemma 2.7:

$$\sum_{\substack{\lambda \in k \\ \mu_0 \in k}} 2c_0\mu_0\lambda^{p-2} g_{1,[\mu_0]}^0 \begin{pmatrix} 1 & 0 \\ [\lambda] & 1 \end{pmatrix} J_{r'} + \sum_{\mu_0 \in k} (c_0^2\mu_0^3 + c_1\mu_0^2) g_{1,[\mu_0]}^0 J_{r'} =$$
$$\sum_{\mu \in I_2} g_{2,\mu}^0 \otimes (2c_0\mu_0\mu_1 + c_0^2\mu_0^3 + c_1\mu_0^2) x^{r-r'} y^{r'} + (-1)^{r'+1} \mathrm{id} \otimes \sum_{\mu_0 \in k} 2c_0\mu_0(\mu_0 x + y)^{r-r'}.$$

Since $r < p$, we see that if $p > 3$ then $r - r' + 1 < p - 1$ and hence the second summand on the right-hand side vanishes. Since the left-hand side is contained in $\tilde{\mathcal{J}}_{r'}$, so is the first summand on the right-hand side. Thus the right-hand side of (14) is contained in $\tilde{\mathcal{J}}_{r'}$ also when r is odd, and the right-hand side of (15) is dealt with similarly. Therefore $J_2' \in (\mathrm{ind}_{KZ}^G \sigma / \tilde{\mathcal{J}}_{r'(\sigma)})^{I(1)}$. It now remains to prove that the KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma / \tilde{\mathcal{J}}_{r'(\sigma)}$ generated by J_2' is irreducible, since the rest of the proposition will then follow immediately from Corollary 2.6.

We argue as in the proof of Proposition 2.10. Indeed, we compute that for all $\lambda \in I_1$ we have

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} J_2' = \sum_{\nu \in I_1} g_{1,\nu}^1 \otimes (-1)^{r'+1} \lambda^{p-1+r-2(r'+1)} (x - \lambda^{p-2}\nu_0^2 y)^{r'+1} y^{r-r'-1} +$$
$$\sum_{\nu \in I_2} g_{2,\nu}^0 \otimes (1 - \lambda\nu_0)^{p-1+r-2(r'+1)} x^{r-r'-1} (y - (\lambda\nu_1^2(1-\lambda\nu_0)^{p-2} + \nu_0(1-\lambda\nu_0)P'(\lambda,\nu_0))x)^{r'+1}.$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J_2' = \sum_{\nu \in I_2} g_{2,\nu}^0 \otimes (-1)^{w+r-r'-1} \nu_0^{p-1+r-2(r'+1)} x^{r-r'-1} (\nu_0^{p-2}\nu_1^2 x + y)^{r'+1} +$$

$$\sum_{\nu\in I_1} g^1_{1,\nu}\otimes(-1)^w x^{r'+1}y^{r-r'-1}.$$

The quantity $P'(\lambda,\nu_0)$ arises from addition in rings of Witt vectors as in [Sch3, Lemma 2.2]. All that concerns us about $P'(\lambda,\nu_0)$ is that it is independent of ν_1, and in fact vanishes unless $e=2$. From this, and from an inspection of the exponents of ν_1 in the expressions above and Remark 2.8, it follows that the following congruence holds modulo $\tilde{\mathcal{J}}_{r'}$:

$$\begin{aligned}\begin{pmatrix}1&0\\ \lambda&1\end{pmatrix}J'_2 \quad\equiv\quad & \sum_{\nu\in I_2} g^0_{2,\nu}\otimes(1-\lambda\nu_0)^{p-1+r-2(r'+1)}x^{r-r'-1}(y-\lambda\nu_1^2(1-\lambda\nu_0)^{p-2}x)^{r'+1}+\\ & \sum_{\nu\in I_1} g^1_{1,\nu}\otimes(-1)^{r'+1}\lambda^{p-1+r-2(r'+1)}(x-\lambda^{p-2}\nu_0^2y)^{r'+1}y^{r-r'-1}.\end{aligned}$$

It is easy to verify that these formulae satisfy a linear relation:

$$\sum_{\lambda\in k}\lambda^{p-2}\begin{pmatrix}1&0\\ [\lambda]&1\end{pmatrix}J'_2+(-1)^w\begin{pmatrix}0&1\\ 1&0\end{pmatrix}J'_2\equiv 0 \mod \tilde{\mathcal{J}}_{r'}. \tag{16}$$

Exactly as in the proof of Proposition 2.10, we observe that by Frobenius reciprocity the KZ-submodule of $\mathrm{ind}^G_{KZ}\sigma/\tilde{\mathcal{J}}_{r'}$ generated by J'_2 is isomorphic to a quotient of $\mathrm{Ind}^{KZ}_{IZ}\chi$, where I acts on the image of J'_2 via the character χ. Since $\mathrm{Ind}^{KZ}_{IZ}\chi$ has length two, the existence of the linear relation (16) implies that this quotient is irreducible. By Lemma 2.2, the character χ is $\mathrm{diag}(a,d)\mapsto(\overline{a}\overline{d})^{w+r'+1}\overline{a}^{p-1+r-2(r'+1)}$, where $a,d\in\mathcal{O}^*$. If r is odd, then the unique irreducible quotient of $\mathrm{Ind}^{KZ}_{IZ}\chi$ is the Serre weight $\det^{w+r'+1}\mathrm{Sym}^{p-2}\overline{\mathbb{F}}_p^2$. If r is even, then χ factors through the determinant and the principal series $\mathrm{Ind}^{KZ}_{IZ}\chi$ is the direct sum of two irreducible constituents, namely $\det^{w+r'+1}$ and $\det^{w+r'+1}\otimes\mathrm{Sym}^{p-1}\overline{\mathbb{F}}_p^2$. Now (16) and Lemma 2.4 show that the KZ-submodule generated by J'_2 is isomorphic to the second of these. □

Corollary 2.12. *Suppose that $\sigma=\det^w\otimes\mathrm{Sym}^r\overline{\mathbb{F}}_p^2$ is a Serre weight such that $r\in\{1,2\}$. Then for all $n\geq 1$, the image of $J^0_n(\sigma)$ in the quotient $\mathrm{ind}^G_{KZ}\sigma/T(\mathrm{ind}^G_{KZ}\sigma)$ is an $I(1)$-invariant and generates an irreducible KZ-submodule.*

Proof. The hypothesis on r implies that $r'=0$, hence that $J^0_n(\sigma)=J'_{n+1}$ for all $n\geq 1$ and that $\tilde{\mathcal{J}}_{r'}=T(\mathrm{ind}^G_{KZ}\sigma)$ in view of the observation made after (8). Our claim is now immediate from Proposition 2.11. □

Remark 2.13. We note for future use that if $r\in\{0,1\}$, then the image of $J^0_0(\sigma)=J_1(\sigma)$ in the universal supersingular quotient $\mathrm{ind}^G_{KZ}\sigma/T(\mathrm{ind}^G_{KZ}\sigma)$ is not an $I(1)$-invariant. Indeed, we observe that

$$\begin{pmatrix}1&0\\ -\pi&1\end{pmatrix}J^0_1(\sigma)-J^0_1(\sigma)=\sum_{\mu\in I_1}g^0_{1,\mu}\otimes\mu_0^2x^r. \tag{17}$$

If $r=1$, then this difference is obviously not contained in $T(\mathrm{ind}^G_{KZ}\sigma)$ because $2>r$. If $r=2$, then we see from Lemma 2.1 that the only element of $T(\mathrm{ind}^G_{KZ}\sigma)$ whose leading term is equal to the right-hand side of (17) is $T(\mathrm{id}\otimes y^2)$. But

$$T(\mathrm{id}\otimes y^2)=\sum_{\mu\in I_1}g^0_{1,\mu}\otimes\mu_0^2x^r+\alpha\otimes y^2$$

and so again the right-hand side of (17) is not contained in $T(\mathrm{ind}^G_{KZ}\sigma)$. However, a modification of $J_1(\sigma)$ is an $I(1)$-invariant if $r=2$, as we will now see.

Lemma 2.14. *Let $\sigma = \det^w \otimes \mathrm{Sym}^2 \overline{\mathbb{F}}_p^2$, and for each $n \geq 0$ consider the elements*

$$\begin{aligned} \hat{J}_0^0(\sigma) &= \sum_{\mu \in I_1} g_{1,\mu}^0 \otimes xy - \alpha \otimes xy \\ \hat{J}_0^1(\sigma) &= \sum_{\mu \in I_1} g_{1,\mu}^1 \otimes xy - \mathrm{id} \otimes xy. \end{aligned}$$

The image of $\hat{J}_0^0(\sigma)$ in $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ is an $I(1)$-invariant, and it generates a one-dimensional KZ-submodule isomorphic to the Serre weight $\det^{w+1}$.

Proof. The proof is an explicit verification of the sort that is by now familiar to us. □

Remark 2.15. It is easy to check that

$$\sum_{\mu \in I_n} g_{n,\mu}^0 \hat{J}_0^0(\sigma) = J_n^0(\sigma)$$

for all $n \geq 1$, and hence that Corollary 2.6 does not enable us to produce any new $I(1)$-invariants from $\hat{J}_0^0(\sigma)$. Note that $\hat{J}_0^1(\sigma) = \beta \hat{J}_0^0(\sigma)$ is an $I(1)$-invariant.

2.4. **A duality.** We will now prove a generalization of Breuil's result [Bre1, Corollaire 4.1.3] on intertwining isomorphisms between universal supersingular representations for $F = \mathbb{Q}_p$. It will be used to produce more $I(1)$-invariants.

