Gaussian-Legendre Quadrature

The basic formula:

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

where the *nodes* x_i, $i = 1, \ldots, n$ are the n roots of the nth Legendre polynomial $P_n(x)$ and the *weights* w_i are given by

$$w_i = \int_a^b w(x) L_i(x) dx$$

where $L_i(x)$ is the Lagrange cardinal function

$$L_i(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)}{(x_i - x_1)(x_i - x_2) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)}$$

The first few sets of nodes and weights can be found exactly:

	node	weight
$n = 1$	0	2
$n = 2$	$\pm\frac{1}{\sqrt{3}}$	1
$n = 3$	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
$n = 4$	$\pm\frac{1}{35}\sqrt{525 + 70\sqrt{30}}$	$\frac{1}{2} - \frac{\sqrt{5}}{6\sqrt{6}}$
	$\pm\frac{1}{35}\sqrt{525 - 70\sqrt{30}}$	$\frac{1}{2} + \frac{\sqrt{5}}{6\sqrt{6}}$
$n = 5$	0	$\frac{128}{225}$
	$\pm\frac{1}{21}\sqrt{245 + 14\sqrt{70}}$	$\frac{161}{450} - \frac{13\sqrt{70}}{900}$
	$\pm\frac{1}{21}\sqrt{245 - 14\sqrt{70}}$	$\frac{161}{450} + \frac{13\sqrt{70}}{900}$

The nodes are always symmetrically placed around 0, and the sum of the weights is always 2 (can you prove this?).