בס"ד

Numerical Methods for Mathematical Finance, 88-636
Moed Aleph, Semester Aleph, 5773

Exam length: 90 minutes
You may use all reference materials and a pocket calculator.
Answer all the questions. All questions carry equal weight.
Explain all your answers thoroughly.

1. The stochastic process $X(t)$ satisfies the stochastic differential equation

$$dX = aX(2 - X)dt + \sigma dW$$

and $X(0) = 0$. It is required to compute

$$p(T) = P(\max_{0<t<T} X(t) < 1).$$

(a, σ and T are all positive constants.)

   (a) Write down the Euler-Maruyama method for simulating the process $X(t)$, and explain how you would use this in a Monte Carlo simulation to find $p(T)$ for some specific time T.

   (b) Do you expect $p(T)$ to be an increasing or decreasing function of the parameters a, σ and T ?

   (c) What are sources of error in the calculation of $p(T)$?

   (d) How does the calculation need to be modified if the SDE is changed to $dX = -2a\log(1 - X)dt + \sigma dW$? (Note $\log(1 - X)$ is only defined for $X < 1$.)

2. The price $S(t)$ of a certain asset follows a geometric Brownian motion

$$dS = S(rdt + \sigma dW) .$$

A "can't lose" contract with expiration T and barrier $B \geq S(0)$ pays the holder, at time T, $S(T)$ if $S(T) \geq S(0)$ and $\max_{0<t<T} S(t) \leq B$, and $S(0)$ otherwise.

   (a) Explain why the (current) value of this contract rises with B. What is the value if $B = S(0)$? Explain why for large B the value is $S(0)\exp(-rT)$ plus the value of a call with strike $S(0)$.

   (b) Explain how you would perform a calculation to determine the value of the contract for fixed r, σ, T and B. You should not write explicit Matlab code, but you should explain all the necessary considerations in writing such a program.

   (c) Explain how you would perform a calculation to determine the value B^* of the barrier for which the value of the contract is equal to $S(0)$.

   (d) The value of σ is not known exactly. How would you go about estimating the accuracy required in σ for the value of B^* to have an error of no more than 5% ?

3. Give a brief explanation of the Euler and Crank Nicolson methods for solution of

$$u_t = u_{xx} + (2 + \sin x)u_x \ , \qquad 0 < x < 2\pi, \quad t > 0$$

assuming Dirichlet boundary conditions $u(0, t)$ and $u(2\pi, t)$ are specified.
If instead of Dirichlet boundary conditions, *periodic boundary conditions*, i.e. that $u(0, t) = u(2\pi, t)$ and $u_x(0, t) = u_x(2\pi, t)$, are specified, how would you implement these in the Euler method? Why is the Crank Nicolson method ruined?

Good luck!