Lemma 2.16. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be any Serre weight. Then the image of $\alpha \otimes y^r$ in the quotient $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ is invariant under the action of $I(1)$ and generates an irreducible KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ that is isomorphic to $\sigma' = \det^{w+r} \otimes \mathrm{Sym}^{p-r-1} \overline{\mathbb{F}}_p^2$.*

Proof. This is proved for $e = 1$ in [Bre1, Prop. 4.1.2] (the same argument works in general and is slightly different from the proof given below) and stated in [Sch3, Lemma 3.6], but we will prove it here for the reader's convenience.

Since obviously $\mathrm{id} \otimes x^r \in (\mathrm{ind}_{KZ}^G \sigma)^{I(1)}$ and β normalizes $I(1)$, we find that $\alpha \otimes y^r = \beta(\mathrm{id} \otimes x^r) \in (\mathrm{ind}_{KZ}^G \sigma)^{I(1)}$. Moreover, it was shown in Proposition 2.5 that the KZ-subspace of $\mathrm{ind}_{KZ}^G \sigma$ generated by $\alpha \otimes y^r$ is isomorphic to the principal series $\mathrm{Ind}_I^K \chi_{\sigma'}$. On the other hand, it is clear from inspecting Lemma 2.1 that the right-hand sides of (6) are linearly dependent modulo $T(\mathrm{ind}_{KZ}^G \sigma)$, and hence the KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ generated by the image of $\alpha \otimes y^r$ is necessarily irreducible. If $r \notin \{0, p-1\}$, then the Serre weight to which this module is isomorphic is already uniquely determined by the action of I on $\alpha \otimes y^r$.

In the remaining cases, the action of I allows for two possibilities for the submodule generated by the image of $\alpha \otimes y^r$: it could be $\det^w \otimes \mathrm{Sym}^{p-1} \overline{\mathbb{F}}_p^2$ or $\det^w$. If $r = p-1$, then we observe by [Sch3, Lemma 2.1] that any linear combination of the right-hand sides of (6) has the form $\sum_{\mu \in I_1} g_{1,\mu}^0 \otimes P(\mu_0) x^r$, where $P(\mu_0)$ is a polynomial of degree at most $p-1$ in the variable μ_0, and hence is the leading term of an element of $T(S_0^0)$. It thus follows from Lemma 2.1 of this paper that all the elements appearing on the right-hand side of (6) are congruent modulo $T(\mathrm{ind}_{KZ}^G \sigma)$ to a scalar multiple of $\alpha \otimes y^r$, and hence the KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ is one-dimensional. If $r = 0$, then it is easy to see that $g_{1,[\lambda_1]}^0 \otimes 1$ and $g_{1,[\lambda_2]}^0 \otimes 1$ are linearly independent modulo $T(\mathrm{ind}_{KZ}^G \sigma)$ if $\lambda_1 \neq \lambda_2$. It follows that the KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ generated by $\alpha \otimes y^r$ has dimension greater than 1, and hence it must be isomorphic to σ'. □

Corollary 2.17. *If $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ is a Serre weight, set $\sigma' = \det^{w+r} \otimes \mathrm{Sym}^{p-r-1} \overline{\mathbb{F}}_p^2$. The map $\mathrm{ind}_{KZ}^G \sigma' \to \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ arising by Frobenius reciprocity from the image of $\alpha \otimes y^r$ induces a natural isomorphism of G-modules*

$$\mathrm{ind}_{KZ}^G \sigma' / T(\mathrm{ind}_{KZ}^G \sigma') \simeq \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma). \tag{18}$$

Proof. Indeed, by Lemma 2.16, the image of $\alpha \otimes y^r$ in $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ generates an irreducible KZ-submodule isomorphic to $\sigma' = \det^{w+r} \otimes \mathrm{Sym}^{p-r-1} \overline{\mathbb{F}}_p^2$. The associated inclusion $\sigma' \hookrightarrow \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ of KZ-modules gives rise to a map $\Phi_\sigma : \mathrm{ind}_{KZ}^G \sigma' \to \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ of G-modules by Frobenius reciprocity. We write elements of the underlying vector space of σ' as homogeneous polynomials of degree $p-r-1$ in the variables x' and y'. Observe that $\Phi_\sigma(\mathrm{id} \otimes (x')^{p-r-1}) = \alpha \otimes y^r$, where by our usual abuse of notation the right-hand side of this equality denotes the image of $\alpha \otimes y^r$ in $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$. Moreover, if $r < p-1$, then we compute that

$$\begin{aligned}\Phi_\sigma(T(\mathrm{id} \otimes (x')^{p-r-1})) = \quad & \Phi_\sigma\left(\sum_{\mu \in I_1} g_{1,\mu}^0 \otimes (x')^{p-r-1}\right) = \Phi_\sigma\left(\sum_{\mu \in I_1} g_{1,\mu}^0 (\mathrm{id} \otimes (x')^{p-r-1})\right) = \\ & \sum_{\mu \in I_1} g_{1,\mu}^0 (\alpha \otimes y^r) = \mathrm{id} \otimes \sum_{\mu \in k} (\mu x + y)^r = 0\end{aligned}$$

In the remaining case $r = p-1$, we find similarly that

$$\Phi_\sigma(T(\mathrm{id} \otimes 1)) = \Phi_\sigma\left(\sum_{\mu \in I_1} g_{1,\mu}^0 \otimes 1 + \alpha \otimes 1\right) = 0.$$

Therefore Φ_σ factors through $\mathrm{ind}_{KZ}^G \sigma' / T(\mathrm{ind}_{KZ}^G \sigma')$ in all cases. This induces a map $\mathrm{ind}_{KZ}^G \sigma' / T(\mathrm{ind}_{KZ}^G \sigma') \to \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$, which we also denote by Φ_σ, and it provides the requisite isomorphism. Indeed, one easily checks that the endomorphism of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ given by the composition $\Phi_\sigma \circ \Phi_{\sigma'}$ is multiplication by a non-zero scalar, and therefore that Φ_σ is an isomorphism. □

Remark 2.18. It follows from the description of the K-socle of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ given in Corollary 3.9 that, up to scalar multiplication, there is a unique isomorphism as in (18).

Proposition 2.19. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight. If $r < p-3$, then the image of $Q_0^0(\sigma) = \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \mu_1^{r+1} x^r$ is an $I(1)$-invariant in the quotient $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ and generates an irreducible KZ-submodule isomorphic to $\det^{w+r+1} \otimes \mathrm{Sym}^{p-r-3} \overline{\mathbb{F}}_p^2$.*

If $r \in \{p-3, p-2\}$, then the image of $Q_1^0(\sigma) = \sum_{\mu \in I_3} g_{3,\mu}^0 \otimes \mu_2^{r+1} x^r$ is an $I(1)$-invariant in the quotient $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$. It generates an irreducible KZ-submodule, which is isomorphic to $\det^{w-1} \otimes \mathrm{Sym}^{p-1} \overline{\mathbb{F}}_p^2$ if $r = p-3$, and to $\det^w \otimes \mathrm{Sym}^1 \overline{\mathbb{F}}_p^2 = \sigma$ if $r = p-2$.

Proof. Continuing to use the notation introduced in the proof of Corollary 2.17, we observe that if $n \geq 1$, then

$$\begin{aligned}\Phi_\sigma(J_n^0(\sigma')) = \quad & \Phi_\sigma\left(-\frac{1}{p-r-1}\sum_{\substack{\mu\in I_{n+1}\\ \lambda\in k^*}}\lambda^{p-2}g_{n+1,\mu}^0\begin{pmatrix}1 & 0\\ [\lambda] & 1\end{pmatrix}(\mathrm{id}\otimes(x')^{p-r-1})\right) = \\ & -\frac{1}{p-r-1}\sum_{\substack{\mu\in I_{n+1}\\ \lambda\in k^*}}\lambda^{p-2}g_{n+1,\mu}^0\begin{pmatrix}1 & 0\\ [\lambda] & 1\end{pmatrix}(\alpha\otimes y^r) = \\ & \frac{(-1)^{r+1}}{p-r-1}\sum_{\substack{\mu\in I_{n+1}\\ \lambda\in k^*}}g_{n+2,\mu+\pi^{n+1}[\lambda]}^0\otimes\lambda^{r+1}x^r = \frac{(-1)^{r+1}}{p-r-1}Q_n^0(\sigma). \qquad (19)\end{aligned}$$

Since Φ_σ is an isomorphism of G-modules, our claim now follows from the corresponding results about $J_n^0(\sigma')$, namely from Lemma 2.9 and Proposition 2.10 (in the case $i = 1$) if $p < r - 3$ and from Corollary 2.12 when $r \in \{p-3, p-2\}$. □

Remark 2.20. In the case $r < p-3$, the previous proposition follows from [Sch3, Lemma 1.1]. Note that in that statement of that lemma the weaker condition $r < p-2$ is assumed; this is a typo, as one easily checks that the proof requires $r < p-3$ to go through. Moreover, a simple computation as in (19) shows that $\Phi_\sigma(Q_n^0(\sigma')) = -rJ_n^0(\sigma)$. So in fact we could have deduced the result that $J_n^0(\sigma) \in (\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))^{I(1)}$ and a characterization of the KZ-module generated by it from the results of [Sch3] by the method of the proof of Proposition 2.19. However, the cases $i > 1$ of Proposition 2.10 do not follow from the results of [Sch3]. Indeed, the quotients V_i appearing in [Sch3, Section 3] are obtained from V_{i-1} by factoring by a proper submodule of the G-module generated by the $I(1)$-invariant $\tilde{X}_i^0$; thus they are not dual to the quotients $\tilde{\mathcal{J}}_i$.

