Legendre Polynomials, $P_n(y)$

Generated using the three-term recursion: $(n+1)P_{n+1}(y) - (2n+1)yP_n(y) + nP_{n-1}(y) = 0$

Orthogonality relations:

$$\int_{-1}^{1} P_n(y)P_m(y)dy = 0 \ , \qquad n \neq m$$

$$\int_{-1}^{1} (P_n(y))^2 dy = \frac{2}{2n+1}$$

First few:

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= y \\
P_2 &= \frac{3}{2}\,y^2 - \frac{1}{2} \\
P_3 &= \frac{5}{2}\,y^3 - \frac{3}{2}\,y \\
P_4 &= \frac{35}{8}\,y^4 - \frac{15}{4}\,y^2 + \frac{3}{8} \\
P_5 &= \frac{63}{8}\,y^5 - \frac{35}{4}\,y^3 + \frac{15}{8}\,y \\
P_6 &= \frac{231}{16}\,y^6 - \frac{315}{16}\,y^4 + \frac{105}{16}\,y^2 - \frac{5}{16} \\
P_7 &= \frac{429}{16}\,y^7 - \frac{693}{16}\,y^5 + \frac{315}{16}\,y^3 - \frac{35}{16}\,y \\
P_8 &= \frac{6435}{128}\,y^8 - \frac{3003}{32}\,y^6 + \frac{3465}{64}\,y^4 - \frac{315}{32}\,y^2 + \frac{35}{128}
\end{aligned}$$

Note: $P_n(y)$ even if n even, odd if n odd.

For orthogonal polynomials with respect to the weight function 1 on the interval $[a, b]$ use

$$P_n\left(\frac{2x-b-a}{b-a}\right) \qquad n = 0, 1, \ldots$$

These satisfy

$$\int_a^b P_n\left(\frac{2x-b-a}{b-a}\right) P_m\left(\frac{2x-b-a}{b-a}\right) dx = 0 \ , \qquad n \neq m$$

$$\int_a^b P_n\left(\frac{2x-b-a}{b-a}\right)^2 dx = \frac{b-a}{2n+1}$$

How to approximate a function $f(x)$ **on the interval** $[a, b]$

Suppose we want to find a polynomial p(x) of degree N which approximates $f(x)$ in the sense that

$$\int_a^b (f(x) - p(x))^2 dx$$

is minimum. The answer:

$$p(x) = \sum_{n=0}^{N} a_n P_n\left(\frac{2x-b-a}{b-a}\right)$$

where

$$a_n = \frac{2n+1}{b-a} \int_a^b f(x) P_n\left(\frac{2x-b-a}{b-a}\right) dx \ .$$

If $f(x)$ is a simple function you can use Maple to get the integral.