Question 2 on Targil 2

Ito's lemma: if $dX = a(X,t)dt + b(X,t)dW$ and $g = g(X,t)$ then

$$dg = (g_t + ag_X + \frac{1}{2}b^2 g_{XX})dt + bg_X dW$$

Taking $X = W$ (i.e. $a = 0, b = 1$) we have that for $g(W,t)$

$$dg = (g_t + \frac{1}{2}g_{WW})dt + g_W dW$$

Taking $g = W^{n+1}$ we have

$$d(W^{n+1}) = \frac{1}{2}n(n+1)W^{n-1}dt + (n+1)W^n dW$$

Integrating, dividing by $n+1$ and rearranging we have

$$\underline{\int_0^t W(s)^n \, dW(s) = \frac{W(t)^{n+1}}{n+1} - \frac{n}{2}\int_0^t W(s)^{n-1} \, ds}$$

Taking the expectation of both sides and using the fact that the expectation of an Ito integral is zero, we obtain

$$\underline{\mathbf{E}[W(t)^{n+1}] = \frac{n(n+1)}{2}\int_0^t \mathbf{E}[W(s)^{n-1}] \, ds}$$

Thus (taking $n = 3$)

$$\underline{\mathbf{E}[W(t)^4]} = 6\int_0^t \mathbf{E}[W(s)^2] \, ds = 6\int_0^t s \, ds \underline{= 3t^2}$$

and (taking $n = 5$)

$$\underline{\mathbf{E}[W(t)^6]} = 15\int_0^t \mathbf{E}[W(s)^4] \, ds = 45\int_0^t s^2 \, ds \underline{= 15t^3}$$

In general let us try $\mathbf{E}[W(t)^{2N}] = a_n t^N$. This works for $N = 0,1,2,3$ if $a_0 = a_1 = 1$, $a_2 = 3$, $a_3 = 15$ and in general, provided

$$a_N t^N = \frac{1}{2}(2N-1)(2N)\int_0^t a_{N-1}s^{N-1}ds = (2N-1)a_{N-1}t^N$$

Thus we have $a_N = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2N-1)$ Answer:

$$\underline{\mathbf{E}[W(t)^{2N}] = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2N-1)t^N}$$