Remark 2.21. Our ambient assumption that $F/\mathbb{Q}_p$ is totally ramified is crucial to the duality of Corollary 2.17. Indeed, Y. Hendel is generalizing some of the results of this paper to arbitrary extensions $F/\mathbb{Q}_p$ for his Bar-Ilan M.Sc. thesis. He shows that Corollary 2.17 fails in general. If σ is a Serre weight for G and $v_\sigma \in V_\sigma$ is a non-zero $I(1)$-invariant, then the K-submodule of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ generated by $\beta(\mathrm{id}\otimes v_\sigma)$ has length $2^f - 1$, where the residue field of F has cardinality p^f. Moreover, σ' does not in general appear in the K-socle of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$.

If e is the ramification index of $F/\mathbb{Q}_p$, then the multiset $\mathcal{D}(\rho)$ of modular Serre weights for a generic Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is the union of e^f perturbations of a set $\mathcal{D}(\rho_0)$ of 2^f modular Serre weights of a representation ρ_0 of $\mathrm{Gal}(\overline{F}/F_0)$, where F_0 is the maximal unramified subextension of $F/\mathbb{Q}_p$ [Sch2]. The proof of the relevant cases of the weight part of Serre's conjecture is completed in [GLS1]. In general, the analogue of the construction of [Sch3] produces quotients whose K-socles contain other Serre weights from $\mathcal{D}(\rho_0)$, whereas the various perturbations of σ arise from analogues of the quotients $\tilde{\mathcal{J}}_i$.

Lemma 2.22. *Suppose that* $\sigma = \det^w \otimes \mathrm{Sym}^{p-3}\overline{\mathbb{F}}_p^2$. *Then the image of*

$$\hat{Q}_0^0(\sigma) = \sum_{\mu\in I_2} g_{2,\mu}^0 \otimes \mu_1^{p-2}x^{p-3} - \sum_{\mu\in I_1} g_{1,\mu}^1 \otimes \mu_0^{p-2}y^{p-3}$$

in $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ *is an* $I(1)$*-invariant and generates a one-dimensional* K*-module.*

Proof. This follows from Lemma 2.14 via the isomorphism Φ_σ. $\square$

2.5. **The space of $I(1)$-invariants of the universal supersingular representations.** We are finally ready to exhibit an explicit basis of the space $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))^{I(1)}$ for any Serre weight.

Definition 2.23. Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight. We define a set $\mathcal{B}_\sigma$ of elements of $\mathrm{ind}_{KZ}^G\sigma$ as follows:

$$\mathcal{B}_\sigma = \begin{cases} \bigcup_{n\geq 0}\{J_n^0, J_n^1, Q_n^0, Q_n^1\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : 2 < r < p-3 \\ \bigcup_{n\geq 1}\{J_n^0, J_n^1\} \cup \{\hat{J}_0^0, \hat{J}_0^1\} \cup \bigcup_{n\geq 0}\{Q_n^0, Q_n^1\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = 2 \\ \bigcup_{n\geq 1}\{J_n^0, J_n^1\} \cup \bigcup_{n\geq 0}\{Q_n^0, Q_n^1\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = 1 \\ \bigcup_{n\geq 0}\{Q_n^0, Q_n^1\} \cup \{\mathrm{id}\otimes 1, \alpha\otimes 1\} & : r = 0 \\ \bigcup_{n\geq 0}\{J_n^0, J_n^1\} \cup \bigcup_{n\geq 1}\{Q_n^0, Q_n^1\} \cup \{\hat{Q}_0^0, \hat{Q}_0^1\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = p-3 \\ \bigcup_{n\geq 0}\{J_n^0, J_n^1\} \cup \bigcup_{n\geq 1}\{Q_n^0, Q_n^1\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = p-2 \\ \bigcup_{n\geq 0}\{J_n^0, J_n^1\} \cup \{\mathrm{id}\otimes x^{p-1}, \alpha\otimes y^{p-1}\} & : r = p-1. \end{cases}$$

Here we recall that the elements J_n^* and Q_n^* were defined in (11), where $* \in \{0,1\}$, that $\hat{J}_0^*$ was defined in Lemma 2.14, and that $\hat{Q}_0^0$ was defined in Lemma 2.22. Keeping our usual conventions, we define $\hat{Q}_0^1 = \beta\hat{Q}_0^0$.

Theorem 2.24. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight. The images in $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ of the elements of $\mathcal{B}_\sigma$ are all distinct. Moreover, they constitute a basis of the subspace $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))^{I(1)}$.*

Proof. We can deduce from an inspection of Lemma 2.1 that the elements of $\mathcal{B}_\sigma$ are linearly independent modulo $T(\mathrm{ind}_{KZ}^G\sigma)$. Indeed, it is not hard to show that the leading term, in the sense of the proof below, of a linear combination of elements of $\mathcal{B}_\sigma$ can never be the leading term of an element of $T(\mathrm{ind}_{KZ}^G\sigma)$. Observe that if $r > 2$, then $J_0^0 = J_1$ is contained in $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))^{I(1)}$ by Lemma 2.9, and the KZ-submodule of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ that it generates is irreducible and isomorphic to $\det^{w+1}\otimes\mathrm{Sym}^{r-2}\overline{\mathbb{F}}_p^2$ by Proposition 2.10. Similarly, if $r < p-3$ then the image of Q_0^0 in $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ is an $I(1)$-invariant and generates an irreducible KZ-submodule isomorphic to $\det^{w+r+1}\otimes\mathrm{Sym}^{p-r-3}\overline{\mathbb{F}}_p^2$ by [Sch3, Lemma 3.1] and [Sch3, Proposition 3.3], respectively, or by Proposition 2.19 of this paper. Note that Q_0^0 is denoted $\tilde{X}_1^0$ in [Sch3].

Observe that $*_n^0 = \sum_{\mu\in I_n} g_{n,\mu}^0 *_0^0$ for $* \in \{J, Q\}$ and all $n \geq 0$. Therefore, the images in $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ of all the J_n^0 (when $r > 2$) and Q_n^0 (when $r < p-3$) are $I(1)$-invariants by Corollary 2.6. Now, $J_n^1 = \beta J_n^0$ and $Q_n^1 = \beta Q_n^0$ for all $n \geq 0$. The J_n^1 and Q_n^1 are therefore also contained in $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))^{I(1)}$ because β normalizes $I(1)$. Finally, $\mathrm{id}\otimes x^r$ and $\alpha\otimes y^r$ are already $I(1)$-invariants in $\mathrm{ind}_{KZ}^G\sigma$, so their images in the quotient $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ are obviously invariant under $I(1)$.

If $r \in \{1,2\}$, then the J_n^0 are $I(1)$-invariants for all $n \geq 1$ by Corollary 2.12, and hence so are the $J_n^1 = \beta J_n^0$. In addition, if $r = 2$, then $\hat{J}_0^0$ and $\hat{J}_0^1$ are $I(1)$-invariants by Lemma 2.14 and the remark following it. If $r \in \{p-2, p-3\}$, then the desired statements about Q_n^0 and Q_n^1, for $n \geq 1$, are deduced as above from the second part of Proposition 2.19. Finally, $\hat{Q}_0^0$ and $\hat{Q}_0^1$ are $I(1)$-invariants when $r = p-3$ by Lemma 2.22.

It remains to show that $\mathcal{B}_\sigma$ spans the space $(\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))^{I(1)}$. For this, we follow the method of [Bre1, Prop. 3.2.1]. Indeed, suppose that $f \in B_n \subset \mathrm{ind}_{KZ}^G$ satisfies $\gamma f - f \in T(\mathrm{ind}_{KZ}^G\sigma)$ for all $\gamma \in I(1)$. We aim to exhibit f as a linear combination of elements of $\mathcal{B}_\sigma$. Write $f = f_n + f'$, where $f_n \in S_n$ and $f' \in B_{n-1}$. Further subdivide $f_n = f_n^0 + f_n^1$, where $f_n^i \in S_n^i$. Then $f_n^0 = \sum_{\mu\in I_n} g_{n,\mu}^0 \otimes v_\mu$ for vectors $v_\mu \in V_\sigma$. Suppose, first of all, that $n \geq 2$. A simple computation shows that

$$\begin{pmatrix} 1 & \pi^n \\ 0 & 1 \end{pmatrix}(g_{n,\mu}^0 \otimes v_\mu) - (g_{n,\mu}^0 \otimes v_\mu) = g_{n,\mu}^0 \otimes \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v_\mu - v_\mu\right). \tag{20}$$

It follows from Lemma 2.1 that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v_\mu - v_\mu \in \overline{\mathbb{F}}_p x^r$ for all $\mu \in I_n$, and therefore that $v_\mu \in \overline{\mathbb{F}}_p x^r + \overline{\mathbb{F}}_p x^{r-1}y$. By [Sch3, Lemma 2.1] we can write

$$v_\mu = c_\mu x^r + d_\mu x^{r-1} y,$$

where $c_\mu = c(\mu_0, \ldots, \mu_{n-1})$ and $d_\mu = d(\mu_0, \ldots, \mu_{n-1})$ are polynomials in which each variable μ_i appears with degree at most $p-1$.

