Targil 5

1. **Question 1**

$$\begin{aligned}
f(x+2h) &= f(x)+(2h)f'(x)+\frac{(2h)^2}{2}f''(x)+\frac{(2h)^3}{6}f'''(x)+\frac{(2h)^4}{24}f''''(x)\\
f(x+h) &= f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(x)+\frac{h^4}{24}f''''(x)\\
f(x) &= f(x)\\
f(x-h) &= f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x)+\frac{h^4}{24}f''''(x)\\
f(x-2h) &= f(x)-(2h)f'(x)+\frac{(2h)^2}{2}f''(x)-\frac{(2h)^3}{6}f'''(x)+\frac{(2h)^4}{24}f''''(x)
\end{aligned}$$

So:

(a)

$$\frac{\alpha f(x+2h)+f(x+h)-(1+\alpha)f(x)}{(1+2\alpha)h} = f'(x)+\frac{\frac{1}{2}+2\alpha}{1+2\alpha}hf''(x)+\frac{\frac{1}{6}+\frac{4}{3}\alpha}{1+2\alpha}h^2f'''(x)$$

Thus the LHS is an approximation of $f'(x)$ with error $O(h)$ unless $\alpha=-\frac{1}{4}$ in which case the error is $O(h^2)$.

(b)

$$\begin{aligned}
\frac{f(x+h)-f(x-h)}{2h} &= f'(x)+\frac{1}{6}h^2f'''(x)+O(h^4)\\
\frac{f(x+2h)-f(x-2h)}{4h} &= f'(x)+\frac{2}{3}h^2f'''(x)+O(h^4)
\end{aligned}$$

and thus

$$\begin{aligned}
&\alpha\left(\frac{f(x+2h)-f(x-2h)}{2h}\right)+\frac{f(x+h)-f(x-h)}{2h}\\
=\ &(1+2\alpha)f'(x)+\left(\frac{1}{6}+\frac{4}{3}\alpha\right)h^2f'''(x)+O(h^4)
\end{aligned}$$

i.e.

$$\begin{aligned}
&\frac{\alpha f(x+2h)+f(x+h)-f(x-h)-\alpha f(x-2h)}{2(1+2\alpha)h}\\
=\ &f'(x)+\frac{\frac{1}{6}+\frac{4}{3}\alpha}{1+2\alpha}h^2f'''(x)+O(h^4)
\end{aligned}$$

Thus the LHS is an approximation of $f'(X)$ with error $O(h^2)$ unless $\alpha=-\frac{1}{8}$ in which case the error is $O(h^4)$.

(c) Similarly:

$$\begin{aligned}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} &= f''(x)+\frac{1}{12}h^2f''''(x)+O(h^4)\\ \frac{f(x+2h)-2f(x)+f(x-2h)}{4h^2} &= f''(x)+\frac{1}{3}h^2f''''(x)+O(h^4)\end{aligned}$$

So

$$\begin{aligned}&\frac{\alpha f(x+2h)+f(x+h)-2(1+\alpha)f(x)+f(x-h)+\alpha f(x-2h)}{(1+4\alpha)h^2}\\ =\ & f''(x)+\frac{\frac{1}{12}+\frac{4}{3}\alpha}{1+4\alpha}h^2f''''(x)+O(h^4)\end{aligned}$$

An approximation to $f''(x)$ with error $O(h^2)$ unless $\alpha=-\frac{1}{16}$ in which case the error is $O(h^4)$.

2. **Question 2** In the Euler method we consider only the values of the function $u(x,t)$ at lattice points $u(ih,jk)$ where h is the stepsize in x, k is the stepsize in t and i,j are integers (with suitable ranges). We construct approximations u_{ij} to $u(ih,jk)$.

   (a) The index i runs from 0 to N and $h=\frac{1}{N}$. The index j runs from 0. The initial condition gives $u_{i0}=ih$. The boundary conditions give $u_{0j}=0$ and $u_{Nj}=1$. For $j\geq 0$ and $1\leq i\leq N-1$

$$\begin{aligned}u_{i,j+1} &= u_{i,j}+\frac{k}{h^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)+k\cdot\alpha\cdot ih\cdot\frac{u_{i+1,j}-u_{i-1,j}}{2h}\\ &= \left(\frac{k}{h^2}+\alpha ki\right)u_{i+1,j}+\left(1-\frac{2k}{h^2}\right)u_{i,j}+\left(\frac{k}{h^2}-\alpha ki\right)u_{i-1,j}\end{aligned}$$

   Note the use of the symmetric approximation to u_x, with error $O(h^2)$.

   (b) The index i runs from 0 to N and $h=\frac{1}{N}$. The index j runs from 0. The initial condition gives $u_{i0}=f(ih)$. The boundary conditions give $u_{0j}=u_{Nj}=0$. For $j\geq 0$ and $1\leq i\leq N-1$

$$u_{i,j+1}=u_{i,j}+\frac{k}{h^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)+kg(ih,jk)$$

   (c) You can either divide the interval into $2N$ steps, taking $h=\frac{1}{N}$, and the index i running from $-N$ to N. Or divide the interval into N steps, take $h=\frac{2}{N}$ and take i running from 0 to N. I will do the first. The initial condition gives $u_{i0}=\frac{1}{2}i^2h^2$ (for $-N\leq i\leq N$). The equation gives

$$u_{i,j+1}=u_{i,j}+\frac{k}{h^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)$$

   for $-N+1\leq i\leq N-1$ and $j\geq 0$. Finally, we find u_{ij} for $i=\pm N$ and $j>0$ by using the following discretization of the boundary conditions:

$$\begin{aligned}1 &= \frac{1}{h}\left(\frac{1}{2}u_{N-2,j}-2u_{N-1,j}+\frac{3}{2}u_{N,j}\right)\\ -1 &= \frac{1}{h}\left(-\frac{1}{2}u_{-N+2,j}+2u_{-N+1,j}-\frac{3}{2}u_{-N,j}\right)\end{aligned}$$

for $j > 0$. (Compare with question 1a.) Note it is a mistake to use the simpler approximation

$$\begin{aligned} 1 &= \frac{1}{h}\left(u_{N,j} - u_{N-1,j}\right) \\ -1 &= \frac{1}{h}\left(u_{-N+1,j} - u_{-N,j}\right) \end{aligned}$$

This has error $O(h)$ and wrecks the $O(h^2)$ error of the Euler method.