“Analyza Shimushit” 88-234

My plan for this course follows. The course — which is the only exposure most of our math majors will get to applied math — is intended to introduce the student to a variety of topics which are on the one hand central topics in analysis, and on the other hand have important applications. In particular, I expect to devote about a third of the course to each of ordinary differential equations, Fourier analysis, and the partial differential equations of mathematical physics.

In section 1, the first lecture should cover sections 1.1–1.2. The second lecture should cover 1.3 and 1.4. A full lecture is needed for each of 1.5 and 1.6. I am currently not using 1.7.

In section 2, sections 2.1–2.3, covering Fourier series, should be covered in two lectures, as should sections 2.4–2.6 covering Fourier transforms. Section 2.4, explaining the Fourier integral as a limit of Fourier series, can be taught in the same lecture as 2.3 if necessary. Note that the computation of examples in sections 2.1–2.3 takes a lot of time.

In section 3, I expect to devote a whole lecture to each of 3.1 and 3.2, one lecture to 3.3 and 3.4 together (they will be presented as simple extensions of the technique of 3.2), and one lecture to the material of 3.5 and 3.6 (the list of material here is too much for one lecture).

Targilim and Exams. I will distribute 4 sheets of targilim to the students, containing 3–4 questions relevant to each lecture. The targilim can be grouped into two categories: those that implement a “method” learnt in the course (these are indicated with a * in the course summary that follows), and those that encourage the student to explore a new branch of the theory. Both the written targilim and the exams I will give will consist of about 75% of the former and 25% of the latter. The students will be encouraged to write out and submit solutions to the targilim, and the targil grade will comprise 20% of the final grade.

1. Ordinary Differential Equations

1.1 Introduction. Definitions of differential equation and order. Linear differential equations: homogeneous, inhomogeneous, singular points. Integration of $y' = f(x)$. Separable equations $y' = f(x)g(y)$ *. Illustration of the existence and uniqueness theorem and the effect of singular points.

1.2 The linear first order equation $a_0(x)y' + a_1(x)y = b(x)$ *. Illustrations of the existence and uniqueness theorem, the effect of singular points and the rule that the general solution of an inhomogeneous linear equation is the sum of a particular integral and the complementary function (the general solution of the associated homogeneous equation).

1.3 Prelude: integration of $y'' = f(x)$ and the rough rule that the general solution of an nth order equation has n parameters. The homogeneous second order constant coefficient equation *. Solution in the general case (without proof that it is the general solution at this stage).

1.4 Inhomogeneous second order constant coefficient equations. Proof that the general solution of an inhomogeneous differential equation is the sum of a particular integral and the complimentary function. Reduction of order for the general second order linear equation: given any nontrivial solution of the associated homogeneous equation, the equation can be reduced to an equation of first order *. Use to complete the proof left from 1.3 and to solve inhomogeneous problems. Solution of simple inhomogeneous second order constant coefficient equations by the method of unknown coefficients *.

1.5 Mechanical oscillations (the equation $y'' + 2ky' + \Omega^2 y = F \cos \omega t$). For $F = 0$, physical description of the solution in the cases $k = 0, \quad k < \Omega, \quad k = \Omega, \quad k > \Omega$. The forced response: mechanical impedance, resonance at $\omega = \Omega$ for small k. The complementary function as a transient. LCR circuits.

1.6 Green's functions for the initial value problem of a second order linear equation. The Wronskian and linear independence of solutions (2nd order equation only). The method of variation of coefficients for solving an inhomogeneous equation when the complementary function is known *. The formula for the solution of $y'' + a_1(x)y' + a_2(x)y = b(x), \; y(0) = y'(0) = 0$, when two linearly independent solutions of the homogeneous equation are known. The Green's function. An integral operator as the inverse of a differential operator.

1.7 Existence and Uniqueness. Statement of the existence and uniqueness theorem for the problem $y' = f(x, y)$, $y(x_0) = y_0$ (no proof). Singular points: definition, some simple types. Reduction of an nth order equation to a n-dimensional system. Statement of existence and uniqueness for the system $\vec{y}' = \vec{f}(x, \vec{y})$, $\vec{y}(x_0) = \vec{y_0}$.

2. Fourier Analysis

2.1 Fourier series. Periodic functions, piecewise continuous functions, "normalized" p.c. functions ($f(x) = \frac{1}{2}(f(x,+) + f(x,-))$ everywhere). Definition of Fourier series, justification of choice of coefficients. Fourier's theorem (without proof). Examples of computation of series *.

2.2 Convergence of Fourier series. Speed of convergence of series for a function with r continuous derivatives. Lack of uniform convergence — demonstration of Gibbs phenomenon.

2.3 Half range series and series on arbitrary intervals. Sine and cosine series for a function defined on $(0, \pi)$ and their properties *. Full Fourier series for a function defined on $(c - L, c + L)$ *; sine and cosine series for a function defined on $(c, c + L)$.

2.4 The $L \to \infty$ limit and the Fourier integral. Derivation of Fourier's integral theorem by considering the $L \to \infty$ limit of Fourier series. Definition of the Fourier transform and precise statement of Fourier's integral theorem.

2.5 Properties of Fourier transforms. Computation of transforms (without use of the residue theorem), examples *. Fourier transforms of derivatives, convolutions, $e^{-a|t|}$ ($\Re(a) > 0$) etc. Sine and cosine transforms.

2.6 Solution of inhomogeneous constant coefficient second order linear differential equations via Fourier transforms *. Nature of the solution; relation with Green's function techniques.

3. The Partial Differential Equations of Mathematical Physics

3.1 Introduction. Laplace's equation, the wave equation and the diffusion equation in one, two or three spatial dimensions. Brief discussion of applications. The need for different types of initial/boundary conditions for different equations. Examples: D'Alembert's general solution for the wave equation in one spatial dimension, uniqueness of solution for Laplace's equation in two spatial dimensions

3.2 Separation of variables in Cartesian coordinates. Basic idea. Examples: wave and diffusion equations on a bounded one dimensional spatial region, with simple homogeneous boundary conditions *; Laplace's equation on a bounded rectangular two dimensional region *. Emphasis on the appearance of a two point problem for a second order ODE.

3.3 The diffusion equation on a bounded one dimensional region with slightly more complicated homogeneous boundary conditions. Generalized Fourier series.

3.4 Simple Fourier transform techniques for PDEs: problems on infinite and semi-infinite spatial regions *.

3.5 Separation of variables in polar coordinates for three dimensional Laplace equation (axisymmetric case only). Legendre's equation, and Legendre polynomials as solutions. (Power series expansion for solution of Legendre's equation; polynomial solutions for integer l — if time.) Comparison of types of problem met in separation of variables in Cartesian and polar coordinates.

3.6 Legendre Polynomials. Rodrigue's formula, generating function, recursion relations, orthogonality, completeness, expansions in Legendre functions *.