We will start by showing that the polynomial d is constant. If $r = 0$, then obviously $d = 0$. Otherwise, we shall prove by induction on $i \geq 0$ that d_μ is independent of the variable μ_{n-i}. For $i = 0$ there is nothing to prove. Now, assume that d_μ is known to be independent of μ_{n-j} for all $j < i$. Noting that

$$\begin{pmatrix} 1 & \pi^{n-i} \\ 0 & 1 \end{pmatrix} g_{n,\mu}^0 = g_{n,\mu'}^0 \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

where $z \in \mathcal{O}$ and $\mu' \in I_n$ is such that $\mu'_{n-i} = \mu_{n-i} + 1$ and $\mu'_m = \mu_m$ for all $m < n-i$, we observe that

$$\begin{pmatrix} 1 & \pi^{n-i} \\ 0 & 1 \end{pmatrix} f_n^0 - f_n^0 = \sum_{\mu\in I_n} g_{n,\mu}^0 \otimes (c'_\mu x^r + (d_\mu - d_{\mu'}) x^{r-1} y),$$

for some scalars $c'_\mu \in \overline{\mathbb{F}}_p$. By Lemma 2.1 we must have $d_\mu - d_{\mu'} = 0$, and it now follows from the induction hypothesis that d_μ is also independent of the variable μ_{n-i}. Thus the polynomial d is constant. Replacing f by $f - dJ_{n-1}^0$, we may assume without loss of generality that $d = 0$.

We again let σ be an arbitrary Serre weight and observe that

$$\begin{pmatrix} 1 & \pi^{n-1} \\ 0 & 1 \end{pmatrix} f_n^0 - f_n^0 = \sum_{\mu\in I_n} g_{n,\mu}^0 \otimes (c(\mu_0, \ldots, \mu_{n-2}, \mu_{n-1} - 1) - c(\mu_0, \ldots, \mu_{n-2}, \mu_{n-1})) x^r.$$

Since the difference $c(\mu_0, \ldots, \mu_{n-2}, \mu_{n-1} - 1) - c(\mu_0, \ldots, \mu_{n-2}, \mu_{n-1})$ must have degree at most r by Lemma 2.1, it follows that the polynomial c has degree at most $r+1$ in the variable μ_{n-1}. Subtracting a suitable element of $T(\mathrm{ind}_{KZ}^G\sigma)$ from f (cf. Remark 2.8), we may assume that for all $\mu \in I_n$ we have $c_\mu = 0$ if $r = p-1$ and $c_\mu = \tilde{c}_\mu \mu_{n-1}^{r+1}$, where $\tilde{c}$ is a polynomial depending only on the variables $\mu_0, \ldots, \mu_{n-2}$, if $r < p-1$. Observe that we can perform this replacement without changing f_n^1. We now need to show that the polynomial $\tilde{c}$ is constant and shall do it by means of an induction similar to that applied above to d. Indeed, suppose that $\tilde{c}$ is independent of μ_{n-j} for all $j < i$; we have just proved this in the base case $i = 2$. Then

$$\begin{pmatrix} 1 & \pi^{n-i} \\ 0 & 1 \end{pmatrix} f_n^0 - f_n^0 = \sum_{\mu\in I_n} g_{n,\mu}^0 \otimes (\tilde{c}_\mu - \tilde{c}_{\mu'}) \mu_{n-1}^{r+1} x^r,$$

where $\mu'_{n-i} = \mu_{n-i} + 1$ and $\mu'_m = \mu_m$ for all $m < n - i$. Since we must have $\tilde{c}_\mu = \tilde{c}_{\mu'}$ by Lemma 2.1, the same inductive argument as above shows that $\tilde{c}$ must be constant. Thus we have shown that, after modifying f by adding an element of $T(\mathrm{ind}_{KZ}^G \sigma)$, the term f_n^0 is necessarily a linear combination of J_{n-1}^0 and Q_{n-2}^0. Applying the same reasoning to βf, we easily see that f_n^1 is a linear combination of J_{n-1}^1 and Q_{n-2}^1. If $r < p-3$, then we may replace f by $f - f_n$ and iterate the argument to reduce to the case of $f \in B_1$. If $r \in \{p-3, p-2\}$, then we may still reduce to the case $f \in B_2$ using this method.

Suppose now that $r < p - 3$ and that $f \in S_1$. If in addition $r > 2$, then exactly as in the argument above we may reduce to the case where

$$f = c_0 \sum_{\mu \in I_1} g_{1,\mu}^0 \otimes \mu_0^{r+1} x^r + c_1 \sum_{\mu \in I_1} g_{1,\mu}^1 \otimes \mu_0^{r+1} y^r + \mathrm{id} \otimes v^0 + \alpha \otimes v^1 \tag{21}$$

for some scalars $c_0, c_1 \in \overline{\mathbb{F}}_p$ and vectors $v^0, v^1 \in V_\sigma$. If $r = 2$, or if $r = p-3$, then we can still reduce to the case (21) by subtracting scalar multiples of $\hat{J}_0^0$ and $\hat{J}_0^1$, or of $\hat{Q}_0^0$ and $\hat{Q}_0^1$, from f. We verify that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} f - f = c_0 \sum_{\mu \in I_1} g_{1,\mu}^0 \otimes ((\mu_0 - 1)^{r+1} - \mu_0^{r+1}) x^r + \mathrm{id} \otimes \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v^0 - v^0 \right), \tag{22}$$

and if $c_0 \neq 0$ then it is easy to see from Lemma 2.1 that the only element of $T(\mathrm{ind}_{KZ}^G \sigma)$ with the same leading term as the right-hand side of (22) is

$$T\left(\mathrm{id} \otimes (-1)^{r+1} c_0 \sum_{i=0}^{r} \binom{r+1}{i} x^{r-i} y^i \right) =$$
$$c_0 \sum_{\mu \in I_1} g_{1,\mu}^0 \otimes ((\mu_0 - 1)^{r+1} - \mu_0^{r+1}) x^r - \alpha \otimes c_0 (r+1)(\mu_0 x + y)^r,$$

but clearly this is not equal to the right-hand side of (22). Therefore $c_0 = 0$. A similar consideration of the action of the matrix $\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}$ concludes that $c_1 = 0$. Again applying the same two matrices, it is easy to see that v^0 and v^1 must be scalar multiples of x^r and y^r, respectively.

This argument also works for $r = 0$. However, if $r = 1$ then we can only reduce to the situation

$$f = c_0 \sum_{\mu \in I_1} g_{1,\mu}^0 \otimes \mu_0^2 x + c_1 \sum_{\mu \in I_1} g_{1,\mu}^1 \otimes \mu_0^2 y + c'_0 J_0^0(\sigma) + c'_1 J_0^1(\sigma) + \mathrm{id} \otimes v^0 + \alpha \otimes v^1,$$

where $c'_0, c'_1, c_0, c_1 \in \overline{\mathbb{F}}_p$ and $v^0, v^1 \in V_\sigma$. In this case, we observe that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} f - f = c_0 \sum_{\mu \in I_1} g_{1,\mu}^0 \otimes (1 - 2\mu_0) x - c'_1 \sum_{\mu \in I_1} g_{1,\mu}^1 \otimes \mu_0^2 y + \mathrm{id} \otimes \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v^0 - v^0 \right),$$

and it is evident that since the right-hand side of this expression lies in $T(\mathrm{ind}_{KZ}^G \sigma)$ we must have $c'_1 = 0$. Similarly, we show that $c'_0 = 0$ by applying the matrix $\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}$. Now the argument can be completed just as in the previous cases.

It remains to consider the cases $r \in \{p-2, p-1\}$. Since $p > 3$, our claim follows immediately from the isomorphism of Corollary 2.17. $\square$

3. Socles

Using the information about the $I(1)$-invariants of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ that we obtained in the previous section, we can precisely describe the K-socle of the universal supersingular G-modules.

Lemma 3.1. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight, and let $\tau : \mathrm{ind}_{KZ}^G\sigma \to W$ be a non-zero map of G-modules. Then the K-submodule of W generated by $\tau(\alpha \otimes y^r)$ is irreducible if and only if τ factors through the quotient $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$.*

Proof. Note that $\tau(\alpha \otimes y^r) \neq 0$ since $\alpha \otimes y^r$ generates $\mathrm{ind}_{KZ}^G\sigma$, and hence one direction is just Lemma 2.16. To prove the converse, assume that $\tau(\alpha \otimes y^r)$ generates an irreducible K-submodule, which by Lemma 2.16 must be $\det^{w+r} \otimes \mathrm{Sym}^{p-r-1}\overline{\mathbb{F}}_p^2$. If $r > 0$, then

$$\begin{aligned}\tau(T(\mathrm{id} \otimes x^r)) = \quad & \tau\left(\sum_{\mu \in I_1} g_{1,\mu}^0 \otimes x^r\right) = \\ & \tau\left((-1)^r \sum_{\lambda \in k^*} \begin{pmatrix} 1 & 0 \\ [\lambda] & 1 \end{pmatrix} (\alpha \otimes y^r) + (-1)^w \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\alpha \otimes y^r)\right) = \\ & (-1)^r \left(\sum_{\lambda \in k^*} \begin{pmatrix} 1 & 0 \\ [\lambda] & 1 \end{pmatrix} \tau(\alpha \otimes y^r) + (-1)^{w+r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tau(\alpha \otimes y^r)\right) = 0,\end{aligned}$$

where the last equality is Lemma 2.4. The case $r = 0$ is dealt with analogously. Thus $T(\mathrm{ind}_{KZ}^G\sigma) \subseteq \ker\tau$ as required. □

Lemma 3.2. *Let $\sigma = \det^w \otimes \mathrm{Sym}^2 \overline{\mathbb{F}}_p^2$ be a Serre weight. Then for all $n \geq 1$, the images in $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ of*

$$\hat{J}_n^0(\sigma) = \begin{cases} \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes xy - \sum_{\mu \in I_1} g_{1,\mu}^1 \otimes xy + \mathrm{id} \otimes xy & : n = 1 \\ \sum_{\mu \in I_{n+1}} g_{n+1,\mu}^0 \otimes xy - \sum_{\mu \in I_n} g_{n,\mu}^1 \otimes xy = J_n^0 - J_{n-1}^1 & : n \geq 2 \end{cases}$$

generate one-dimensional K-submodules that are isomorphic to the Serre weight $\det^{w+1}$.

Proof. By Lemma 2.14 and Frobenius reciprocity, there is a map of G-modules

$$\Psi : \mathrm{ind}_{KZ}^G \det^{w+1} \to \mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$$

that maps $\mathrm{id} \otimes 1 \in \mathrm{ind}_{KZ}^G \det^{w+1}$ to $\hat{J}_0^0(\sigma)$. Applying the formula (7) from the proof of Proposition 2.5, we see that $\hat{J}_1^0 = \Psi(T(\mathrm{id} \otimes 1))$ and that $\hat{J}_n^0 = \Psi(T^n(\mathrm{id} \otimes 1) - T^{n-2}(\mathrm{id} \otimes 1))$ if $n \geq 2$. Since $\mathrm{id} \otimes 1$ clearly generates a one-dimensional K-submodule of $\mathrm{ind}_{KZ}^G \det^{w+1}$, and since both Ψ and T are maps of G-modules, and since the image of $\hat{J}_n^0$ is non-zero, our claim follows. □

The following lemma is established by the same argument.

Lemma 3.3. *Let $\sigma = \det^w \otimes \mathrm{Sym}^{p-3} \overline{\mathbb{F}}_p^2$ be a Serre weight. The images in $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ of the elements*

$$\hat{Q}_n^0(\sigma) = Q_n^0 - Q_{n-1}^1$$

generate one-dimensional K-submodules that are isomorphic to the Serre weight $\det^{w-1}$.

Given a character $\chi : I \to \overline{\mathbb{F}}_p^*$, define

$$(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))^{I,\chi} = \{x \in (\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))^{I(1)} : ix = \chi(i)x \text{ for all } i \in I\}. \quad (23)$$

Lemma 3.4. *If χ factors through the determinant, then*

$$(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))^{I,\chi} \subseteq \mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)).$$

Proof. Let $x \in (\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))^{I,\chi}$. By Frobenius reciprocity we obtain a map $\mathrm{Ind}_I^K \chi \to \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ of K-modules whose image is the K-submodule generated by x. Now $\mathrm{Ind}_I^K \chi$ is semisimple under our hypothesis on χ by Lemma 2.3, and hence x is contained in a semisimple K-module. □

Corollary 3.5. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight such that $r \in \{2, p-3\}$. If $r = 2$, then the linear span of $\bigcup_{n \geq 1} \{J_n^0, J_n^1\} \cup \{\hat{J}_0^0, \hat{J}_0^1\}$ lies in $\mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))$. If $r = p-3$, then the linear span of $\bigcup_{n \geq 1} \{Q_n^0, Q_n^1\} \cup \{\hat{Q}_0^0, \hat{Q}_0^1\}$ lies in $\mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))$.*

Proof. We will prove the claim for $r = 2$, since the claim for $r = p-3$ will then follow by the isomorphism Φ_σ of Corollary 2.17. By Theorem 2.24 the set $\bigcup_{n \geq 1} \{J_n^0, J_n^1\} \cup \{\hat{J}_0^0, \hat{J}_0^1\}$ consists of $I(1)$-invariants, and by Lemma 2.2 its elements satisfy the hypotheses of the preceding lemma.

Alternatively, it follows from Corollary 2.12 and Lemma 3.2 that

$$\bigcup_{n \geq 1} J_n^0 \cup \bigcup_{n \geq 0} \hat{J}_n^0 \subset \mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)). \quad (24)$$

But this set clearly spans the same subspace as $\bigcup_{n \geq 1} \{J_n^0, J_n^1\} \cup \{\hat{J}_0^0, \hat{J}_0^1\}$. □

Given a character $\chi : I \to \overline{\mathbb{F}}_p^*$, define $\mathcal{W}_\chi$ to be the set of all Serre weights σ satisfying $\chi_\sigma = \chi$, and define $\mathcal{S}_\chi = \bigoplus \{\sigma : \sigma \in \mathcal{W}_\chi\}$. Note that $\mathcal{S}_\chi$ is just a single Serre weight unless χ factors through the determinant, in which case it is the direct sum of two Serre weights.

Lemma 3.6. *Let $\chi : I \to \overline{\mathbb{F}}_p^*$ be a character that does not factor through the determinant. If $A \in \mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))$ is an eigenvector of the I-action on which I acts via χ, then A generates an irreducible K-submodule that is isomorphic to the unique Serre weight in $\mathcal{S}_\chi$.*

Proof. By Frobenius reciprocity, we obtain a K-module map $\mathrm{Ind}_I^K \chi \to \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ whose image is the K-module generated by A. Since this image is contained in the K-socle of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ it is semisimple. Hence it is irreducible, since by Lemma 2.3 $\mathrm{Ind}_I^K \chi$ is a non-semisimple module of length two. □

If $x \in \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$, then we denote by V_x the K-submodule of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ generated by x. Also, we define the following sets of natural numbers:

$$\mathbf{J}_\sigma = \begin{cases} \mathbb{N} & : r > 2 \\ \mathbb{N}_{\geq 1} & : r \in \{1, 2\} \\ \varnothing & : r = 0. \end{cases} \qquad \mathbf{Q}_\sigma = \begin{cases} \mathbb{N} & : r < p-3 \\ \mathbb{N}_{\geq 1} & : r \in \{p-3, p-2\} \\ \varnothing & : r = p-1. \end{cases}$$

Proposition 3.7. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight, and let $A \in \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ be a non-zero element of the linear span of the set $\mathcal{C}_\sigma$, where*

$$\mathcal{C}_\sigma = \begin{cases} \{J_n^1(\sigma) : n \in \mathbf{J}_\sigma\} \cup \{Q_n^1(\sigma) : n \in \mathbf{Q}_\sigma\} & : r \notin \{1, p-2\} \\ \{J_n^1(\sigma) : n \in \mathbf{J}_\sigma\} \cup \{Q_n^1(\sigma) : n \in \mathbf{Q}_\sigma\} \cup \{\mathrm{id} \otimes x\} & : r = 1 \\ \{J_n^1(\sigma) : n \in \mathbf{J}_\sigma\} \cup \{Q_n^1(\sigma) : n \in \mathbf{Q}_\sigma\} \cup \{\alpha \otimes y^{p-2}\} & : r = p-2. \end{cases} \tag{25}$$

In the latter two cases, suppose that A is not a scalar multiple of $\mathrm{id} \otimes x$ or $\alpha \otimes y^{p-2}$, respectively. Then the submodule $V_A \subset \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ is reducible.

Proof. Suppose first that $r \notin \{1, p-2\}$. Write $A = A_J + A_Q$, where $A_* \in \mathrm{span}\{*_n^1\}$ for $* \in \{J, Q\}$. Note that $A_J, A_Q \in (\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))^{I(1)}$, since this is true for all elements of the spanning set (25). By Lemma 2.2, A_J and A_Q are I-eigenvectors, and I acts on them via the characters χ_J and χ_Q, respectively, where

$$\begin{aligned} \chi_J(\mathrm{diag}(a,d)) &= \overline{a}^{w+1} \overline{d}^{w+r-1} \\ \chi_Q(\mathrm{diag}(a,d)) &= \overline{a}^{w+r+1} \overline{d}^{w-1}. \end{aligned} \tag{26}$$

Clearly these characters are distinct when $0 < r < p-1$, namely in the cases where $\mathbf{J}_\sigma$ and $\mathbf{Q}_\sigma$ are both non-empty. Observe that for $* \in \{J, Q\}$ we have

$$\sum_{a,d \in k^*} \chi_*^{-1} \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} A = (p-1)^2 A_* = A_*$$

and hence that $V_{A_*} \subset V_A$ and we may assume without loss of generality that $A = A_J$ or $A = A_Q$.

So suppose that $A = A_*$. Observe that the image of βA is an $I(1)$-invariant lying in the span of $\{*_n^0 : n \in *_\sigma\}$. All the $*_n^0$ lie in $\mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))$, and I acts on them all via the same character; this follows from Proposition 2.10 and Corollary 2.12 when $* = J$ and from Proposition 2.19 when $* = Q$. By Lemma 3.6, βA generates an irreducible KZ-module isomorphic, say, to the Serre weight η. Let $0 \neq v_\eta \in \eta^{I(1)}$. By Frobenius reciprocity this produces a map of G-modules $\Phi : \mathrm{ind}_{KZ}^G \eta \to \mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ which, up to multiplication by a scalar, satisfies $\Phi(\mathrm{id} \otimes v_\eta) = \beta A$. Hence $\Phi(\alpha \otimes w v_\eta) = \Phi(\beta(\mathrm{id} \otimes v_\eta)) = A$. Write $\beta A = \sum_{n \in *_\sigma} c_n *_n^0 (\sigma)$, for $c_n \in \overline{\mathbb{F}}_p$. Observe that

$$\Phi(T(\mathrm{id} \otimes v_\eta)) = \Phi(\sum_{\mu \in I_1} g_{1,\mu}^0 (\mathrm{id} \otimes v_\eta)) = \sum_{\mu \in I_1} g_{1,\mu}^0 \cdot \beta A = \sum_n c_n *_{n+1}^0 (\sigma) \neq 0,$$

from which it follows by Lemma 3.1 that the KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ generated by A is reducible. This completes the proof if $r \notin \{1, p-2\}$.

If $r = 1$, then the image of $\mathrm{id} \otimes x$ in $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ generates an irreducible K-module that is isomorphic to those generated by the $J_n^1(\sigma)$. Hence, if we write $A = A_J + A_Q$, where now $A_J \in \mathrm{span}\{\{J_n^1 : n \geq 1\} \cup \{\mathrm{id} \otimes x\}\}$, then it is easy to see that the argument above works. Similarly, if $r = p-2$, then $\alpha \otimes y^{p-2}$ generates an irreducible K-module isomorphic to those generated by the $Q_n^1(\sigma)$, and the same argument works if we modify A_Q. $\square$

Theorem 3.8. *Let* $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ *be a Serre weight. Define the set*

$$\tilde{\mathcal{B}}_\sigma = \begin{cases} \bigcup_{n\geq 0}\{J_n^0, Q_n^0\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : 2 < r < p-3 \\ \bigcup_{n\geq 1}\{J_n^0, J_n^1\} \cup \{\hat{J}_0^0, \hat{J}_0^1\} \cup \bigcup_{n\geq 0}\{Q_n^0\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = 2 \\ \bigcup_{n\geq 1}\{J_n^0\} \cup \bigcup_{n\geq 0}\{Q_n^0\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = 1 \\ \bigcup_{n\geq 0}\{Q_n^0\} \cup \{\mathrm{id}\otimes 1, \alpha\otimes 1\} & : r = 0 \\ \bigcup_{n\geq 0}\{J_n^0\} \cup \bigcup_{n\geq 1}\{Q_n^0, Q_n^1\} \cup \{\hat{Q}_0^0, \hat{Q}_0^1\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = p-3 \\ \bigcup_{n\geq 0}\{J_n^0\} \cup \bigcup_{n\geq 1}\{Q_n^0\} \cup \{\mathrm{id}\otimes x^r, \alpha\otimes y^r\} & : r = p-2 \\ \bigcup_{n\geq 0}\{J_n^0\} \cup \{\mathrm{id}\otimes x^{p-1}, \alpha\otimes y^{p-1}\} & : r = p-1. \end{cases}$$

Define $\mathcal{A}_\sigma = \mathrm{span}\left\{\tilde{\mathcal{B}}_\sigma\right\}$. *Then*

$$(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))^{I(1)} \cap \mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)) = \mathcal{A}_\sigma. \tag{27}$$

Proof. It is immediate from Theorem 2.24 that $\mathcal{A}_\sigma \subset (\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))^{I(1)}$ for all Serre weights σ. Moreover, it follows from Lemma 2.16 and from Corollary 2.6 applied to the results of Proposition 2.10, Corollary 2.12, and Proposition 2.19, and from Corollary 3.5, that $\mathcal{A}_\sigma$ generates a semisimple K-module $\langle \mathcal{A}_\sigma \rangle$ and hence $\mathcal{A}_\sigma \subset \mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma))$.

Let $\mathcal{S}$ denote the left-hand side of (27). It remains to show that $\mathcal{S} \subseteq \mathcal{A}_\sigma$. Suppose not. Then there exists a K-submodule $\mathcal{N}$ of the socle such that $\mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)) = \langle \mathcal{A}_\sigma \rangle \oplus \mathcal{N}$. Let $x \in \mathcal{N}^{I(1)}$ be a non-zero $I(1)$-invariant; it is a standard fact that such an element exists. Writing x in terms of the basis $\mathcal{B}_\sigma$ of Theorem 2.24, we see that, since $x \notin \langle \mathcal{A}_\sigma \rangle$, some of the elements J_n^1 or Q_n^1 must appear in x with non-zero coefficients. Moreover, if $r = 2$ (respectively, if $r = p-3$), then some of the Q_n^1 (respectively, J_n^1) must appear with non-zero coefficients. Thus we may uniquely write $x = x_J + x_Q + x'$, where I acts on x_J and x_Q by the characters χ_J and χ_Q of (26), respectively, and x' is a linear combination of elements of $\mathcal{B}_\sigma$ that are eigenvectors for other characters of I. We know that at least one of x_J and x_Q (and specifically x_Q if $r = 2$ and x_J is $r = p-3$) is non-zero. As in the proof of Proposition 3.7, we may replace x by x_J or x_Q, as suitable. By Lemma 3.6, this element generates an irreducible K-module. But that contradicts Proposition 3.7. □

Corollary 3.9. *Let* $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ *be a Serre weight. The K-socle of* $\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)$ *is isomorphic to:*

$$\begin{cases} \mathcal{S}_{\chi_\sigma} \oplus \mathcal{S}_{\chi_\sigma^s} \oplus \bigoplus_{\mathbb{N}} \left(\mathcal{S}_{\chi_\sigma \varepsilon} \oplus \mathcal{S}_{\chi_\sigma^s \varepsilon}\right) & : 1 < r < p-2 \\ \mathcal{S}_{\chi_\sigma} \oplus \bigoplus_{\mathbb{N}} \mathcal{S}_{\chi_\sigma \varepsilon} & : r \in \{0, p-1\}. \end{cases}$$

Proof. Every irreducible component of the K-socle has a one-dimensional space of $I(1)$-invariants; see, for instance, [BL, Lemma 2]. The K-modules generated by the elements of $\tilde{\mathcal{B}}_\sigma$ were computed explicitly in this paper, and we observe that finite direct sums of these modules have trivial intersections. Therefore it follows from Theorem 3.8 that

$$\mathrm{soc}_K(\mathrm{ind}_{KZ}^G \sigma / T(\mathrm{ind}_{KZ}^G \sigma)) = \bigoplus_{x \in \tilde{\mathcal{B}}_\sigma} V_x.$$

The Serre weights to which the various V_x are isomorphic were determined in the propositions referenced in the proof of Theorem 3.8. □

Remark 3.10. We note in passing that $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ has no non-zero irreducible G-submodules. Indeed, let $0 \neq U \subset \mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ be a G-submodule. Then $U_{|K}$ must contain an irreducible K-submodule τ by the compactness of K, and hence there is a map $\Psi : \mathrm{ind}_{KZ}^G\tau \to U$. Since $\mathrm{ind}_{KZ}^G\tau$ is generated by a single $I(1)$-invariant, the image of Ψ is also generated by a single $I(1)$-invariant. However, it is easy to see from a case-by-case analysis of Theorem 2.24 that for any $f \in (\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma))^{I(1)}$ one can construct another $I(1)$-invariant f' such that the G-submodule of $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ generated by f' is strictly contained in that generated by f, and this implies the reducibility of the image of Ψ and hence of U.

For instance, if $2 < r < p-3$, then f is a linear combination of the elements of $\mathcal{B}_\sigma$. As in the proof of Proposition 3.7, we can replace f by an $I(1)$-invariant f'' that lies in the K-submodule generated by f and consists of the components of f that are eigenvectors for a given character of I. If f'' is a scalar multiple of $\mathrm{id}\otimes x^r$ or of $\alpha\otimes y^r$, then $U = \mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$ and is reducible. If $f'' = \sum_{n=0}^N c_n J_n^0$, then we may take

$$f' = \sum_{\mu\in I_1} g_{1,\mu}^0 f'' = \sum_{n=0}^N c_n J_{n+1}^0,$$

with a similar construction if f'' is a linear combination of the J_n^1 or the Q_n^0 or the Q_n^1.

4. Deeper into the composition series

In the second section of this paper we made a few steps towards the ultimate goal of understanding the socle filtrations as G-modules of the universal supersingular representations $\mathrm{ind}_{KZ}^G\sigma/T(\mathrm{ind}_{KZ}^G\sigma)$. We introduced the chain $\tilde{\mathcal{J}}_1 \subset \tilde{\mathcal{J}}_2 \subset \cdots \subset \tilde{\mathcal{J}}_{r'(\sigma)}$ of G-submodules, but it is clear that the successive quotients $\tilde{\mathcal{J}}_{i+1}/\tilde{\mathcal{J}}_i$ are themselves non-admissible and of infinite length. Therefore we are far from understanding even the first few steps of the socle filtration. In this section we will construct a few more G-submodules. They should convince the reader that it quickly becomes complicated to proceed by explicit computations.

In this section we assume $p > 5$; as in the previous sections, it is possible to obtain results for smaller primes using the same methods, but they must be treated separately.

Lemma 4.1. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight such that $r < p-3$. Let $1 \leq i \leq r'(\sigma)$. Then the image of*

$$L_i = \sum_{\mu\in I_2} g_{2,\mu}^0 \otimes \mu_1^{r-2i+1} x^{r-i} y^r \in \mathrm{ind}_{KZ}^G\sigma$$

in the quotient $\mathrm{ind}_{KZ}^G\sigma/\tilde{\mathcal{J}}_i$ is invariant under the action of $I(1)$.

Proof. We use the notation of the proof of Lemma 2.7. Let $\gamma \in I(1)$. Define the polynomials

$$\begin{aligned} P_{2,1}^\gamma(\nu) &= b_1 - d_0\nu_0 + (\nu_0 - b_0)(a_0 - c_0\nu_0) \\ P_{2,2}^\gamma(\nu) &= b_2 + d_1\nu_0 + d_0\nu_1 + \delta_{e,2}\frac{\nu_0^{p^2} - b_0^{p^2} - (\nu_0 - b_0)^{p^2}}{\pi^2}. \end{aligned}$$

Here $\delta_{e,2}$ is the usual Kronecker delta function. By a computation analogous to those in the proof of Lemma 2.9 we see that

$$\gamma L_i = \sum_{\nu\in I_2} g_{2,\mu}^0 \otimes (\nu_1 - P_{2,1}^\gamma(\nu))^{r-2i+1} x^{r-i} (P_{2,2}^\gamma(\nu)x + y)^i.$$

Observe that $P^{\gamma}_{2,1}(\nu)$ depends only on ν_0, whereas $P^{\gamma}_{2,2}(\nu)$ is linear in ν_1. Therefore,

$$\gamma L_i - L_i = \sum_{\nu\in I_2} g^0_{2,\nu} \otimes \sum_{j=0}^{i} Q_j(\nu) x^{r-j} y^j, \tag{28}$$

where for each $0 \leq j < i$ the polynomial $Q_j(\nu)$ has degree $r - 2i + 1 + j$ in the variable ν_1. This is strictly smaller than $r - 2j$, and hence by Remark 2.8 the corresponding summand of (28) lies in $\tilde{\mathcal{J}}_j \subset \tilde{\mathcal{J}}_i$. The remaining summand is

$$\sum_{\nu\in I_2} g^0_{2,\nu} \otimes ((\nu_1 - P^{\gamma}_{2,1}(\nu))^{r-2i+1} - \nu_1^{r-2i+1}) x^{r-i} y^i,$$

and by Lemma 2.7 and Remark 2.8 the image of this in the quotient $\mathrm{ind}^G_{KZ}\sigma/\tilde{\mathcal{J}}_i$ is equal to the image of

$$(-1)^{r-i} \sum_{\nu_0\in I_1} \mathrm{id} \otimes P^{\gamma}_{2,1}(\nu_0) x^i (\nu_0 x + y)^{r-i},$$

but this element vanishes. Indeed, $P^{\gamma}_{2,1}(\nu_0)$ is quadratic in ν_0, and hence the degree of ν_0 in each of the terms of the expression above is at most $r - i + 2$, which is strictly smaller than $p - 1$. Thus, $\gamma L_i - L_i \in \tilde{\mathcal{J}}_i$. □

Throughout this section we maintain the hypothesis that $r \leq p - 3$.

Definition 4.2. Let $0 \leq i \leq r'(\sigma)$. We define $\mathcal{J}_i$ to be the G-submodule of $\mathrm{ind}^G_{KZ}\sigma$ generated by the set $\{J_0, J_1, \ldots, J_i\} \cup \{L_1, \ldots, L_i\}$.

Lemma 4.3. *Let $0 \leq i \leq r'(\sigma)$. Then the following identities hold in the quotient $\mathrm{ind}^G_{KZ}\sigma/\mathcal{J}_{i-1}$:*

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} L_i = {} & \sum_{\nu\in I_2} g^0_{2,\nu} \otimes \left(\sum_{j=0}^{i} (-\lambda)^{i-j} \binom{i}{j} (1-\lambda\nu_0)^{p-r+i+j-3} \nu_1^{r-2j+1} x^{r-j} y^j \right) \\
& + \sum_{\tau\in I_1} g^1_{1,\tau} \otimes \left(\sum_{j=0}^{i} (-1)^{r-j+1} \binom{i}{j} \lambda^{p-r+i+j-3} \tau^{r-2j+1} x^j y^{r-j} \right) \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L_i = {} & \sum_{\nu\in I_2} g^0_{2,\nu} \otimes (-1)^{i-1} \left(\sum_{j=0}^{i} \binom{i}{j} \nu_0^{p-r+i+j-3} \nu_1^{r-2j+1} x^{r-j} y^j \right) \\
& + \sum_{\tau\in I_1} g^1_{1,\tau} \otimes \tau^{r-2i+1} x^i y^{r-i}
\end{aligned}$$

Proof. The identities are obtained by direct computation. □

Fix $0 \leq i \leq r'(\sigma)$. Given $0 \leq g \leq p - r + 2i - 3$, we define an element $F^i_g \in \mathrm{ind}^G_{KZ}\sigma$ as follows.

Set $k = p - r + 2i - 3 - g$. If $p - r + i - 3 \leq g \leq p - r + 2i - 3$, then $0 \leq k \leq i$. In this case we define

$$F_g^i = \sum_{\nu \in I_2} g_{2,\nu}^0 \otimes \left(\sum_{j=0}^{i} \binom{i}{j} \binom{p-r+i+j-3}{k} \nu_0^{g-(i-j)} \nu_1^{r-2j+1} x^{r-j} y^j \right) +$$

$$(-1)^{i-1} \sum_{\tau \in I_1} g_{1,\tau}^1 \otimes \binom{i}{k} \tau^{r-2i+2k+1} x^{i-k} y^{r-i+k}.$$

If $i \leq g \leq p - r + i - 3$, then k lies in the same range, and we define

$$F_g^i = \sum_{\nu \in I_2} g_{2,\nu}^0 \otimes \left(\sum_{j=0}^{i} \binom{i}{j} \binom{p-r+i+j-3}{k} \nu_0^{g-(i-j)} \nu_1^{r-2j+1} x^{r-j} y^j \right).$$

Finally, if $0 \leq g \leq i - 1$, then we set

$$F_g^i = \sum_{\nu \in I_2} g_{2,\nu}^0 \otimes \left(\sum_{j=i-g}^{i} \binom{i}{j} \binom{p-r+i+j-3}{k} \nu_0^{g-(i-j)} \nu_1^{r-2j+1} x^{r-j} y^j \right). \tag{29}$$

Corollary 4.4. *Let $0 \leq i \leq r'(\sigma)$. The KZ-submodule of $\mathrm{ind}_{KZ}^G \sigma / \mathcal{J}_{i-1}$ generated by the image of L_i is irreducible and isomorphic to $\det^{w+r-i+1} \otimes \mathrm{Sym}^{p-r+2i-3} \overline{\mathbb{F}}_p^2$. An $\overline{\mathbb{F}}_p$-basis of it is given by the images of the elements $\{F_g^i : 0 \leq g \leq p - r + 2i - 3\}$.*

Proof. Since the image of L_i in $\mathrm{ind}_{KZ}^G \sigma / \mathcal{J}_{i-1}$ is an $I(1)$-invariant by Lemma 4.1, the KZ-module it generates is clearly spanned by the set

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L_i \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} L_i : \lambda \in I_1 \right\}.$$

For any $0 \leq g \leq p - 1$, define

$$F_g^i = (-1)^{g-1} \sum_{\lambda \in I_1} \lambda^{p-1-g} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} L_i. \tag{30}$$

Observe that if $0 \leq g \leq p-r+2i-3$, then this is compatible with the definition of F_g^i stated above. A simple Vandermonde argument shows that the $\overline{\mathbb{F}}_p$-span of $\left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} L_i : \lambda \in I_1 \right\}$ is equal to that of $\{F_g^i : 0 \leq g \leq p-1\}$. Moreover, an inspection of the identities of Lemma 4.3 shows that $F_g^i = 0$ when $p-r+2i-3 < g \leq p-1$ and that the set $\{F_g^i : 0 \leq g \leq p-r+2i-3\}$ is linearly independent. Note that I acts on the $I(1)$-invariant L_i via the character $\chi : I \to \overline{\mathbb{F}}_p^*$ given by:

$$\chi : \begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} \mapsto \bar{a}^{i-1+w} \bar{d}^{r-i+1+w},$$

where $\bar{a}$ and $\bar{d}$ are the reductions modulo π of a and d, respectively. Our running hypothesis implies that χ does not factor through the determinant. By Frobenius reciprocity, we obtain a non-zero map

$$\mathrm{ind}_I^K \chi \to (\mathrm{ind}_{KZ}^G \sigma / \mathcal{J}_{i-1})_{|K} \tag{31}$$

whose image is the KZ-submodule generated by L_i. This map cannot be injective, since we already know that its image has dimension at most $p - r + 2i - 1$, and this is strictly smaller than $p+1$ by our assumption on i. Therefore, the image of (31) must be isomorphic to $\det^{w+r-i+1} \otimes \mathrm{Sym}^{p-r+2i-3}\overline{\mathbb{F}}_p^2$, since this is the unique non-trivial proper quotient of $\mathrm{Ind}_I^K \chi$. $\square$

From now on we assume that $F/\mathbb{Q}_p$ is a *quadratic* totally ramified extension. Indeed, the fact that the congruence

$$\sum_{j=0}^{\infty}[a_j]\pi^j + \sum_{j=0}^{\infty}[b_j]\pi^j \equiv \sum_{j=0}^{\infty}[a_j+b_j]\pi^j \mod \pi^q \tag{32}$$

fails for $q \geq e-1$ has not yet played a role in the computations we have done in this paper. We restrict to the case $e=2$ to consider the simplest case where (32) fails. For all $2 \leq i \leq r'(\sigma)$, we define the following element of $\mathrm{ind}_{KZ}^G \sigma$:

$$M_i = \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \mu_1^{r-2i+2} x^{r-i} y^i + \mathrm{id} \otimes (-1)^{r+i} \frac{i}{r-i+2} \frac{p}{\pi^2} x^{i-2} y^{r-i+2}.$$

Lemma 4.5. *Let $p>5$ be a prime number. Let $F/\mathbb{Q}_p$ be a quadratic totally ramified extension, and let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight such that $r \leq p-3$. For all $2 \leq i \leq r'(\sigma)$, the image of M_i in $\mathrm{ind}_{KZ}^G \sigma / \mathcal{J}_i$ is invariant under $I(1)$.*

Proof. It suffices to prove invariance under elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1+\pi a & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ \pi c & 1 \end{pmatrix}$, where $a,b,c \in \mathcal{O}$, since $I(1)$ is generated by these matrices modulo $I(1) \cap Z$. Now observe that

$$\begin{aligned} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} M_i &= \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes (\mu_1 - b_1)^{r-2i+2} x^{r-i} \left[\left(b_2 + \frac{(\mu_0-b_0)^{p^2} + b_0^{p^2} - \mu_0^{p^2}}{\pi^2}\right) x + y\right]^i + \\ & \quad \mathrm{id} \otimes (-1)^{r+i} \frac{i}{r-i+2} \frac{p}{\pi^2} x^{i-2} (b_0 x + y)^{r-i+2}. \end{aligned}$$

By Remark 2.8, we observe that in the quotient $\mathrm{ind}_{KZ}^G \sigma / \tilde{\mathcal{J}}_i$ we have:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} M_i - M_i = -b_1(r-2i+2)L_i + \tag{33}$$

$$\sum_{\mu \in I_2} g_{2,\mu}^0 \otimes i \left(b_2 + \frac{(\mu_0-b_0)^{p^2} + b_0^{p^2} - \mu_0^{p^2}}{\pi^2}\right) \mu_1^{r-2(i-1)} x^{r-i+1} y^{i-1} + \tag{34}$$

$$\mathrm{id} \otimes (-1)^{r+i} \frac{i}{r-i+2} \frac{p}{\pi^2} \sum_{m=0}^{r-i+1} \binom{r-i+2}{m} b_0^{r-i+2-m} x^{r-m} y^m. \tag{35}$$

By the identities of Lemma 2.7 we observe that in the quotient $\mathrm{ind}_{KZ}^G \sigma / \tilde{\mathcal{J}}_i$ the expression of (34) is equal to

$$\mathrm{id} \otimes (-1)^{r-i} i \sum_{\mu_0 \in I_1} \left(b_2 + \frac{(\mu_0-b_0)^{p^2} + b_0^{p^2} - \mu_0^{p^2}}{\pi^2}\right) x^{i-1} (\mu_0 x + y)^{r-i+1} =$$

$$\mathrm{id} \otimes (-1)^{r-i} \frac{i}{\pi^2} \sum_{\mu_0 \in I_1} \left(\sum_{m=1}^{p-1} \binom{p^2}{mp} \mu_0^{mp} (-b_0)^{(p-m)p}\right) x^{i-1} (\mu_0 x + y)^{r-i+1} =$$

$$\mathrm{id}\otimes(-1)^{r-i+1}i\frac{p}{\pi^2}\sum_{\mu_0\in I_1}\left(\sum_{m=1}^{p-1}\frac{1}{m}\mu_0^m b_0^{p-m}\right)x^{i-1}(\mu_0 x+y)^{r-i+1}=$$
$$\mathrm{id}\otimes(-1)^{r-i}i\frac{p}{\pi^2}\sum_{m=0}^{r-i+1}\binom{r-i+1}{m}\frac{1}{p-r-2+i+m}b_0^{r-i+2-m}x^{r-m}y^m=$$
$$\mathrm{id}\otimes(-1)^{r-i+1}i\frac{p}{\pi^2}\sum_{m=0}^{r-i+1}\binom{r-i+2}{m}\frac{1}{r-i+2}b_0^{r-i+2-m}x^{r-m}y^m,$$

and this last expression is just the negative of (35). All these equalities are modulo $\tilde{\mathcal{J}}_i$, and in the second equality we made use of the observation that

$$\binom{p^2}{pm}\equiv-\frac{p}{m}\binom{p-1}{m}\mod p^2.$$

for all $1\le m\le p-1$. Moreover, $\binom{p-1}{m}\equiv(-1)^{m+1}\mod p$. This proves the claimed invariance.

Now let $a\in\mathcal{O}$ and observe that

$$\begin{pmatrix}1+\pi a&0\\0&1\end{pmatrix}M_i=\sum_{\mu\in I_2}g_{2,\mu}^0\otimes(\mu_1-a_0\mu_0)^{r-2i+2}x^{r-i}((a_1(\mu_1-a_0\mu_0)+a_1\mu_0)x+y)^i+$$
$$\mathrm{id}\otimes(-1)^{r+i}\frac{i}{r-i+2}\frac{p}{\pi^2}x^{i-2}y^{r-i+2}.$$

Hence our assumption about p ensures that modulo $\tilde{\mathcal{J}}_i$ we have

$$\begin{aligned}\begin{pmatrix}1+\pi a&0\\0&1\end{pmatrix}M_i-M_i\equiv&\sum_{\mu\in I_2}g_{2,\mu}^0\otimes -a_0(r-2i+2)\mu_0\mu_1^{r-2i+1}x^{r-i}y^i+\\&\sum_{\mu\in I_2}g_{2,\mu}^0\otimes i(a_1(\mu_1-a_0\mu_0)+a_1\mu_0)(\mu_1-a_0\mu_0)^{r-2(i-1)}x^{r-i+1}y^{i-1}.\end{aligned}\tag{36}$$

By Remark 2.8, the expression in the second line of (36) is congruent modulo $\tilde{\mathcal{J}}_{i-1}$ to ia_1L_{i-1}. On the other hand, by (29) we see that

$$F_1^i=\sum_{\mu\in I_2}g_{2,\mu}^0\otimes\left((p-r+2i-3)\mu_0\mu_1^{r-2i+1}x^{r-i}y^i+i\mu_1^{r-2(i-1)+1}x^{r-i+1}y^{i-1}\right)\in\mathcal{J}_i.$$

Therefore, modulo $\mathcal{J}_i$ the first line of (36) is also congruent to a scalar multiple of L_{i-1} and hence to zero.

In similar fashion we find that $\begin{pmatrix}1&0\\\pi c&1\end{pmatrix}M_i-M_i\in\mathcal{J}_i$ for all $c\in\mathcal{O}$. □

Lemma 4.6. *Let $2 \leq i \leq r'(\sigma)$. If $\lambda \in I_1$, then the following identities hold in the quotient* $\mathrm{ind}_{KZ}^G \sigma / \mathcal{J}_i$*:*

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} M_i &= \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \sum_{j=0}^{i} \left((-\lambda)^{i-j} \binom{i}{j} (1-\lambda\mu_0)^{p-r+i+j-5} \mu_1^{r-2i+2} x^{r-j} y^j \right) + \\
& \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes \sum_{j=0}^{i-1} \left((-1)^{i-j} i \binom{i-1}{j} \lambda^{i-j-1} \mu_0 P(\mu_0) (1-\lambda\mu_0)^{p-r+i+j-3} \mu_1^{r-2j} x^{r-j} y^j \right) + \\
& \sum_{\tau \in I_1} g_{1,\tau}^1 \otimes (-1)^{r+i} \lambda^{p-r+2i-5} \tau^{r-2i+2} (x - \lambda^{p-2}\tau^2 y)^i y^{r-i} + \\
& \mathrm{id} \otimes (-1)^{r+i} \frac{i}{r-i+2} \frac{p}{\pi^2} (x+\lambda y)^{i-2} y^{r-i+2} \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M_i &= \sum_{\mu \in I_2} g_{2,\mu}^0 \otimes (-1)^r \left(\sum_{j=0}^{i} \binom{i}{j} \mu_0^{p-r+i+j-5} \mu_1^{r-2j+2} x^{r-j} y^j \right) + \\
& \sum_{\tau \in I_1} g_{1,\tau}^1 \otimes \tau^{r-2i+2} x^i y^{r-i} + \\
& \mathrm{id} \otimes (-1)^{r+i} \frac{i}{r-i+2} \frac{p}{\pi^2} x^{r-i+2} y^{i-2}.
\end{aligned}$$

Here $P(\mu_0)$ is a polynomial of degree $p-1$ in the variable μ_0 defined by

$$P(\mu_0) = \frac{((1-\lambda\mu_0)^{p-2} - 1)^{p^2} + 1 - (1-\lambda\mu_0)^{(p-2)p^2}}{\pi^2}.$$

Proof. Computation. □

Proposition 4.7. *Let $\sigma = \det^w \otimes \mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ be a Serre weight such that $r < p-3$, and let $2 \leq i \leq r'(\sigma)$. Then the K-submodule of $\mathrm{ind}_{KZ}^G \sigma / \mathcal{J}_i$ generated by the image of M_i is irreducible and isomorphic to* $\det^{w+r-i+2} \otimes \mathrm{Sym}^{p-r+2i-5} \overline{\mathbb{F}}_p^2$.

Proof. By Lemma 4.5 and Frobenius reciprocity, there exists a map $\mathrm{Ind}_I^K \chi \to \mathrm{ind}_{KZ}^G \sigma / \mathcal{J}_i$ for a suitable character χ, whose image is the K-submodule generated by M_i. Since the principal series $\mathrm{Ind}_I^K \chi$ has length two, it suffices to show that this map is not injective to establish that its image is irreducible. Clearly the image is spanned by

$$\left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} M_i : \lambda \in I_1 \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

It is easy to see from Lemma 4.6 that the following linear relation holds:

$$\sum_{\lambda \in I_1} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} M_i = 0.$$

Therefore the K-submodule generated by M_i has dimension strictly smaller than $p+1 = \dim_{\overline{\mathbb{F}}_p}(\mathrm{Ind}_I^K \chi)$, and we are done. □

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